5. Capital Asset Pricing Model and Factor Models

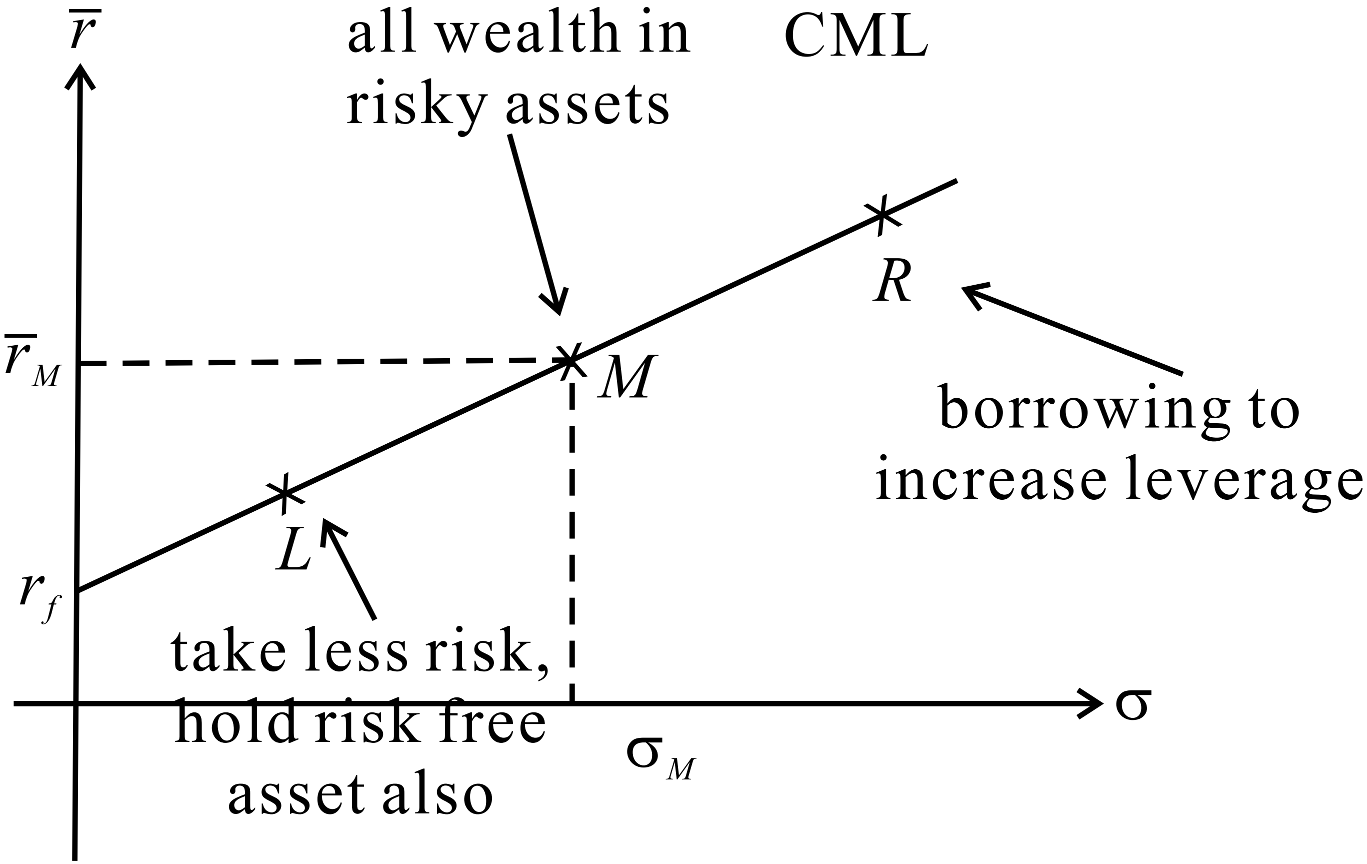
Capital market line (CML)

CML is the tangent line drawn from the risk free point to the feasible region for risky assets. This line shows the relation between $\overline{r}_P$ and σ_P for efficient portfolios (risky assets plus the risk free asset).

The tangency point M represents the *market portfolio*, so named since all rational investors (minimum variance criterion) should hold their risky assets in the same proportions as their weights in the market portfolio.

- Every investor is a mean-variance investor and all have homogeneous expectations on means and variances, then everyone buys the same portfolio. Prices adjust to drive the market to efficiency.

Based on the risk level that an investor can take, she will combine the market portfolio of risky assets with the risk free asset.



Equation of the CML:

$$\overline{r} = r_f + \frac{\overline{r}_M - r_f}{\sigma_M}\sigma,$$

where $\overline{r}$ and σ are the mean and standard deviation of the rate of return of an efficient portfolio.

Slope of the CML $= \dfrac{\overline{r}_M - r_f}{\sigma_M} =$ price of risk of an efficient portfolio.

This indicates how much the expected rate of return must increase if the standard deviation increases by one unit.

Example Consider an oil drilling venture; current share price of the venture = $875, expected to yield $1,000 in one year. The standard deviation of return, $\sigma = 40\%$; and $r_f = 10\%$. Also, $r_M = 17\%$ and $\sigma_M = 12\%$ for the market portfolio.

Question How does this venture compare with the investment on efficient portfolios on the CML?

Given this level of σ, the expected rate of return predicted by the CML is

$$\overline{r} = 0.10 + \frac{0.17 - 0.10}{0.12} \times 0.40 = 33\%.$$

The actual expected rate of return $= \frac{1,000}{875} - 1 = 14\%$, which is well below 33%. This venture does not constitute an efficient portfolio. It bears certain type of risk that does not contribute to the expected rate of return.

Sharpe ratio

One index that is commonly used in performance measure is the Sharpe ratio, defined as

$$\frac{\overline{r}_i - r_f}{\sigma_i} = \frac{\text{excess return above riskfree rate}}{\text{standard deviation}}.$$

We expect Sharpe ratio $\leq$ slope of CML.

Closer the Sharpe ratio to the slope of CML, the better the performance of the fund in terms of return against risk.

In previous example,

$$\begin{aligned} \text{Slope of CML} &= \frac{17\% - 10\%}{12\%} = \frac{7}{12} = 0.583 \\ \text{Sharpe ratio} &= \frac{14\% - 10\%}{40\%} = 0.1 < \text{ Slope of CML.} \end{aligned}$$

Lemma – Capital Asset Pricing Model

Let M be the market portfolio M, then the expected return $\overline{r}_i$ of any asset i satisfies

$$\overline{r}_i - r_f = \beta_i(\overline{r}_M - r_f)$$

where

$$\beta_i = \frac{\sigma_{iM}}{\sigma_M^2}.$$

Here, σ_{iM} is the correlation between the return of risky asset i and the return of market portfolio M.

Remark

If we write $\sigma_{iM} = \rho_{iM}\sigma_i\sigma_M$, then

$$\frac{\overline{r}_i - r_f}{\sigma_i} = \rho_{iM}\frac{\overline{r}_M - r_f}{\sigma_M}.$$

The Sharpe ratio of asset i is given by the product of ρ_{iM} and the slope of CML.

Proof

Consider the portfolio with α portion invested in asset i and $1-\alpha$ portion invested in the market portfolio M. The expected rate of return of this portfolio is

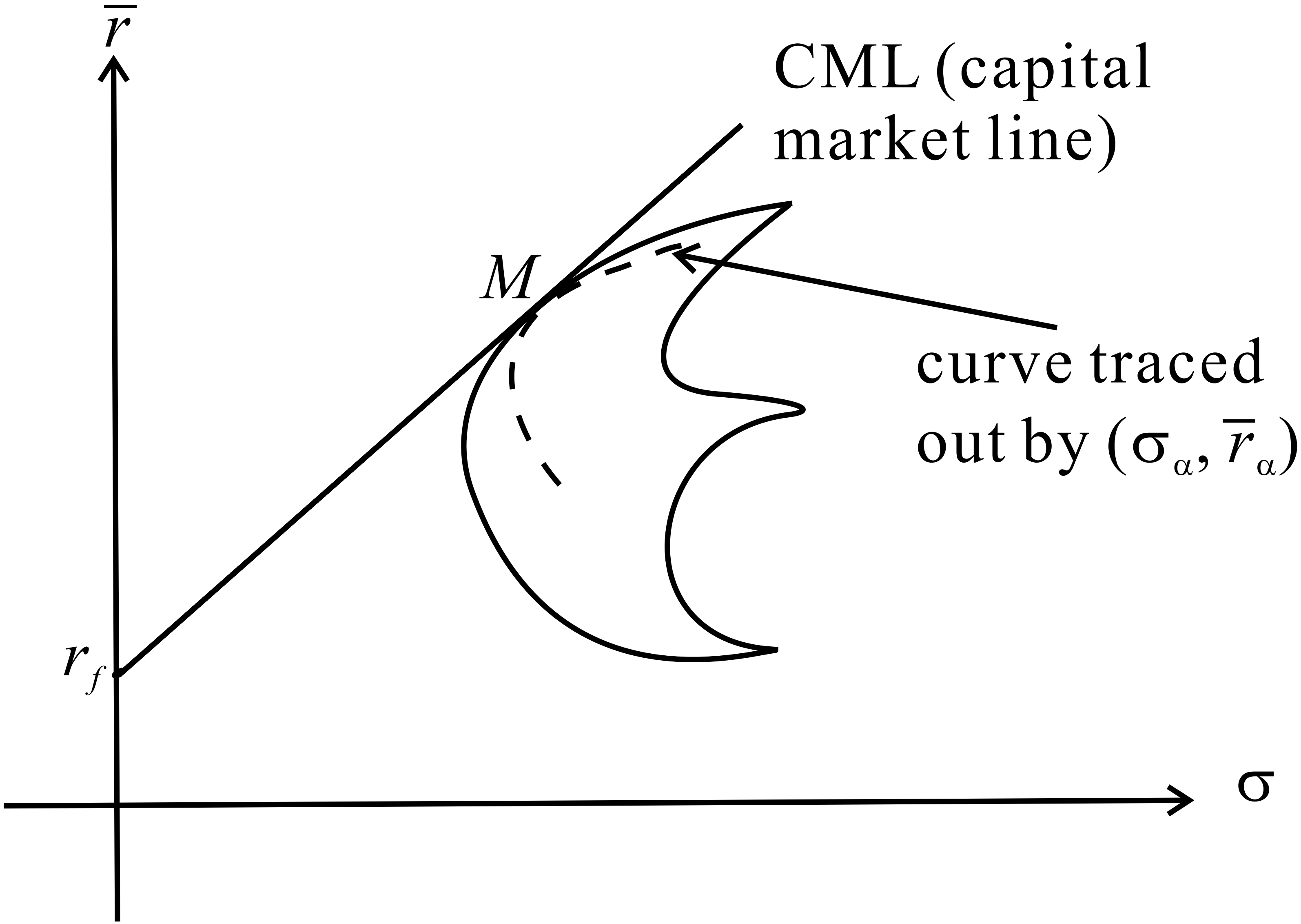
$$\overline{r}_\alpha = \alpha\overline{r}_i + (1-\alpha)\overline{r}_M$$

and its variance is

$$\sigma_\alpha^2 = \alpha^2\sigma_i^2 + 2\alpha(1-\alpha)\sigma_{iM} + (1-\alpha)^2\sigma_M^2.$$

As α varies, $(\sigma_\alpha, \overline{r}_\alpha)$ traces out a curve in the $\sigma - \overline{r}$ diagram. The market portfolio M corresponds to $\alpha = 0$.

