3. Modeling default of single name

Market's assessment of the default risk of the obligor (assuming some form of market efficiency – information is aggregated in the market prices). The sources are

- market prices of bonds and other defaultable securities issued by the obligor
- prices of CDS's referencing this obligor's credit risk

How to construct a clean term structure of credit spreads from observed market prices?

⋆ Based on no-arbitrage pricing principle, a model that is based upon and calibrated to the prices of traded assets is immune to simple arbitrage strategies using these traded assets.

Market instruments used in bond price-based pricing

- At time t, the defaultable and default-free zero-coupon bond prices of all maturities $T \geq t$ are known. These defaultable zero-coupon bonds have no recovery at default.

- Information about the probability of default over all time horizons as assessed by market participants are fully reflected when market prices of default-free and defaultable bonds of all maturities are available.

Risk neutral probabilities

The financial market is modeled by a filtered probability space (Ω, $(\mathcal{F}_t)_{t\geq 0}, \mathcal{F}, Q$), where Q is the risk neutral probability measure.

- All probabilities and expectations are taken under Q. Probabilities are considered as state prices.

  1. For constant interest rates, the discounted Q-probability of an event A at time T is the price of a security that pays off \$1 at time T if A occurs.
  2. Under stochastic interest rates, the price of the contingent claim associated with A is $E[\beta(T)\mathbf{1}_A]$, where $\beta(T)$ is the discount factor. This is based on the risk neutral valuation principle and the money market account $M(T) = \dfrac{1}{\beta(T)} = e^{\int_t^T r_u\,du}$ is used as the numeraire.

Indicator functions

For $A \in \mathcal{F}$, $\mathbf{1}_A(\omega) = \begin{cases} 1 & \text{if } \omega \in A \\ 0 & \text{otherwise} \end{cases}$.

τ = random time of default; $I(t)$ = survival indicator function

$$I(t) = \mathbf{1}_{\{\tau > t\}} = \begin{cases} 1 & \text{if } \tau > t \\ 0 & \text{if } \tau \le t \end{cases} .$$

$B(t,T)$ = price at time t of zero-coupon bond paying off \$1 at T

$\overline{B}(t,T)$ = price of defaultable zero-coupon bond if $\tau > t$;

$$I(t)\overline{B}(t,T) = \begin{cases} \overline{B}(t,T) & \text{if } \tau > t \\ 0 & \text{if } \tau \le t \end{cases} .$$

Monotonicity properties on the bond prices

1. $0 \leq \overline{B}(t,T) < B(t,T), \quad \forall t < T$

2. Starting at $\overline{B}(t,t) = B(t,t) = 1$,

$$B(t,T_1) \geq B(t,T_2) > 0 \quad \text{and} \quad \begin{array}{l} \overline{B}(t,T_1) \geq \overline{B}(t,T_2) \geq 0 \\ \forall t < T_1 < T_2, \tau > t. \end{array}$$

Independence assumption

$\{B(t,T) | t \leq T\}$ and τ are independent under $(\Omega, \mathcal{F}, Q)$ (not the true measure).

Implied probability of survival in $[t,T]-$ based on market prices of bonds

$$B(t,T) = E\left[e^{-\int_t^T r_u\,du}\right] \quad \text{and} \quad \overline{B}(t,T) = E\left[e^{-\int_t^T r_u\,du} I(T)\right].$$

Invoking the independence between defaults and the default-free interest rates

$$\overline{B}(t,T) = E\left[e^{-\int_t^T r_u\,du}\right] E[I(T)] = B(t,T)P(t,T)$$

implied survival probability over $[t,T] = P(t,T) = \dfrac{\overline{B}(t,T)}{B(t,T)}$.

- The *implied default probability* over $[t,T]$, $P_{def}(t,T) = 1 - P(t,T)$.

- Assuming $P(t,T)$ has a right-sided derivative in T, the *implied density of the default time*

$$Q[\tau \in (T, T+dT]|\mathcal{F}_t] = -\frac{\partial}{\partial T} P(t,T)\, dT.$$

- If prices of zero-coupon bonds for all maturities are available, then we can obtain the implied survival probabilities for all maturities (complementary distribution function of the time of default).

Properties on implied survival probabilities, $P(t, T)$

1. $P(t, t) = 1$ and it is non-negative and decreasing in T. Also, $P(t, \infty) = 0$.

2. Normally $P(t, T)$ is continuous in its second argument, except that an important event *secheduled* at some time T_1 has direct influence on the survival of the obligor.

3. Viewed as a function of its first argument t, all survival probabilities for fixed maturity dates will tend to increase.

If we want to focus on the default risk over a given time interval in the future, we should consider conditional survival probabilities.

$$\begin{aligned} & \text{conditional survival probability over } [T_1, T_2] \text{ as seen from } t \\ = \; & P(t, T_1, T_2) = \frac{P(t, T_2)}{P(t, T_1)}, \quad \text{where } t \leq T_1 < T_2. \end{aligned}$$

Implied hazard rate (default probabilities per unit time interval length)

Discrete implied hazard rate of default over $(T, T + \Delta T]$ as seen from time t

$$H(t, T, T + \Delta T)\Delta T = \frac{P(t, T)}{P(t, T + \Delta T)} - 1 = \frac{P_{def}(t, T, T + \Delta T)}{P(t, T, T + \Delta T)},$$

so that

$$P(t, T) = P(t, T + \Delta T)[1 + H(t, T, T + \Delta T)\Delta T].$$

In the limit of $\Delta T \to 0$, the continuous hazard rate at time T as seen at time t is given by

$$h(t, T) = -\frac{\partial}{\partial T} \ln P(t, T).$$

Forward spreads and implied hazard rate of default

For $t \leq T_1 < T_2$, the simply compounded forward rate over the period $(T_1, T_2]$ as seen from t is given by

$$F(t, T_1, T_2) = \frac{B(t, T_1)/B(t, T_2) - 1}{T_2 - T_1}.$$

This is the price of the forward contract with expiration date T_1 on a unit-par zero-coupon bond maturing on T_2. To prove, we consider the compounding of interest rates over successive time intervals.

$$\underbrace{\frac{1}{B(t, T_2)}}_{\text{compounding over } [t, T_2]} = \underbrace{\frac{1}{B(t, T_1)}}_{\text{compounding over } [t, T_1]} \underbrace{[1 + F(t, T_1, T_2)(T_2 - T_1)]}_{\text{simply compounding over } [T_1, T_2]}$$

Defaultable simply compounded forward rate over $[T_1, T_2]$

$$\overline{F}(t, T_1, T_2) = \frac{\overline{B}(t, T_1)/\overline{B}(t, T_2) - 1}{T_2 - T_1}.$$

Instantaneous continuously compounded forward rates

$$f(t, T) = \lim_{\Delta T \to 0} F(t, T, T + \Delta T) = -\frac{\partial}{\partial T} \ln B(t, T)$$

$$\overline{f}(t, T) = \lim_{\Delta T \to 0} \overline{F}(t, T, T + \Delta T) = -\frac{\partial}{\partial T} \ln \overline{B}(t, T).$$

Implied hazard rate of default

Recall

$$
\begin{aligned}
P(t,T_1,T_2) &= \frac{\overline{B}(t,T_2)}{B(t,T_2)}\frac{B(t,T_1)}{\overline{B}(t,T_1)} \\
&= \frac{1+F(t,T_1,T_2)(T_2-T_1)}{1+\overline{F}(t,T_1,T_2)(T_2-T_1)} = 1 - P_{def}(t,T_1,T_2),
\end{aligned}
$$

and upon expanding, we obtain

$$
P_{def}(t,T_1,T_2)\underbrace{[1+\overline{F}(t,T_1,T_2)(T_2-T_1)]}_{\overline{B}(t,T_1)/\overline{B}(t,T_2)} = [\overline{F}(t,T_1,T_2)-F(t,T_1,T_2)](T_2-T_1).
$$

The implied hazard rate of default at time $T > t$ as seen from time t is the spread between the forward rates:

$$h(t,T) = \overline{f}(t,T) - f(t,T)$$

obtained using

$$\begin{aligned}\overline{f}(t,T) - f(t,T) &= -\frac{\partial}{\partial T}\ln\frac{\overline{B}(t,T)}{B(t,T)} \\ &= -\frac{\partial}{\partial T}\ln P(t,T) = h(t,T).\end{aligned}$$

The *local default probability* at time t over the next small time step Δt

$$\frac{1}{\Delta t} Q[\tau \leq t + \Delta t | \mathcal{F}_t \wedge \{\tau > t\}] \approx \overline{r}(t) - r(t) = \lambda(t)$$

where $r(t) = f(t,t)$ is the riskfree short rate and $\overline{r}(t) = \overline{f}(t,t)$ is the defaultable short rate.