The curve cannot cross the CML, otherwise this would violate the property that the CML is an efficient boundary of the feasible region. Hence, as α passes through zero, the curve traced out by $(\sigma_\alpha, \overline{r}_\alpha)$ must be tangent to the CML at M.



Tangency condition Slope of the curve at M = slope of CML.

First, we obtain $\dfrac{d\overline{r}_\alpha}{d\alpha} = \overline{r}_i - \overline{r}_M$ and

$$\frac{d\sigma_\alpha}{d\alpha} = \frac{\alpha\sigma_i^2 + (1-2\alpha)\sigma_{iM} + (\alpha-1)\sigma_M^2}{\sigma_\alpha}$$

so that $\left.\dfrac{d\sigma_\alpha}{d\alpha}\right|_{\alpha=0} = \dfrac{\sigma_{iM} - \sigma_M^2}{\sigma_M}$.

Next, we apply the relation $\dfrac{d\overline{r}_\alpha}{d\sigma_\alpha} = \dfrac{\frac{d\overline{r}_\alpha}{d\alpha}}{\frac{d\sigma_\alpha}{d\alpha}}$ to obtain

$$\left.\frac{d\overline{r}_\alpha}{d\sigma_\alpha}\right|_{\alpha=0} = \frac{(\overline{r}_i - \overline{r}_M)\sigma_M}{\sigma_{iM} - \sigma_M^2}.$$

However, $\left.\dfrac{d\overline{r}_\alpha}{d\sigma_\alpha}\right|_{\alpha=0}$ should be equal to the slope of CML, that is,

$$\frac{(\overline{r}_i - \overline{r}_M)\sigma_M}{\sigma_{iM} - \sigma_M^2} = \frac{\overline{r}_M - r_f}{\sigma_M}.$$

Solving for $\overline{r}_i$, we obtain

$$\overline{r}_i = r_f + \underbrace{\frac{\sigma_{iM}}{\sigma_M^2}}_{\beta_i}(\overline{r}_M - r_f) = r_f + \beta_i(\overline{r}_M - r_f).$$

Now, $\beta_i = \dfrac{\overline{r}_i - r_f}{\overline{r}_M - r_f}$

$$= \frac{\text{expected excess return of asset } i \text{ over } r_f}{\text{expected excess return of market portfolio over } r_f}.$$

Predictability of equilibrium return

The CAPM implies in equilibrium the expected excess return on any single risky asset is proportional to the expected excess return on the market portfolio. The constant of proportionality is β_i.

Alternative proof of CAPM

Consider

$$\sigma_{iM} = \mathrm{cov}(r_i, r_M) = \boldsymbol{e}_i^T \Omega \boldsymbol{w}_M^*,$$

where $\boldsymbol{e}_i = (0 \cdots 1 \cdots 0) = i^{\text{th}}$ co-ordinate vector is the weight of asset i.

Recall $\boldsymbol{w}_M^* = \dfrac{\Omega^{-1}(\boldsymbol{\mu} - r\mathbf{1})}{b - ar}$ so that

$$\sigma_{iM} = \frac{(\boldsymbol{\mu} - r\mathbf{1})_i}{b - ar} = \frac{\overline{r}_i - r}{b - ar}, \text{ provided } b - ar \neq 0. \tag{1}$$

Also, we recall $\mu_P^M = \dfrac{c - br}{b - ar}$ and $\sigma_{P,M}^2 = \dfrac{c - 2rb + r^2 a}{(b - ar)^2}$ so that

$$\mu_P^M - r = \frac{c - br}{b - ar} - r = \frac{c - 2rb + r^2 a}{(b - ar)^2} = (b - ar)\sigma_{P,M}^2. \tag{2}$$

Eliminating $b - ar$ from eqs (1) and (2), we obtain

$$\overline{r}_i - r = \frac{\sigma_{iM}}{\sigma_M^2}(\mu_P^M - r).$$

What is the interpretation of $\dfrac{\sigma_{iM}}{\sigma_M}$, where $\sigma_{iM} = \text{cov}(r_i, r_M)$?

Consider $\sigma_M^2 = \boldsymbol{w}_M^{*T} \Omega \boldsymbol{w}_M^*$, we differentiate with respect to w_i and obtain

$$2\sigma_M \frac{d\sigma_M}{dw_i^M} = 2\boldsymbol{e}_i^T \Omega \boldsymbol{w}_M^* = 2\sigma_{iM}$$

so that

$$\frac{d\sigma_M}{dw_i} = \frac{\sigma_{iM}}{\sigma_M} \quad \text{or} \quad \frac{d\sigma_M}{\sigma_M} = \beta_i \, dw_i^M.$$

This is a measure of how the weight of one asset affecting the risk of the market portfolio.

Beta of a portfolio

Consider a portfolio containing n assets with weights $w_1, w_2, \cdots, w_n$. Since $r_P = \sum_{i=1}^{n} w_i r_i$, we have $\text{cov}(r_P, r_M) = \sum_{i=1}^{n} w_i \text{cov}(r_i, r_M)$ so that

$$\beta_P = \frac{\text{cov}(r_P, r_M)}{\sigma_M^2} = \frac{\sum_{i=1}^{n} w_i \text{cov}(r_i, r_M)}{\sigma_M^2} = \sum_{i=1}^{n} w_i \beta_i.$$

Some special cases of beta values

1. When $\beta_i = 0, \overline{r}_i = r_f$. A risky asset (with $\sigma_i > 0$) that is uncorrelated with the market portfolio will have an expected rate of return equal to the risk free rate. There is no expected excess return over r_f even the investor bears some risk in holding a risky asset with zero beta.

2. When $\beta_i = 1, \overline{r}_i = r_M$. A risky asset which is perfectly correlated with the market portfolio has the same expected rate of return as that of the market portfolio.

3. When $\beta_i > 1$, expected excess rate of return is higher than that of market portfolio - *aggressive asset*. When $\beta_i < 1$, the asset is said to be *defensive*.

4. When $\beta_i < 0, \overline{r}_i < r_f$. Since $\dfrac{d\sigma_M}{\sigma_M} = \beta_i \, dw_i^M$, so a risky asset with *negative beta reduces the variance* of the portfolio. This risk reduction potential of asset with negative β is something like paying premium to reduce risk.

Extension

Let P be any efficient portfolio along the upper tangent line and Q be any portfolio. We also have

$$\overline{R}_Q - r = \beta_{PQ}(\overline{R}_P - r), \qquad (A)$$

that is, P is not necessary to be the market portfolio.

More generally,

$$R_Q - r = \beta_{PQ}(R_P - r) + \epsilon_{PQ} \qquad (B)$$

with $\text{cov}(R_P, \epsilon_{PQ}) = E[\epsilon_{PQ}] = 0$.

The first result (A) can be deduced from the CAPM by observing

$$\begin{aligned}\sigma_{QP} &= \mathrm{cov}(R_Q, \alpha R_M + (1-\alpha)R_f) = \alpha \mathrm{cov}(R_Q, R_M) = \alpha\sigma_{QM}, \quad \alpha > 0 \\ \sigma_P^2 &= \alpha^2\sigma_M^2 \quad \text{and} \quad \overline{R}_P - r = \alpha(\overline{R}_M - r).\end{aligned}$$

Consider

$$\begin{aligned}\overline{R}_Q - r &= \beta_{MQ}(\overline{R}_M - r) = \frac{\sigma_{QM}}{\sigma_M^2}(\overline{R}_M - r) \\ &= \frac{\sigma_{QP}/\alpha}{\sigma_P^2/\alpha^2}(\overline{R}_P - r)/\alpha = \beta_{PQ}(\overline{R}_P - r).\end{aligned}$$

The relationship among R_Q, R_P and r can always be written as

$$R_Q = \alpha_0 + \alpha_1 R_P + \epsilon_{PQ}$$

with $\text{cov}(R_P, \epsilon_{PQ}) = E[\epsilon_{PQ}] = 0$, where α_0 and α_1 are coefficients from the regression of R_Q on R_P.

Observe that

$$\overline{R}_Q = \alpha_0 + \alpha_1 \overline{R}_P$$

and from result (A), we obtain

$$\overline{R}_Q = \beta_{PQ}\overline{R}_P + r(1 - \beta_{PQ})$$

so that

$$\alpha_0 = r(1 - \beta_{PQ}) \quad \text{and} \quad \alpha_1 = \beta_{PQ}.$$

Hence, we obtain result (B).

Zero-beta CAPM

From the CML, there exists a portfolio Z_M whose beta is zero. Consider the CML

$$\overline{r}_Q = r + \beta_{QM}(\overline{r}_M - r),$$

since $\beta_{MZ_M} = 0$, we have $\overline{r}_{Z_M} = r$. Hence the CML can be expressed in terms of market portfolio M and its zero-beta counterpart Z_M as follows

$$\overline{r}_Q = \overline{r}_{Z_M} + \beta_{QM}(\overline{r}_M - \overline{r}_{Z_M}).$$

In this form, the role of the riskfree asset is replaced by the zero-beta portfolio Z_M. In this sense, we allow the absence of riskfree asset.

$\star$ The more general version allows the choice of *any* efficient (mean-variance) portfolio and its zero-beta counterpart.

Finding the non-correlated counterpart

Let P and Q be any two frontier portfolios. Recall that

$$\boldsymbol{w}_P^* = \Omega^{-1}(\lambda_1^P \mathbf{1} + \lambda_2^P \boldsymbol{\mu}) \quad \text{and} \quad \boldsymbol{w}_Q^* = \Omega^{-1}(\lambda_1^Q \mathbf{1} + \lambda_2^Q \boldsymbol{\mu})$$

where

$$\lambda_1^P = \frac{c - b\mu_P}{\Delta}, \quad \lambda_2^P = \frac{a\mu_P - b}{\Delta}, \quad \lambda_1^Q = \frac{c - b\mu_Q}{\Delta}, \quad \lambda_2^Q = \frac{a\mu_Q - b}{\Delta},$$
$$a = \mathbf{1}^T \Omega^{-1} \mathbf{1}, \quad b = \mathbf{1}^T \Omega^{-1} \boldsymbol{\mu}, \quad c = \boldsymbol{\mu}^T \Omega^{-1} \boldsymbol{\mu}, \quad \Delta = ac - b^2.$$

Find the covariance between R_P and R_Q.

$$\begin{aligned} \text{cov}(R_P, R_Q) &= {\boldsymbol{w}_P^*}^T \Omega \boldsymbol{w}_Q^* = \left[\Omega^{-1}(\lambda_1^P \mathbf{1} + \lambda_2^P \boldsymbol{\mu})\right]^T (\lambda_1^Q \mathbf{1} + \lambda_2^Q \boldsymbol{\mu}) \\ &= \lambda_1^P \lambda_1^Q a + (\lambda_1^P \lambda_2^Q + \lambda_1^Q \lambda_2^P) b + \lambda_2^P \lambda_2^Q c \\ &= \frac{a}{\Delta}\left(\mu_P - \frac{b}{a}\right)\left(\mu_Q - \frac{b}{a}\right) + \frac{1}{a}. \end{aligned}$$

Find the portfolio Z such that $\text{cov}(R_P, R_Z) = 0$. We obtain

$$\mu_Z = \frac{b}{a} - \frac{\frac{\Delta}{a^2}}{\mu_P - \frac{b}{a}}.$$

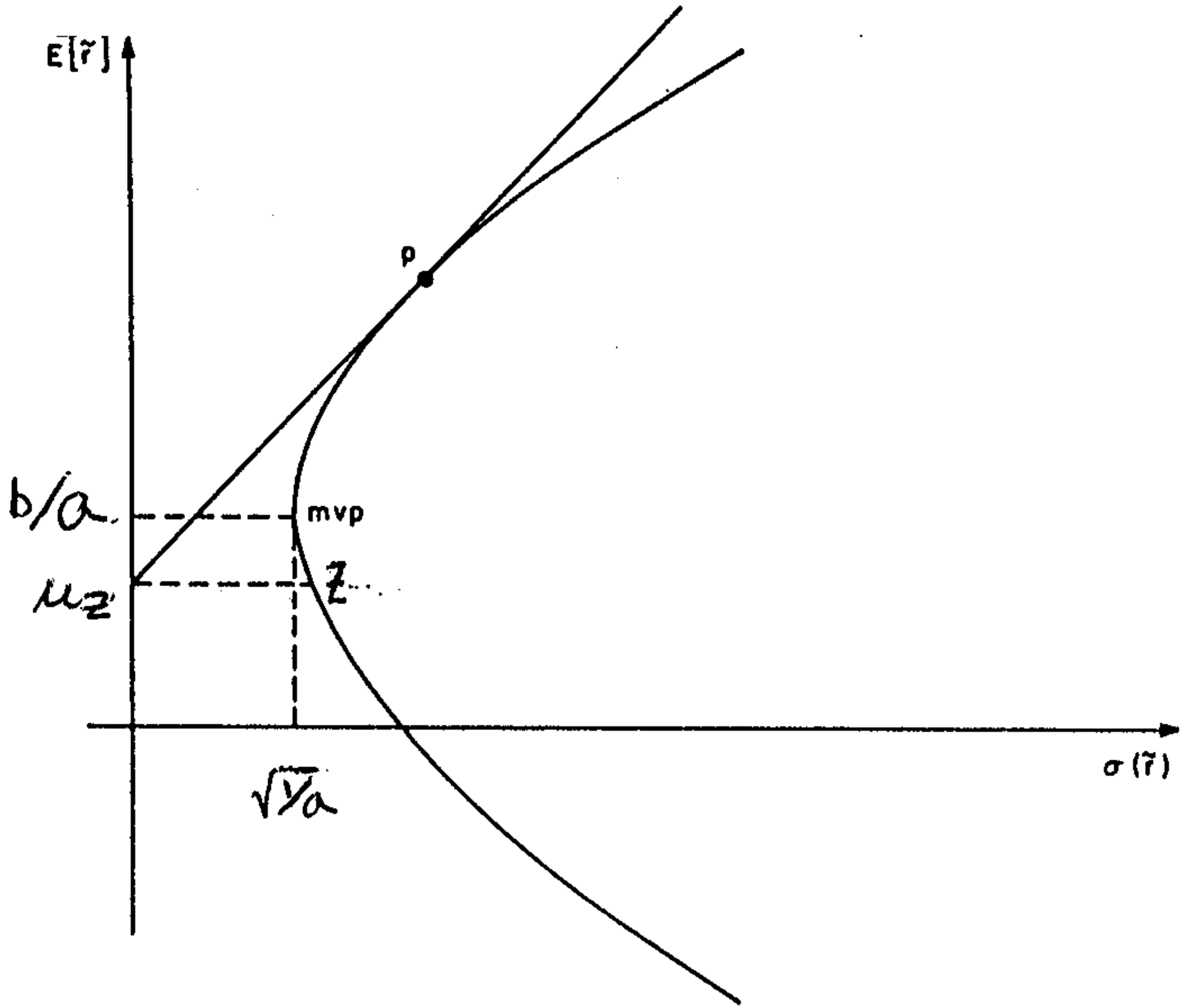
Since $(\mu_P - \mu_g)(\mu_Z - \mu_g) = -\dfrac{\Delta}{a^2} < 0$, where $\mu_g = \dfrac{b}{a}$, if one portfolio is efficient, then the zero-covariance counterpart is non-efficient.

Slope of the tangent at P to the frontier curve:

$$\frac{d\mu_P}{d\sigma_P} = \frac{\Delta \sigma_P}{a\mu_P - b}.$$

It can be shown that

$$\begin{aligned} \mu_P - \frac{d\mu_P}{d\sigma_P}\sigma_P &= \mu_P - \frac{\Delta \sigma_P^2}{a\mu_P - b} \\ &= \mu_P - \frac{a\mu_P^2 - 2b\mu_P + c}{a\mu_P - b} = \frac{b}{a} - \frac{\Delta/a^2}{\mu_P - b/a} = \mu_Z. \end{aligned}$$



The Location of a Zero Covariance Portfolio in the $\sigma(\tilde{r})$–$E[\tilde{r}]$ Space

Let P be a frontier portfolio other than the global minimum variance portfolio and Q be any portfolio, then

$$\begin{aligned}\text{cov}(R_P, R_Q) &= \left[\Omega^{-1}\left(\lambda_1^P \mathbf{1} + \lambda_2^P \boldsymbol{\mu}\right)\right]^T \Omega \boldsymbol{w}_Q \\ &= \lambda_1^P \mathbf{1}^T \boldsymbol{w}_Q + \lambda_2^P \boldsymbol{\mu}^T \boldsymbol{w}_Q = \lambda_1^P + \lambda_2^P \mu_Q.\end{aligned}$$

Solving for μ_Q and substituting $\lambda_1^P = \dfrac{c - b\mu_P}{\Delta}$ and $\lambda_2^P = \dfrac{a\mu_P - b}{\Delta}$

$$\begin{aligned}\mu_Q &= \frac{b\mu_P - c}{a\mu_P - b} + \text{cov}(R_P, R_Q)\frac{\Delta}{a\mu_P - b} \\ &= \frac{b}{a} - \frac{\Delta/a^2}{\mu_P - b/a} + \frac{\text{cov}(R_P, R_Q)}{\sigma_P^2}\left[\frac{1}{a} + \frac{\left(\mu_P - \frac{b}{a}\right)^2}{\Delta/a}\right]\frac{\Delta}{a\mu_P - b} \\ &= \mu_{Z_P} + \beta_{PQ}\left(\mu_P - \frac{b}{a} + \frac{\Delta/a^2}{\mu_P - b/a}\right) \\ &= \mu_{Z_P} + \beta_{PQ}(\mu_P - \mu_{Z_P})\end{aligned}$$

so that

$$\mu_Q - \mu_{Z_P} = \beta_{PQ}(\mu_P - \mu_{Z_P}).$$

Summary

The zero-beta CAPM provides an alternative model of equilibrium returns to the standard CAPM.

- With no borrowing or lending at the riskless rate, an investor can attain his own optimal portfolio by combining *any* mean-variance efficient Portfolio P with its corresponding zero-beta Portfolio Z.
- Portfolio Z observes the properties

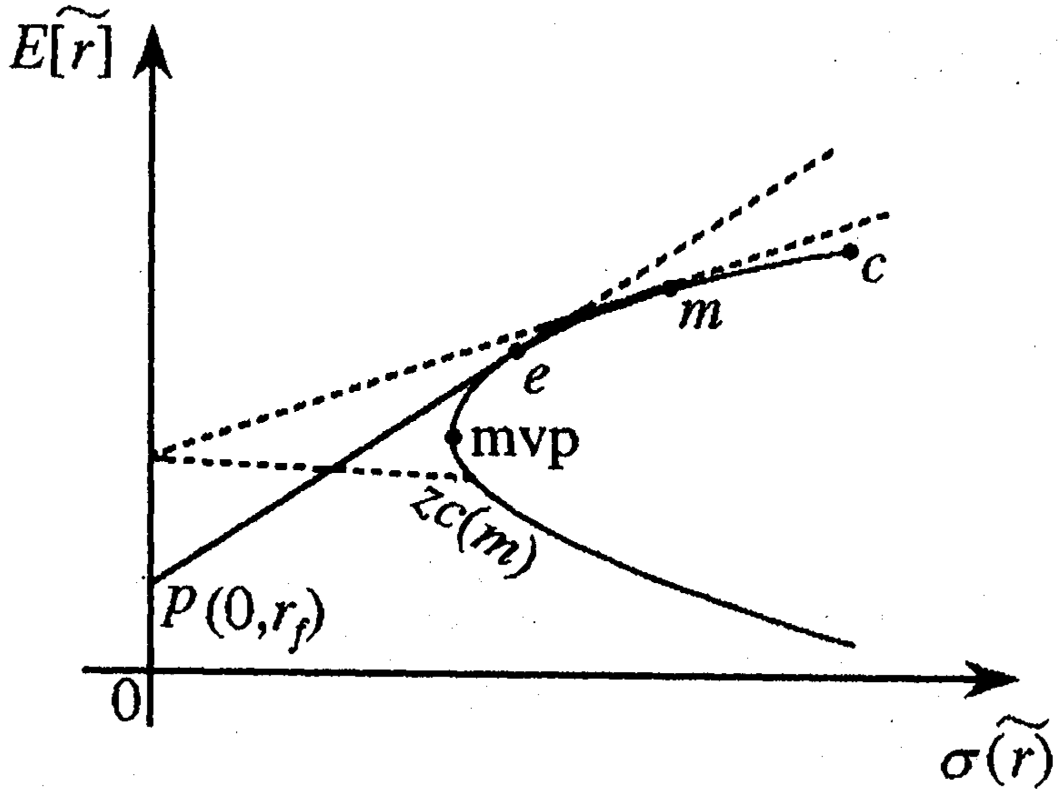
  (i) $\text{cov}(R_P, R_Z) = 0$

  (ii) Z is a frontier portfolio

- The choice of P is not unique so does the combination of portfolio P and Z.
- $$R_Q - R_Z = \beta_{PQ}(R_P - R_Z) + \widetilde{\epsilon}_Q$$

  where $\text{cov}(R_P, \widetilde{\epsilon}_Q) = E[\widetilde{\epsilon}_Q] = 0$.

No shorting selling of riskfree asset (no riskless borrowing)



- Here, e is the tangency portfolio with respect to the riskfree rate r_f and m is the market portfolio.
- The arc emc lies on the efficient frontier curve (without riskfree asset). Portfolio $zc(m)$ is the corresponding zero-correlated portfolio of Portfolio m.

How do we understand the market portfolio under the restriction of borrowing?

- If there is no restriction on lending and borrowing, then every investor must hold the tangency portfolio since all efficient portfolios are a combination of the riskfree asset and the tangency portfolio e. Hence e is the market portfolio.
- When an investor prefers a risk-return trade-off to the right of e, then the chosen portfolio must lie on the frontier curve to the right of e.

Market portfolio is defined as the average of the portfolios held by all investors.

Suppose there are I investors, each possesses wealth $W_0^i, i = 1, 2, \cdots, I$.

- Assume there are $k < I$ investors who choose an efficient portfolio along pe by forming a convex combination of the riskfree asset and tangency portfolio e. Let α^i denote the weight on the tangency portfolio e.
- The remaining $I - k$ investors choose an efficient portfolio along ec.

  Define $\boldsymbol{w}_e$ be the weight of e and $\boldsymbol{w}_{i'}$ be the weight of risky assets adopted by investor i' in the second category.