Recovery value

View an asset with positive recovery as an asset with an additional positive payoff at *default*. The recovery value is the *expected* value of the recovery shortly after the occurrence of a default.

Payment upon default

Define $e(t, T, T+\Delta T)$ to be the value at time $t < T$ of a deterministic payoff of \$1 paid at $T + \Delta T$ if and only if a default happens in $[T, T + \Delta T]$.

$$e(t, T, T + \Delta T) = E_Q \left[\beta(t, T + \Delta T)[I(T) - I(T + \Delta T)] | \mathcal{F}_t \right].$$

Note that

$$I(T) - I(T + \Delta T) = \begin{cases} 1 & \text{if default occurs in } [T, T + \Delta T] \\ 0 & \text{otherwise} \end{cases},$$

$$\begin{aligned} E_Q[\beta(t, T + \Delta T) I(T)] &= E_Q[\beta(t, T + \Delta T)] E_Q[I(T)] \\ &= B(t, T + \Delta T) P(t, T), \end{aligned}$$

$$E_Q[\beta(t, T + \Delta T) I(T + \Delta T)] = \overline{B}(t, T + \Delta T),$$

and

$$B(t, T + \Delta T) = \overline{B}(t, T + \Delta T) / P(t, T + \Delta T).$$

It is seen that

$$\begin{aligned} e(t,T,T+\Delta T) &= B(t,T+\Delta T)P(t,T) - \overline{B}(t,T+\Delta T) \\ &= \overline{B}(t,T+\Delta T)\left[\frac{P(t,T)}{P(t,T+\Delta T)} - 1\right] \\ &= \Delta T \overline{B}(t,T+\Delta T)H(t,T,T+\Delta T) \end{aligned}$$

On taking the limit $\Delta T \to 0$, we obtain

$$\begin{aligned} \text{rate of default compensation} &= e(t,T) = \lim_{\Delta T \to 0} \frac{e(t,T,T+\Delta T)}{\Delta T} \\ &= \overline{B}(t,T)h(t,T) = B(t,T)P(t,T)h(t,T). \end{aligned}$$

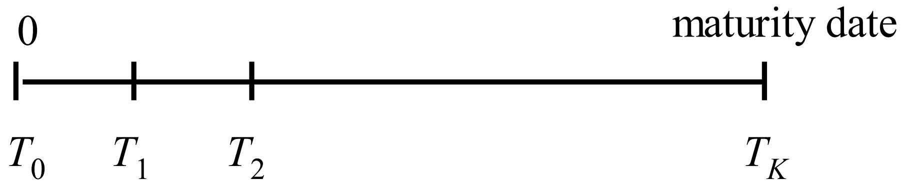
The value of a security that pays $\pi(s)$ if a default occurs at time s for all $t < s < T$ is given by

$$\int_t^T \pi(s)e(t,s)\,ds = \int_t^T \pi(s)\overline{B}(t,s)h(t,s)\,ds.$$

This result holds for deterministic recovery rates.

Building blocks for credit derivatives pricing

Tenor structure



$$\delta_k = T_{k+1} - T_k, 0 \leq k \leq K-1$$

Coupon and repayment dates for bonds, fixing dates for rates, payment and settlement dates for credit derivatives all fall on $T_k, 0 \leq k \leq K$.