  Total value invested on risky asset j

$$V_j = \sum_{i=1}^{k} W_0^i \alpha^i w_e^j + \sum_{i'=k+1}^{I} W_0^{i'} w_{i'}^j.$$

Total value invested on all risky assets

$$\sum_{j=1}^{N} V_j = \sum_{i \leq k} W_0^i \alpha^i + \sum_{i' > k} W_0^{i'}.$$

Let $\boldsymbol{w}_m$ denote the weight of the risky assets in the market portfolio m. The j^{th} component is given by

$$\begin{aligned} w_m^j &\triangleq \frac{V_j}{\sum_{j=1}^{N} V_j} \\ &= \sum_{i=1}^{k} \frac{W_0^i \alpha^i}{\sum_{i \leq k} W_0^i \alpha^i + \sum_{i' > k} W_0^{i'}} w_e^j + \sum_{i'=k+1}^{I} \frac{W_0^{i'}}{\sum_{i \leq k} W_0^i \alpha^i + \sum_{i' > k} W_0^{i'}} w_{i'}^j, \\ & \qquad j = 1, 2, \cdots, N. \end{aligned}$$

Alternatively,

$$\boldsymbol{w}_m = \sum_{i=1}^{k} \gamma_i \boldsymbol{w}_e + \sum_{i'=k+1}^{I} \gamma_{i'} \boldsymbol{w}_{i'}$$

where

$$\begin{aligned}
\gamma_i &= \frac{W_0^i \alpha^i}{\sum_{i \le k} W_0^i \alpha^i + \sum_{i' > k} W_0^{i'}}, \quad i = 1, 2, \cdots, k \\
\gamma_{i'} &= \frac{W_0^{i'}}{\sum_{i \le k} W_0^i \alpha^i + \sum_{i' > k} W_0^{i'}}, \quad i' = k+1, \cdots, I.
\end{aligned}$$

Note that

$$\sum_{i \le k} \gamma_i + \sum_{i' > k} \gamma_{i'} = 1.$$

Since both e and portfolio held by investor are frontier portfolios, their weights admit the form

$$\boldsymbol{w}_e = \boldsymbol{g} + \boldsymbol{h}\mu_e \quad \text{and} \quad \boldsymbol{w}_{i'} = \boldsymbol{g} + \boldsymbol{h}\mu_{i'}.$$

This is because

$$\boldsymbol{w}^e = \Omega^{-1}(\lambda_1 \mathbf{1} + \lambda_2 \boldsymbol{\mu})$$

where

$$\lambda_1 = \frac{c - b\mu_P}{\Delta} \quad \text{and} \quad \lambda_2 = \frac{a\mu_P - b}{\Delta}.$$

Since $\mu_{i'} \geq \mu_e$ for all i',

$$\begin{aligned}
\boldsymbol{w}_m &= \sum_{i \leq k} \gamma_i(\boldsymbol{g} + \boldsymbol{h}\mu_e) + \sum_{i' > k} \gamma_{i'}(\boldsymbol{g} + \boldsymbol{h}\mu_i) \\
&= \boldsymbol{g} + \boldsymbol{h}\left(\sum_{i \leq k} \gamma_i \mu_e + \sum_{i' > k} \gamma_{i'} \mu_{i'}\right) \geq \boldsymbol{g} + \boldsymbol{h}\mu_e
\end{aligned}$$

hence the market portfolio m lies to the right side of e on the minimum variance frontier.

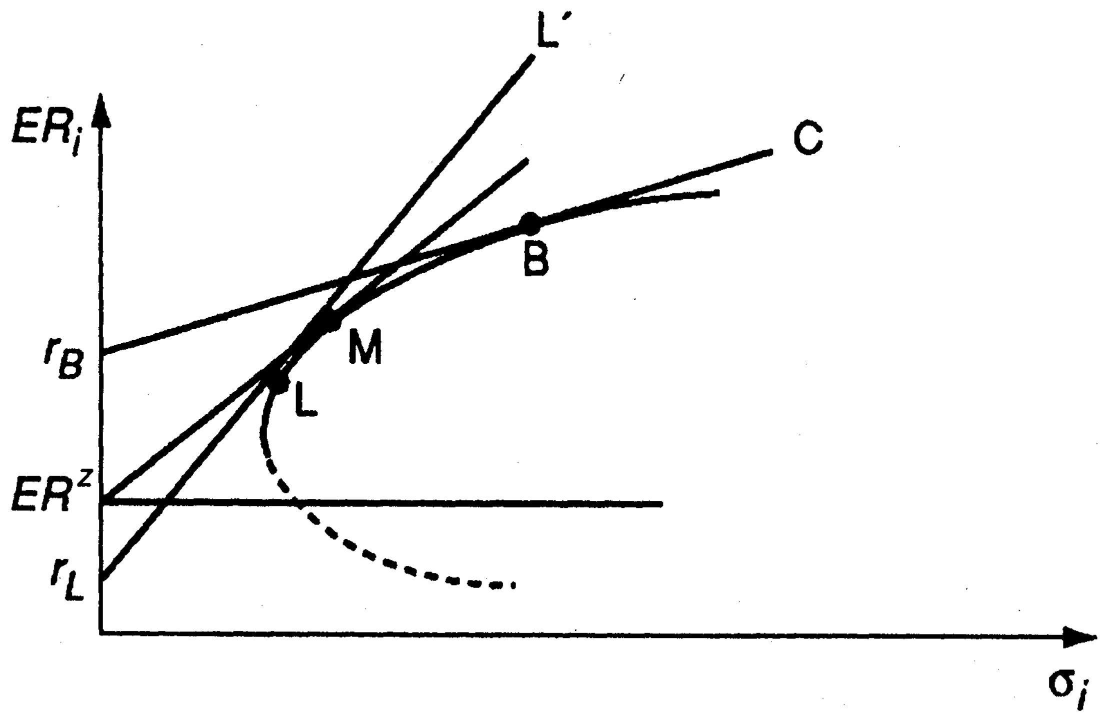
Three-fund Theorem

Riskfree asset, market portfolio m on the right side of e and its zero correlated counterpart $zc(m)$. The tangency portfolio e can be constructed by a convex combination of Portfolio m and its zero correlated counterpart $zc(m)$.

All securities contained in m have an expected return given by

$$\overline{r}_j = \overline{r}_z + \beta_j(\overline{r}_m - \overline{r}_z).$$

Different borrowing and lending rates



Here, M is the market portfolio, B and L are the tangency portfolios corresponding to borrowing rate r_B and lending rate r_L, respectively.

- For a lender, the optimal portfolio is along the straight line segment $r_L L$.

- For a borrower, the optimal portfolio is along BC.

- Note that LL' and $r_B B$ are not feasible.

- If the investor neither borrows nor lends, his optimal portfolio lies at any point along the curved section LMB.

- The market portfolio is a weighted average of the portfolios at L, B and *all* the portfolios along the curved segment LMB.

- We can always construct a zero-beta portfolio z for those who neither borrow nor lend and for such unlevered portfolio the equilibrium return on asset i is

$$\mu_i = \mu_z + (\mu_m - \mu_z)\beta_i, \quad \text{cov}(R_m, R_z) = 0$$

(i) For portfolios held by lenders

$$\mu_q = r_L + (\mu_L - r_L)\beta_{qL}, \quad \beta_{qL} = \text{cov}(R_q, R_L)/\sigma_L^2.$$

(ii) For portfolios held by borrowers

$$\mu_k = r_B + (\mu_B - r_B)\beta_{kB}.$$

Summary

1. All lenders hold the same risky portfolio of assets at L.

2. Unlevered portfolios differ among individuals.

3. All borrowers hold risky assets in the same proportions as at B but the levered portfolio of any individual can be anywhere along BC.

Non-marketable assets

Let V_n be the value of all non-marketable assets, V_m be the value of marketable assets.

New form of the CAPM

$$\mu_i = r + \beta_i^*(\mu_m - r)$$

where

$$\beta_i^* = \frac{\text{cov}(R_i, R_m) + \frac{V_n}{V_m}\text{cov}(R_i, R_n)}{\sigma_m^2 + \frac{V_n}{V_m}\text{cov}(R_m, R_n)}.$$

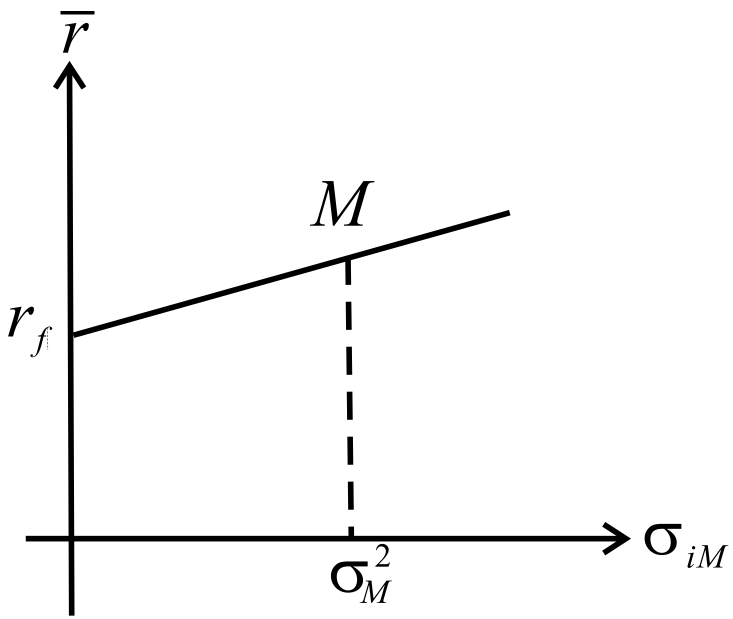
Method of derivation

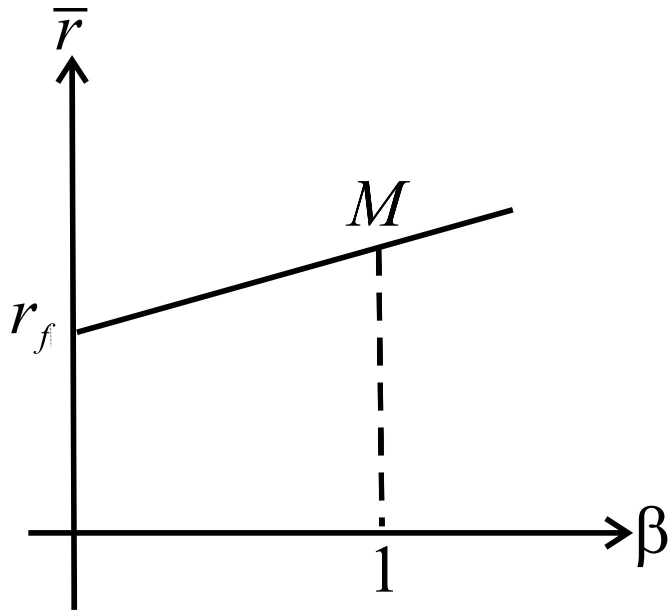
One may follow similar approach of the Liability Model, where the optimal choice on the weights of risky assets is limited to the marketable assets.

Security market line (SML)

From the two relations: $\begin{cases} \overline{r} = r_f + \frac{\overline{r} - r_f}{\sigma_M^2} \sigma_{iM} \\ \overline{r} = r_f + (\overline{r}_M - r_f)\beta_i \end{cases}$,

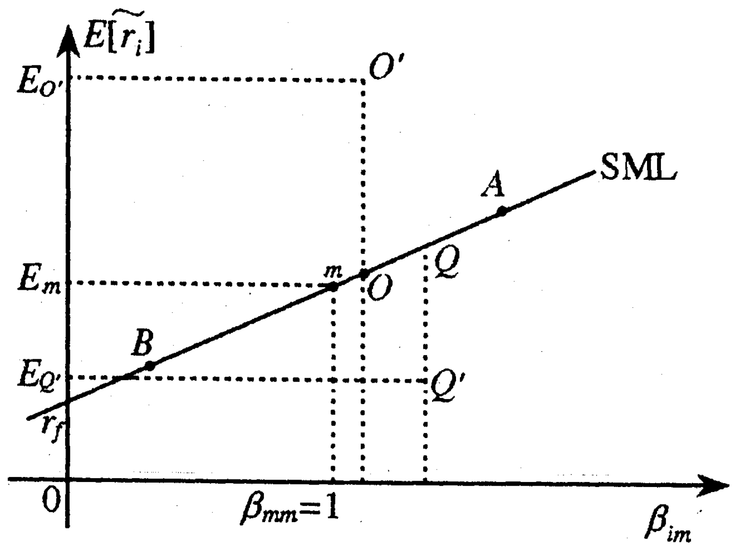
we can plot either $\overline{r}$ against σ_{iM} or $\overline{r}$ against β_i.





Under the equilibrium conditions assumed by the CAPM, every asset should fall on the SML. The SML expresses the risk reward structure of assets according to the CAPM.

- Point O represents under priced security. This is because the expected return is higher than the return with reference to the risk. In this case, the demand for such security will increase and this results in price increase and lowering of expected return.



Decomposition of risks

Suppose we write the random rate of return of asset i formally as

$$r_i = r_f + \beta_i(r_M - r_f) + \epsilon_i.$$

The CAPM tells us something about ϵ_i.

(i) Taking the expectation on both sides

$$E[r_i] = r_f + \beta_i(\overline{r}_M - r_f) + E[\epsilon_i]$$

while $\overline{r}_i = r_f + \beta_i(\overline{r}_M - r_f)$ so that $E[\epsilon_i] = 0$.

(ii) Taking the covariance of r_i with r_M

$$\begin{aligned} \text{cov}(r_i, r_M) &= \overbrace{\text{cov}(r_f, r_M)}^{\text{zero}} + \beta_i \left[\text{cov}(r_M, r_M) - \underbrace{\text{cov}(r_f, r_M)}_{\text{zero}} \right] \\ &\quad + \text{cov}(\epsilon_i, r_M) \end{aligned}$$

so that

$$\text{cov}(\epsilon_i, r_M) = 0.$$

(iii) Consider the variance of r_i

$$\text{var}(r_i) = \beta_i^2 \underbrace{\text{cov}(r_M - r_f, r_M - r_f)}_{\text{var}(r_M)} + \text{var}(\epsilon_i)$$

so that

$$\sigma_i^2 = \beta_i^2 \sigma_M^2 + \text{var}(\epsilon_i).$$

Systematic risk $= \beta_i^2 \sigma_M^2$, this risk cannot be reduced by diversification because every asset with nonzero beta contains this risk.

Portfolios on the CML – efficient portfolios

Consider a portfolio formed by the combination of the market portfolio and the risk free asset. This portfolio is an efficient portfolio (one fund theorem) and it lies on the CML with a beta value equal to β_0 (say). Its rate of return can be expressed as

$$r_p = (1 - \beta_0) r_f + \beta_0 r_M = r_f + \beta_0 (r_M - r_f)$$

so that $\epsilon_p = 0$. The portfolio variance is $\beta_0^2 \sigma_M^2$. This portfolio has only systematic risk (zero non-systematic risk).

Suppose the portfolio lies both on the CML and SML, then

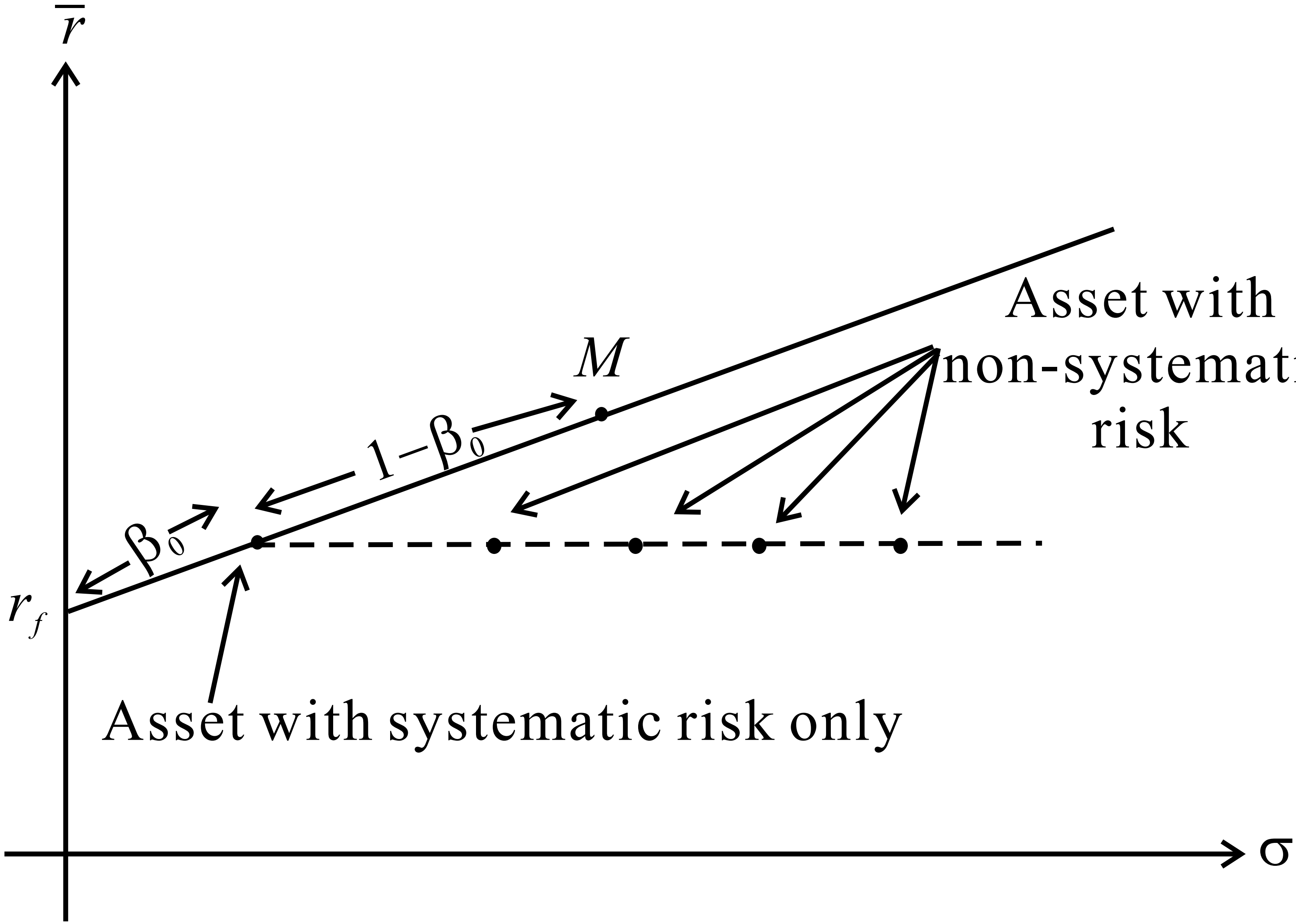
$$\begin{cases} \overline{r}_p = r_f + \beta(\overline{r}_M - r_f) \quad \Rightarrow \quad \beta = \frac{\rho_{iM}\sigma_M\sigma_p}{\sigma_M^2} = \frac{\sigma_p}{\sigma_M} \\ \overline{r}_p = r_f + \frac{\overline{r}_M - r_f}{\sigma_M}\sigma \quad \Leftrightarrow \quad \rho_{iM} = 1. \end{cases}$$

Portfolios not on the CML – non-efficient portfolios

For other portfolios with the same value of β_0 but not lying on the CML, they lie below the CML since they are non-efficient portfolios. With the same value of β_0, they all have the same expected rate of return given by

$$\overline{r} = r_f + \beta_0(\overline{r}_M - r_f)$$

but the portfolio variance is greater than or equal to $\beta_0^2\sigma_M^2$. The extra part of the portfolio variance is var(ϵ_i).



equation of CML: $\overline{r} = r_f + \frac{\overline{r}_M - r_f}{\sigma_M}\sigma$

Note that ϵ_i is uncorrelated with r_M as revealed by $\text{cov}(\epsilon_i, r_M) = 0$. The term $\text{var}(\epsilon_i)$ is called the *non-systematic* or *specific* risk. This risk can be reduced by diversification.

Consider

$$\begin{aligned} \mu_P &= \sum_{i=1}^{n} w_i \overline{r}_i = \sum_{i=1}^{n} (1 - \beta_{iM}) w_i r_f + \sum_{i=1}^{n} \beta_{iM} w_i \overline{r}_M \\ \sigma_P^2 &= \sum_{i,j=1}^{n} w_i w_j \beta_{iM} \beta_{jM} \sigma_M^2 + \sum_{i=1}^{n} w_i^2 \sigma_{\epsilon_i}^2. \end{aligned}$$

Let $\beta_{PM} = \sum_{i=1}^{n} w_i \beta_{iM}$ and $\alpha_P = \sum_{i=1}^{n} w_i (1 - \beta_{iM}) r_f$, then

$$\begin{aligned} \mu_P &= \alpha_P + \beta_{PM} \mu_M \\ \sigma_P^2 &= \beta_{\rho M}^2 \sigma_M^2 + \sum_{i=1}^{n} w_i^2 \sigma_{\epsilon_i}^2. \end{aligned}$$

Suppose we take $w_i = 1/n$ so that

$$\sigma_P^2 = \beta_{PM}^2 \sigma_M^2 + \frac{1}{n^2} \sum_{i=1}^{n} \sigma_{\epsilon_i}^2 = \beta_{PM}^2 \sigma_M^2 + \overline{\sigma}^2/n,$$

where $\overline{\sigma}^2$ is the average of $\sigma_{\epsilon_1}^2, \cdots, \sigma_{\epsilon_n}^2$. When n is sufficiently large

$$\sigma_P \to \left(\sum_{i=1}^{n} w_i \beta_{iM} \right) \sigma_M = \beta_{PM} \sigma_M.$$

- We may view β_{iM} as the contribution of asset i to the portfolio variance.

- From $\sigma_i^2 = \beta_{iM}^2 \sigma_M^2 + \sigma_{\epsilon_i}^2$, the contribution from $\sigma_{\epsilon_i}^2$ to portfolio variance goes to zero as $n \to \infty$.