Fundamental quantities of the model

- Term structure of default-free interest rates $F(0,T)$
- Term structure of implied hazard rates $H(0,T)$
- Expected recovery rate π (rate of recovery as percentage of par)

From $B(0,T_i) = \dfrac{B(0,T_{i-1})}{1+\delta_{i-1}F(0,T_{i-1}T_i)}, i = 1,2,\cdots,k$, and $B(0,T_0) = B(0,0) = 1$, we obtain

$$B(0,T_k) = \prod_{i=1}^{k} \frac{1}{1+\delta_{i-1}F(0,T_{i-1},T_i)}.$$

Similarly, from $P(0,T_i) = \dfrac{P(0,T_{i-1})}{1+\delta_{i-1}H(0,T_{i-1},T_i)}$, we deduce that

$$
\begin{aligned}
\overline{B}(0,T_k) &= B(0,T_k)P(0,T_k) = B(0,T_k)\prod_{i=1}^{k}\frac{1}{1+\delta_{i-1}H(0,T_{i-1},T_i)}. \\
e(0,T_k,T_{k+1}) &= \delta_k H(0,T_k,T_{k+1})\overline{B}(0,T_{k+1}) \\
&= \text{value of \$1 at } T_{k+1} \text{ if a default} \\
&\quad\ \text{has occurred in } (T_k,T_{k+1}].
\end{aligned}
$$

Taking the limit $\delta_i \to 0$, for all $i = 0, 1, \cdots, k$

$$\begin{aligned} B(0, T_k) &= \exp\left(-\int_0^{T_k} f(0, s)\, ds\right) \\ \overline{B}(0, T_k) &= \exp\left(-\int_0^{T_k} [h(0, s) + f(0, s)]\, ds\right) \\ e(0, T_k) &= h(0, T_k)\overline{B}(0, T_k). \end{aligned}$$

Alternatively, the above relations can be obtained by integrating

$$\begin{aligned} f(0, T) &= -\frac{\partial}{\partial T} \ln B(0, T) \quad \text{with} \quad B(0, 0) = 1 \\ \overline{f}(0, T) &= h(0, T) + f(0, T) = -\frac{\partial}{\partial T} \ln \overline{B}(0, T) \quad \text{with} \quad \overline{B}(0, 0) = 1. \end{aligned}$$

Defaultable fixed coupon bond

$$\begin{aligned}\overline{c}(0) = &\sum_{n=1}^{K} \overline{c}_n \overline{B}(0, T_n) && \text{(coupon)} && \overline{c}_n = \overline{c}\delta_{n-1} \\ &+ \overline{B}(0, T_K) && \text{(principal)} \\ &+ \pi \sum_{k=1}^{K} e(0, T_{k-1}, T_k) && \text{(recovery)}\end{aligned}$$

The recovery payment can be written as

$$\pi \sum_{k=1}^{K} e(0, T_{k-1}, T_k) = \sum_{k=1}^{K} \pi\delta_{k-1} H(0, T_{k-1}, T_k)\overline{B}(0, T_k).$$

The recovery payments can be considered as an additional coupon payment stream of $\pi\delta_{k-1}H(0, T_{k-1}, T_k)$.

Defaultable floater

Recall that $L(T_{n-1}, T_n)$ is the reference LIBOR rate applied over $[T_{n-1}, T_n]$ at T_{n-1} so that $1 + L(T_{n-1}, T_n)\delta_{n-1}$ is the growth factor over $[T_{n-1}, T_n]$. Application of no-arbitrage argument gives

$$B(T_{n-1}, T_n) = \frac{1}{1 + L(T_{n-1}, T_n)\delta_{n-1}}.$$

- The coupon payment at T_n equals LIBOR plus a spread

$$\delta_{n-1}\left[L(T_{n-1},T_n)+s^{par}\right]=\left[\frac{1}{B(T_{n-1},T_n)}-1\right]+s^{par}\delta_{n-1}.$$

- Consider the payment of $\dfrac{1}{B(T_{n-1},T_n)}$ at T_n, its value at T_{n-1} is $\dfrac{\overline{B}(T_{n-1},T_n)}{B(T_{n-1},T_n)}=P(T_{n-1},T_n)$. Why? We use the defaultable discount factor $\overline{B}(T_{n-1},T_n)$ since the coupon payment may be defaultable over $[T_{n-1},T_n]$.

- Seen at $t = 0$, the value becomes

$$\begin{aligned} & \overline{B}(0, T_{n-1}) P(0, T_{n-1}, T_n) \\ = \; & B(0, T_{n-1}) P(0, T_{n-1}) P(0, T_{n-1}, T_n) \\ = \; & B(0, T_{n-1}) P(0, T_n). \end{aligned}$$

Combining with the fixed part of the coupon payment and observing the relation

$$\begin{aligned} [B(0, T_{n-1}) - B(0, T_n)] P(0, T_n) &= \left[\frac{B(0, T_{n-1})}{B(0, T_n)} - 1 \right] \overline{B}(0, T_n) \\ &= \delta_{n-1} F(0, T_{n-1}, T_n) \overline{B}(0, T_n), \end{aligned}$$

the model price of the defaultable floating rate bond is

$$\begin{aligned} \overline{c}(0) = & \sum_{n=1}^{K} \delta_{n-1} F(0, T_{n-1}, T_n) \overline{B}(0, T_n) + s^{par} \sum_{n=1}^{K} \delta_{n-1} \overline{B}(0, T_n) \\ & + \overline{B}(0, T_K) + \pi \sum_{k=1}^{K} e(0, T_{k-1}, T_k). \end{aligned}$$

Credit default swap

Fixed leg Payment of $\delta_{n-1}\overline{s}$ at T_n if no default until T_n.

The value of the fixed leg is

$$\overline{s}\sum_{n=1}^{N}\delta_{n-1}\overline{B}(0,T_n).$$

Floating leg Payment of $1-\pi$ at T_n if default in $(T_{n-1},T_n]$ occurs. The value of the floating leg is

$$\begin{aligned}&(1-\pi)\sum_{n=1}^{N}e(0,T_{n-1},T_n)\\ = \;&(1-\pi)\sum_{n=1}^{N}\delta_{n-1}H(0,T_{n-1},T_n)\overline{B}(0,T_n).\end{aligned}$$

The market CDS spread is chosen such that the fixed leg and floating leg of the CDS have the same value. Hence

$$\overline{s} = (1 - \pi)\frac{\sum_{n=1}^{N} \delta_{n-1} H(0, T_{n-1}, T_n)\overline{B}(0, T_n)}{\sum_{n=1}^{N} \delta_{n-1}\overline{B}(0, T_n)}.$$

Define the weights

$$w_n = \frac{\delta_{n-1}\overline{B}(0,T_n)}{\sum_{k=1}^{N}\delta_{k-1}\overline{B}(0,T_k)}, \quad n = 1,2,\cdots,N, \quad \text{and} \quad \sum_{n=1}^{N} w_n = 1,$$

then the fair swap premium rate is given by

$$\overline{s} = (1-\pi)\sum_{n=1}^{N} w_n H(0,T_{n-1},T_n).$$

1. $\overline{s}$ depends only on the defaultable and default free discount rates, which are given by the market bond prices. CDS is an example of a cash product.

2. It is similar to the calculation of fixed rate in the interest rate swap

$$s = \sum_{n=1}^{N} w'_n F(0, T_{n-1}, T_n)$$

where

$$w'_n = \frac{\delta_{n-1} B(0, T_n)}{\sum_{k=1}^{N} \delta_{k-1} B(0, T_k)}, \quad n = 1, 2, \cdots, N.$$

Marked-to-market value

original CDS spread $= \overline{s}'$; new CDS spread $= \overline{s}$

Let $\Pi = \text{CDS}_{old} - \text{CDS}_{new}$, and observe that $\text{CDS}_{new} = 0$, then

$$\text{marked-to-market value} = \text{CDS}_{old} = \Pi = (\overline{s} - \overline{s}') \sum_{n=1}^{N} \overline{B}(0, T_n)\delta_{n-1}.$$

Why? If an offsetting trade is entered at the current CDS rate $\overline{s}$, only the fee difference $(\overline{s} - \overline{s}')$ will be received over the life of the CDS. Should a default occurs, the protection payments will cancel out, and the fee difference payment will be cancelled, too. The fee difference stream is defaultable and must be discounted with $\overline{B}(0, T_n)$.

- CDS's are useful instruments to gain exposure against spread movements, not just against default arrival risk.