Example

Consider the following set of data for 3 risky assets, market portfolio and risk free asset:

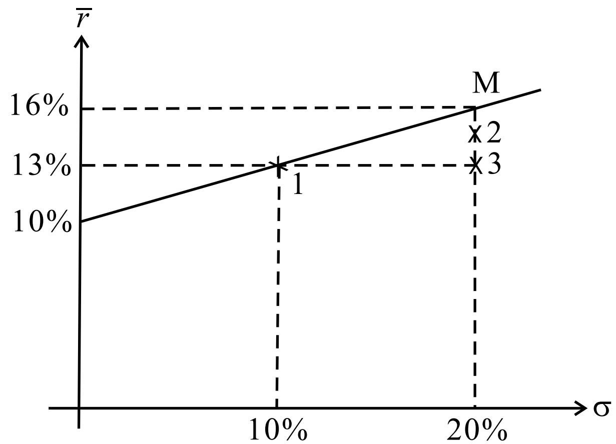
portfolio/security	σ	ρ_{iM}	β	actual expected rate of return $= \dfrac{E[P_1 + D_1]}{P_0} - 1.0$
1	10%	1.0	0.5	13%
2	20%	0.9	0.9	15.4%
3	20%	0.5	0.5	13%
market portfolio	20%	1.0	1.0	16%
risk free asset	0	0.0	0.0	10%

Use of CML

The CML identifies expected rates of return which are available on *efficient portfolios* of all possible risk levels. Portfolios 2 and 3 lie below the CML. The market portfolio, the risk free asset and Portfolio 1 all lie on the CML. Hence, Portfolio 1 is efficient while Portfolios 2 and 3 are non-efficient.

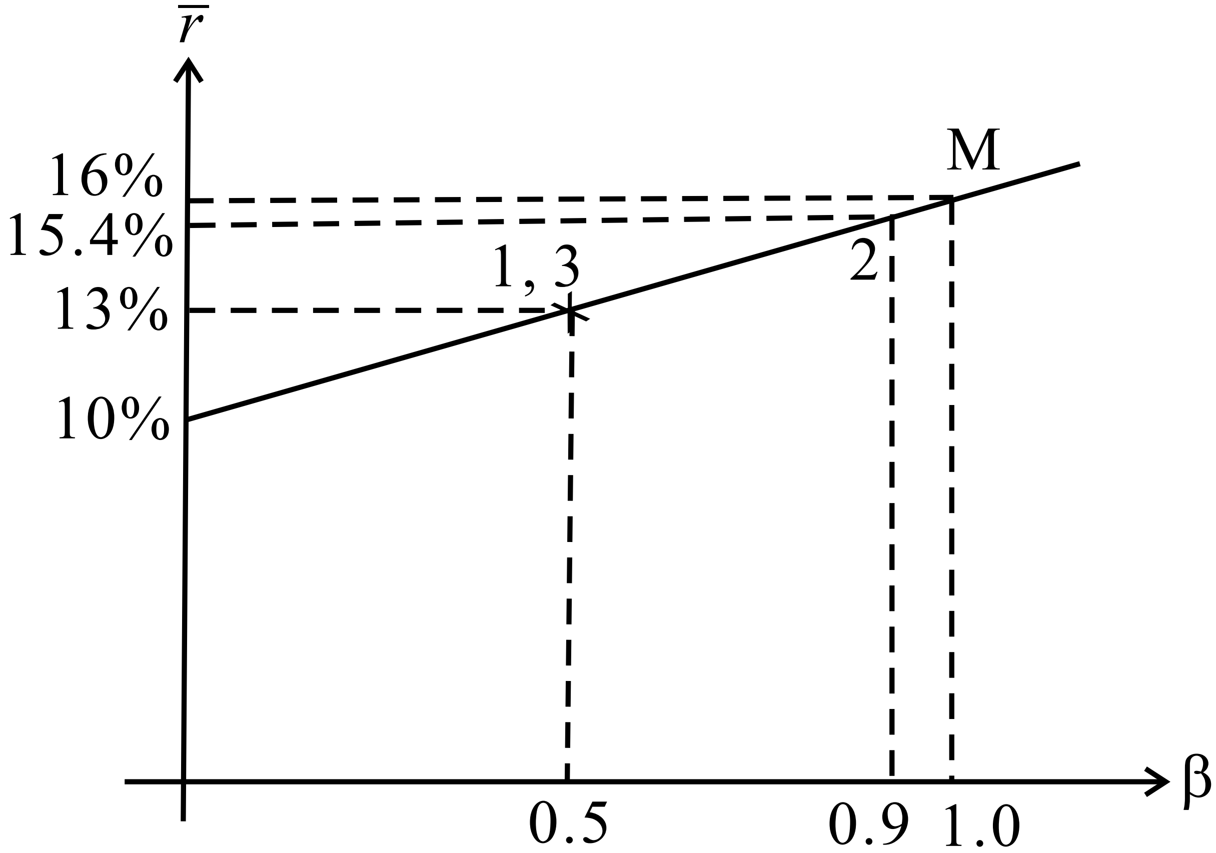
$$\text{At } \sigma = 10\%, \overline{r} = \underbrace{10\%}_{r_f} + \underbrace{10\%}_{\sigma} \times \underbrace{\frac{(16-10)\%}{20\%}}_{(\overline{r}_M - r_f)/\sigma_M} = 13\%.$$

$$\text{At } \sigma = 20\%, \overline{r} = 10\% + 20\% \times \frac{(16-10)\%}{20\%} = 16\%.$$



Use of SML

The SML asks whether the portfolio provides a return equal to what equilibrium conditions suggest should be earned.



Impact of ρ_{iM}

Portfolio 1 has unit value of ρ_{iM}, that is, it is perfectly correlated with the market portfolio. Hence, Portfolio 1 has zero non-systematic risk.

Portfolios 2 and 3 both have ρ_{iM} less than one.

Portfolio 2 has ρ_{iM} closer to one and so it lies closer to the CML.

The expected rates of return of the portfolios for the given values of beta are given by

$$\begin{aligned} \overline{r}_1 &= \overline{r}_3 = \underbrace{10\%}_{r_f} + \underbrace{0.5}_{\beta} \times (\underbrace{16\% - 10\%}_{\overline{r}_M - r_f}) = 13\% \\ \overline{r}_2 &= 10\% + 0.9 \times (16\% - 10\%) = 15.4\%. \end{aligned}$$

These expected rates of return suggested by the SML agree with the actual expected rates of return. Hence, each investment is fairly priced.

Summary

The CAPM predicts that the excess return on any stock (portfolio) adjusted for the risk on that stock (portfolio) should be the same

$$\frac{E[r_i] - r_f}{\beta_i} = \frac{E[r_j] - r_f}{\beta_j}.$$

Recall the somewhat restrictive assumptions of the standard CAPM

- all agents have homogeneous expectations
- agents maximize expected return relative to the standard deviation
- agents can borrow or lend unlimited amounts at the riskfree rate
- the market is in equilibrium at all times.

In real world, it is possible that over short periods the market is not in equilibrium and profitable opportunities arises.

CAPM as a pricing formula

Suppose an asset is purchased at P and later sold at Q. The rate of return is $\dfrac{Q-P}{P}$, P is known and Q is random. Using the CAPM,

$$\frac{\overline{Q}-P}{P} = r_f + \beta(\overline{r}_M - r_f) \text{ so that } P = \frac{\overline{Q}}{1 + r_f + \beta(\overline{r}_M - r_f)}.$$

The factor $\dfrac{1}{1 + r_f + \beta(\overline{r}_M - r_f)}$ can be regarded as the risk adjusted discount rate.

Example (Investment in a mutual fund)

A mutual fund invests 10% of its funds at the risk free rate of 7% and the remaining 90% at a widely diversified portfolio that closely approximates the market portfolio, and $\overline{r}_M = 15\%$. The beta of the fund is then equal to 0.9.

Suppose the expected value of one share of the fund one year later is \$110, what should be the fair price of one share of the fund now?

According to the pricing form of the CAPM, the current fair price of one share $= \dfrac{\$110}{1 + 7\% + 0.9 \times (15 - 8)\%} = \dfrac{\$110}{1.142} = \$96.3.$

Linearity of pricing?

Note that $\beta = \text{cov}\left(\frac{Q}{P} - 1, r_M\right) \Big/ \sigma_M^2$ so that $\beta = \frac{\text{cov}(Q, r_M)}{P\sigma_M^2}$. We then have

$$1 = \frac{\overline{Q}}{P(1 + r_f) + \text{cov}(Q, r_M)(\overline{r}_M - r_f)/\sigma_M^2}$$

so that

$$P = \frac{1}{1 + r_f}\left[\overline{Q} - \frac{\text{cov}(Q, r_M)(\overline{r}_M - r_f)}{\sigma_M^2}\right].$$

The bracket term is called the certainty equivalent of Q. In this form, the linearity of Q is more apparent! Note that the riskfree discount factor $\frac{1}{1 + r_f}$ is applied. Net present value is

$$-P + \frac{1}{1 + r_f}\left[\overline{Q} - \frac{\text{cov}(Q, r_M)(\overline{r}_M - r_f)}{\sigma_M^2}\right].$$

Use of CAPM for choice of risky asset

Let r_j denote the equilibrium rate of return of risky asset i as deduced from CAPM and S be its equilibrium price. Let P_0 be the market price of asset j and $\widetilde{P}_e$ be the random return. Let r_j' be the rate of return deduced from the market price.

$$\begin{aligned} E[r_j'] &= E\left[\frac{\widetilde{P}_e - P_0}{P_0}\right] = E\left[\frac{\widetilde{P}_e}{S}\right]\frac{S}{P_0} - 1 \\ &= (E[r_j] + 1)\frac{S}{P_0} - 1 \\ &= \left[1 + r_f + \beta_{jm}(\mu_m - r_f)\right]\frac{S}{P_0} - 1 \end{aligned}$$

so that

$$\begin{aligned} E[r_j'] - r_f &= (r_f + 1)\left(\frac{S}{P_0} - 1\right) + \frac{\mathrm{cov}(r_j, r_m)}{\sigma_m^2}(\mu_m - r_f)\frac{S}{P_0} \\ &= (r_f + 1)\left(\frac{S}{P_0} - 1\right) + \frac{\mathrm{cov}(\widetilde{P}_e/P_0, r_m)}{\sigma_m^2}(\mu_m - r_f) \\ &= \alpha_j + \beta_{jm}'(\mu_m - r_f) \end{aligned}$$

where

$$\alpha_j = (1 + r_f)\left(\frac{S}{P_0} - 1\right) \quad \text{and} \quad \beta_{jm}' = \mathrm{cov}(\widetilde{P}_e/P_0, r_m)/\sigma_m^2.$$

Here, β_{jm}' is the beta as deduced from the market price.

In the literature, α_j is called the abnormal return.

(i) $P_0 = S \quad \Leftrightarrow \quad \alpha_j = 0$ appropriate price

(ii) $P_0 > S \quad \Leftrightarrow \quad \alpha_j < 0$ price too high

(iii) $P_0 < S \quad \Leftrightarrow \quad \alpha_j > 0$ price too low.

Take the historical data and perform regression

$$r'_{jt} - r_f = \alpha_j + \beta'_{jm}(r_{mt} - r_f) + \epsilon_{jt}, \quad t = 1, 2, \cdots, T.$$

If the estimate $\widehat{\alpha}_j$ differs significantly from zero, this indicates mis-pricing.

Market proxy

Take a subset of N risky assets from the financial market and assume their beta values to be $\boldsymbol{\beta}_m = (\beta_{1m} \quad \beta_{2m} \quad \beta_{Nm})^T$. We would like to construct the market proxy $\widehat{m}$ from these N risky assets such that the beta of $\widehat{m}$ is one and $\boldsymbol{\beta}_{\widehat{m}} = \boldsymbol{\beta}_m$.

Consider the following minimization problem

$$\min_{\substack{\boldsymbol{w} \\ \boldsymbol{\beta}_m^T \boldsymbol{w} = 1}} \quad \boldsymbol{w}^T \Omega \boldsymbol{w}$$

where Ω is the covariance matrix of the random returns of the N assets. The first order conditions give

$$\Omega \boldsymbol{w} - \lambda \boldsymbol{\beta}_m = 0 \quad \text{and} \quad \boldsymbol{\beta}_m^T \boldsymbol{w} = 1.$$

We obtain

$$\lambda = \frac{1}{\boldsymbol{\beta}_m^T \Omega^{-1} \boldsymbol{\beta}_m} \quad \text{and} \quad \boldsymbol{w}^* = \frac{\Omega^{-1}\boldsymbol{\beta}_m}{\boldsymbol{\beta}_m^T \Omega^{-1} \boldsymbol{\beta}_m}.$$

We set the market proxy to be $\boldsymbol{w}^*$. If suffices to check that $\boldsymbol{\beta}_{\widehat{m}} = \boldsymbol{\beta}_m$. Consider

$$\beta_{j\widehat{m}} = \frac{\text{cov}(r_j, r_{\widehat{m}})}{\sigma_{\widehat{m}}^2} = \frac{\boldsymbol{e}_j^T \Omega \boldsymbol{w}^*}{\boldsymbol{w}^{*T} \Omega \boldsymbol{w}^*}, \text{ where } \boldsymbol{e}_j = (0 \cdots 1 \cdots 0)^T,$$

so that

$$\boldsymbol{\beta}_{\widehat{m}} = \frac{\Omega \boldsymbol{w}^*}{\boldsymbol{w}^{*T} \Omega \boldsymbol{w}^*} = \frac{\Omega(\lambda \Omega^{-1} \boldsymbol{\beta}_m)}{\lambda^2 (\Omega^{-1}\boldsymbol{\beta}_m)^T \Omega (\Omega^{-1}\boldsymbol{\beta}_m)} = \frac{\boldsymbol{\beta}_m}{\lambda \boldsymbol{\beta}_m^T \Omega^{-1} \boldsymbol{\beta}_m} = \boldsymbol{\beta}_m.$$

Suppose the residual terms e_j in the linear regression of r_j and $r_{\widehat{m}}$ is uncorrelated with the rate of return of the actual market portfolio r_m, we can formally write

$$r_j = \alpha_j + \beta_j r_{\widehat{m}} + e_j$$

where $E[e_j] = \text{cov}(r_m, e_j) = 0$, then

$$\begin{aligned}\beta_{jm} &= \frac{\text{cov}(r_j, r_m)}{\sigma_m^2} = \frac{\text{cov}(\alpha_j + \beta_j r_{\widehat{m}} + e_j, r_m)}{\sigma_m^2} \\ &= \beta_j \underbrace{\frac{\text{cov}(r_{\widehat{m}}, r_m)}{\sigma_m^2}}_{\text{equals 1 since } \widehat{m} \text{ has unit beta}} = \beta_j.\end{aligned}$$

Lastly, we solve for α_j using

$$\overline{r}_j = \alpha_j + \beta_{jm}\overline{r_{\widehat{m}}} = r_f + \beta_{jm}(\overline{r}_m - r_f)$$

so that

$$\alpha_j = r_f + \beta_{jm}(\overline{r}_m - \overline{r}_{\widehat{m}}) - \beta_{jm} r_f.$$

The alternative representation of the equation of SML is

$$\overline{r}_j = r_f + \beta_{jm}(\overline{r_{\widehat{m}}} - r_f) + \beta_{jm}(\overline{r}_m - \overline{r}_{\widehat{m}}), \text{ where } \overline{r}_{\widehat{m}} = \boldsymbol{w}^{*T}\boldsymbol{\mu}.$$

Generalization of CAPM under utility framework

Let w_i and R_i be the weights and random return of risky asset i in the portfolio of n risky assets and one riskfree asset. Let R_f be the return of the riskfree asset. Let W_0 be the initial wealth and $\widetilde{W}$ be the random wealth one period later, then

$$\widetilde{W} = W_0 \left(w_0 R_f + \sum_{i=1}^{n} w_i R_i \right),$$

where w_0 is the weight of the riskfree asset and $\displaystyle\sum_{i=0}^{n} w_i = 1$.

The solution of the portfolio optimization problem

$$\max_{\sum_{i=1}^{n} w_i = 1} E\left[u(\widetilde{W})\right], \quad \text{assuming } u' > 0 \quad \text{and} \quad u'' < 0,$$

is given by

$$E\left[u'(\widetilde{W})(R_i - R_f)\right] = 0, \quad i = 1, 2, \cdots, n.$$

Alternative formulation

Define the following set of control variables θ_s and the corresponding utility function V, where

$$\theta_s = \sum_i w_i z_{si}, \quad s = 1, 2, \cdots, S,$$

$$V(\theta_1, \cdots, \theta_S) = \sum_{s=1}^{S} \pi_s u(\theta_s).$$

Here, π_s denote the probability of occurrence of state s. Define

$$Q = \left\{ (\theta_1 \cdots \theta_S) \in \mathbb{R}^S : \quad \theta_s = \sum_i w_i z_{si}, s = 1, \cdots, S \quad \text{and} \quad \sum_i w_i = 1 \right\}.$$

The portfolio optimization is equivalent to

$$\max_{\substack{\theta_s \\ s.t. (\theta_1 \cdots \theta_S)^T \in Q}} \quad V(\theta_1, \cdots, \theta_S).$$

Since Q is a convex set in $\mathbb{R}^S$ and $V(\theta_1, \cdots, \theta_S)$ is a strictly concave function, the above is a standard convex optimization problem with strictly concave objective function. It is well known that the solution exists and it is also unique.

Define Z be the payoff function where

$$\boldsymbol{\theta}^* = Z\boldsymbol{w}^*.$$

If there is no redundant security, then Z has full column rank. The left inverse exists and it is uniquely given by

$$Z^- = (Z^T Z)^{-1} Z^T$$

and $\boldsymbol{w}^*$ is uniquely given by

$$\boldsymbol{w}^* = Z^- \boldsymbol{\theta}^*.$$

Generalized CAPM formula

From the first order conditions, we obtain

$$E\left[u'(\widetilde{W})R_i\right] = R_f E\left[u'(\widetilde{W})\right]$$

so that

$$\begin{aligned}\text{cov}(u'(\widetilde{W}), R_i) &= E[u'(\widetilde{W})R_i] - E[u'(\widetilde{W})]E[R_i] \\ &= -E[u'(\widetilde{W})](E[R_i] - R_f). \qquad (i)\end{aligned}$$

Multiplying both sides by w_i and summing, we obtain

$$\begin{aligned}\text{cov}(u'(\widetilde{W}), \widetilde{W}/W_0) &= \text{cov}(u'(\widetilde{W}), R_f) + \sum_{i=1}^n \text{cov}(u'(\widetilde{W}), w_i R_i) \\ &= -E[u'(\widetilde{W})]\left\{E\left[\sum_{i=1}^n w_i R_i\right] - R_f\left(\sum_{i=1}^n w_i\right)\right\} \\ &= -E[u'(\widetilde{W})](E[\widetilde{W}/W_0] - R_f), \qquad (ii)\end{aligned}$$

since $\sum_{i=1}^{n} w_i R_i = \frac{\widetilde{W}}{W_0} - w_0 R_f$ and $\sum_{i=0}^{n} w_i = 1$. Write $R_p = \widetilde{W}/W_0$, which is the random return of the portfolio.

Combining (i) and (ii), we obtain

$$\frac{\text{cov}(u'(\widetilde{W}), R_i)}{\text{cov}(u'(\widetilde{W}), R_p)} = \frac{E[R_i] - R_f}{E[R_p] - R_f}.$$

When u is a quadratic utility, suppose $u(W) = \alpha + \beta W + \gamma W^2$, we have $u'(W) = \beta + 2\gamma W$. Since $u'(W)$ is a linear function, the constant term does not contribute to the covariances and the multiplying factor 2γ can be cancelled, we then recover the standard CAPM formula (in terms of rates of return)

$$\frac{\text{cov}(r_p, r_i)}{\text{cov}(r_p, r_p)} = \frac{E[r_i] - r_f}{E[r_p] - r_f}.$$

Difficulties with the CAPM

1. Application of the mean-variance theory requires the determination of the parameter values: mean values of the asset returns and the covariances among them. Suppose there are n assets, then there are n mean values, n variances and $\frac{n(n-1)}{2}$ covariances. For example, when $n = 1,000$, the number of parameter values required $= 501,500$.

2. In the CAPM, there is really only one factor that influences the expected return, namely, the covariance between the asset return and the return on the market portfolio.

   The assumption of investors utilizing a mean variance framework is replaced by an assumption of the process generating security returns.

Merit of the Arbitrage Pricing Theory (APT)

APT is based on the law of one price: portfolios with the same payoff have the same price. APT requires that the returns on any stock be linearly related to a number of factors. It implies that the return on a security can be broken down into an expected return and an unexpected (or surprise) component. The APT allows one to go from a multi-index model to a description of equilibrium.

Factor models

Randomness displayed by the returns of n assets often can be traced back to a smaller number of underlying basic sources of randomness (factors). Hopefully, this leads to a simpler covariance structure.

Specifying the influences affecting the return-generating process

1. Inflation

   Inflation impacts both the level of the discount rate and the size of the future cash flows.

2. Risk premia

   Differences between the return on safe bonds and more risky bonds are used to measure the market's reaction to risk.

3. Industrial production

   Changes in industrial production affect the opportunities facing investors and the real value of cash flow.

Single-factor model

Rates of return r_i and the factor are related by

$$r_i = a_i + b_i f + e_i \quad i = 1, 2, \cdots, n.$$

Here, f is a random quantity, a_i and b_i are fixed constants, e_i's are random errors (without loss of generality, take $E[e_i] = 0$). Further, we assume

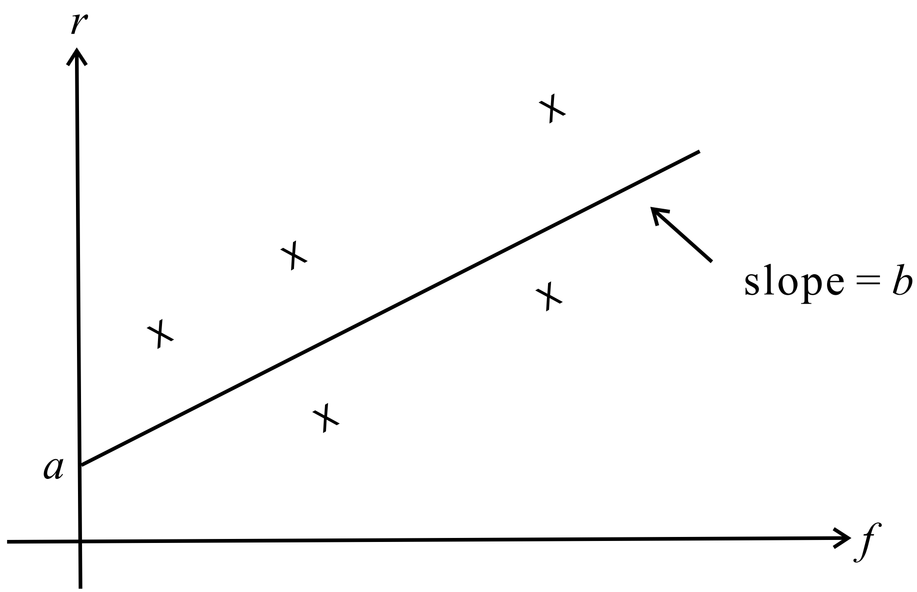
$$E[(f - \overline{f})e_i] = 0 \quad \text{and} \quad E[e_i e_j] = 0, \quad i \neq j.$$

We can deduce

$$\text{cov}(e_i, f) = E[e_i f] - E[e_i]E[f] = 0.$$

The variances of e_i's are known, which are denoted by $\sigma_{e_i}^2$.

b_i = factor loading; which measures the sensitivity of the return to the factor.



⋆ Different data sets (past one month or two months data) may lead to different estimated values.

$$\begin{aligned} \overline{r}_i &= a_i + b_i \overline{f} \\ \sigma_i^2 &= b_i^2 \sigma_f^2 + \sigma_{e_i}^2 \\ \sigma_{ij} &= b_i b_j \sigma_f^2, \quad i \neq j \\ b_i &= \text{cov}(r_i, f)/\sigma_f^2. \end{aligned}$$

Only a_i's, b_i's, $\sigma_{e_i}^2$'s, $\overline{f}$ and σ_f^2 are required. There are $(3n+2)$ parameters.

Portfolio parameter

Let w_i denote the weight for asset $i, i = 1, 2, \cdots, n$.

$$r_p = \sum_{i=1}^n w_i a_i + \sum_{i=1}^n w_i b_i f + \sum_{i=1}^n w_i e_i$$

so that $r_p = a + bf + e$, where

$$a = \sum_{i=1}^n w_i a_i, \quad b = \sum_{i=1}^n w_i b_i \quad \text{and} \quad e = \sum_{i=1}^n w_i e_i.$$

Further, since $E[e_i] = 0, E[(f - \overline{f})e_i] = 0$ so that $E[e] = 0$ and $E[(f - \overline{f})e] = 0$; e and f are uncorrelated. Also, $\sigma_e^2 = \sum_{i=1}^n w_i^2 \sigma_{e_i}^2$. Overall variance of portfolio $= \sigma^2 = b^2 \sigma_f^2 + \sigma_e^2$.

For simplicity, we take $\sigma_{e_i}^2 = S^2$ and $w_i = 1/n$ so that $\sigma_e^2 = \dfrac{S^2}{n}$.

As $n \to \infty, \sigma_e^2 \to 0$. The overall variance of portfolio σ^2 tends to decrease as n increases since σ_e^2 goes to zero, but σ^2 does not go to zero since $b^2\sigma_f^2$ remains finite.

The risk due to e_i is said to be *diversifiable* since its contribution to overall risk is essentially zero in a well-diversified portfolio. This is because e_i's are independent and so each can be reduced by diversification.

The risk due to $b_i f$ is said to be systematic since it is present even in a diversified portfolio.

CAPM as a factor model

Express the model in terms of excess returns $r_i - r_f$ and $r_M - r_f$.

$$r_i - r_f = \alpha_i + \beta_i(r_M - r_f) + e_i.$$

With $e_i = 0$, this corresponds to the characteristic line

$$r_i - r_f = \alpha_i + \beta_i(r_M - r_f). \tag{1}$$

Taking the expectation on both sides

$$\overline{r}_i - r_f = \alpha_i + \beta_i(\overline{r}_M - r_f).$$

With $\alpha_i = 0$, the above relation reduces to the CAPM. We assume that e_i is uncorrelated with the market return r_M. Taking the covariance of both sides of (1) with r_M

$$\sigma_{iM} = \beta_i \sigma_M^2 \quad \text{or} \quad \beta_i = \frac{\sigma_{iM}}{\sigma_M^2}.$$

- The characteristic line is more general than the CAPM since it allows α_i to be non-zero.

- The factor model does not assume any utility function or that agents consider only the mean and variance of prospective portfolios.

Remarks

1. The presence of non-zero α_i can be regarded as a measure of the amount that asset i is mispriced. A stock with positive α_i is considered performing better than it should.

2. The general CAPM model is based on an arbitrary covariance structure while the one-factor model assumes very simple covariance structure.

Single-factor, residual-risk-free models

Assume zero idiosyncratic (asset-specific) risk,

$$r_i = a_i + b_i f, \quad i = 1, 2, \cdots, n,$$

where $E[f] = 0$ so that $\overline{r}_i = a_i$.

Consider two assets which have two different b_i's, what should be the relation between their expected returns under the assumption of no arbitrage?

Consider a portfolio with weight w in asset i and weight $1 - w$ in asset j. The portfolio return is

$$r_p = w(a_i - a_j) + a_j + [w(b_i - b_j) + b_j]f.$$

By choosing $w^* = \dfrac{b_j}{b_j - b_i}$, the portfolio becomes riskfree and

$$r_p^* = \frac{b_j(a_i - a_j)}{b_j - b_i} + a_j.$$

This must be equal to the return of the riskfree asset, denoted by r_0. We write the relation as

$$\frac{a_j - r_0}{b_j} = \frac{a_i - r_0}{b_i} = \lambda.$$

↑ set

Hence, $\overline{r}_i = r_0 + b_i\lambda$, where λ is the factor risk premium. Note that when two assets have the same b, they have the same expected return.

Example 1 (Four stocks and one index)

Historical rates of return for four stocks over 10 years, record of industrial price index over the same period.

Estimate of $\overline{r}_i$ is $\widehat{\overline{r}}_i = \frac{1}{10}\sum_{k=1}^{10} r_i^k$.

$$\mathrm{var}(r_i) = \frac{1}{9}\sum_{k=1}^{10}(r_i^k - \widehat{\overline{r}}_i)^2$$

$$\mathrm{cov}(r_i, f) = \frac{1}{9}\sum_{k=1}^{10}(r_i^k - \widehat{\overline{r}}_i)(f^k - \widehat{\overline{f}}).$$

Once the covariances are estimated, b_i and a_i are obtained:

$$b_i = \frac{\mathrm{cov}(r_i, f)}{\mathrm{var}(f)} \quad \text{and} \quad a_i = \widehat{\overline{r}}_i - b_i \widehat{\overline{f}}.$$

We estimate the variance of the error under the assumption that these errors are uncorrelated with each other and with the index. The formula to be used is

$$\text{var}(e_i) = \text{var}(r_i) - b_i^2 \text{var}(f).$$

- Unfortunately, the error variances are almost as large as the variances of the stock returns.
- There is a high non-systematic risk, so the choice of this factor does not explain much of the variation in returns.
- Further, $\text{cov}(e_i, e_j)$ are not small so that the errors are highly correlated. We have $\text{cov}(e_1, e_2) = 44$ and $\text{cov}(e_2, e_3) = 91$. Recall that the factor model was constructed under the assumption of zero error covariances.

Year	**Stock 1**	**Stock 2**	**Stock 3**	**Stock 4**	**Index**
1	11.91	29.59	23.27	27.24	12.30
2	18.37	15.25	19.47	17.05	5.50
3	3.64	3.53	−6.58	10.20	4.30
4	24.37	17.67	15.08	20.26	6.70
5	30.42	12.74	16.24	19.84	9.70
6	−1.45	−2.56	−15.05	1.51	8.30
7	20.11	25.46	17.80	12.24	5.60
8	9.28	6.92	18.82	16.12	5.70
9	17.63	9.73	3.05	22.93	5.70
10	15.71	25.09	16.94	3.49	3.60
aver	15.00	14.34	10.90	15.09	6.74
var	90.28	107.24	162.19	68.27	6.99
cov	2.34	4.99	5.45	11.13	6.99
b	0.33	0.71	0.78	1.59	1.00
a	12.74	9.53	5.65	4.36	0.00
e-var	89.49	103.68	157.95	50.55	

The record of the rates of return for four stocks and an index of industrial prices are shown. The averages and variances are all computed, as well as the covariance of each with the index. From these quantities, the b_i''s and the a_i's are calculated. Finally, the computed error variances are also shown. The index does not explain the stock price variations very well.

Two-factor extension

Consider the two-factor model

$$r_i = a_i + b_{i1} f_1 + b_{i2} f_2, \quad i = 1, 2, \cdots, n,$$

where the factor f_1 and f_2 are chosen such that

$$E[f_1 f_2] = 0, E[f_1^2] = E[f_2^2] = 1, E[f_1] = E[f_2] = 0.$$

We assume $\mathbf{1}, \boldsymbol{b}_1 = \begin{pmatrix} b_{11} \\ b_{21} \\ b_{31} \end{pmatrix}$ and $\boldsymbol{b}_2 = \begin{pmatrix} b_{12} \\ b_{22} \\ b_{32} \end{pmatrix}$ to be linearly independent. Form the portfolio with weights w_1, w_2 and w_3 so that

$$r_p = \sum_{i=1}^{3} w_i a_i + f_1 \sum_{i=1}^{3} w_i b_{i1} + f_2 \sum_{i=1}^{3} w_i b_{i2}.$$

Since $\mathbf{1}, \boldsymbol{b}_1$ and $\boldsymbol{b}_2$ are independent, the following system of equations

$$\begin{pmatrix} 1 & 1 & 1 \\ b_{11} & b_{21} & b_{31} \\ b_{12} & b_{22} & b_{32} \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

always has unique solution. In this case, the portfolio becomes riskfree so

$$r_p = \sum_{i=1}^{3} w_i a_i = r_0$$

or

$$\sum_{i=1}^{3} (a_i - r_0) w_i = 0.$$

Hence, there is a non-trivial solution to

$$\begin{pmatrix} a_1 - r_0 & a_2 - r_0 & a_3 - r_0 \\ b_{11} & b_{21} & b_{31} \\ b_{12} & b_{22} & b_{32} \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The above coefficient matrix must be singular so that

$$a_i - r_0 = \lambda_1 b_{i1} + \lambda_2 b_{i2}$$

for some λ_1 and λ_2. The risk premium on asset i is given by

Absence of riskfree asset

$$\overline{r}_i - r_0 = \lambda_1 b_{i1} + \lambda_2 b_{i2}, \quad i = 1, 2, \cdots, n.$$

If no riskfree asset exists naturally, then we replace r_0 by λ_0. Once λ_0, λ_1 and λ_2 are known, the expected return of an asset is completely determined by the factor loadings b_{i1} and b_{i2}. Theoretically, a riskless asset can be constructed from any two risky assets so that λ_0 can be determined.

Prices of risk, λ_1 and λ_2

– interpreted as the excess expected return per unit risk associated with the factors f_1 and f_2.

⋆ Given any two portfolios P and M with $\frac{b_{P1}}{b_{P2}} \neq \frac{b_{M1}}{b_{M2}}$, we can solve for λ_1 and λ_2 in terms of the expected return on these two portfolios: $\overline{r}_M - r_0$ and $\overline{r}_P - r_0$. One can show that

$$\overline{r}_i = r_0 + b'_{i1}(\overline{r}_M - r_0) + b'_{i2}(\overline{r}_P - r_0)$$

where

$$b'_{i1} = \frac{b_{i1}b_{P2} - b_{i2}b_{P1}}{b_{M1}b_{P2} - b_{M2}b_{P1}}, \quad b'_{i2} = \frac{b_{i2}b_{M1} - b_{i1}b_{M2}}{b_{M1}b_{P2} - b_{M2}b_{P1}}.$$

Linkage between CAPM and factor model

Consider a two-factor model

$$r_i = a_i + b_{i1} f_1 + b_{i2} f_2 + e_i,$$

the covariance of the i^{th} asset with the market portfolio is given by

$$\text{cov}(r_M, r_i) = b_{i1}\text{cov}(r_M, f_1) + b_{i2}\text{cov}(r_M, f_2) + \text{cov}(r_M, e_i).$$

It is reasonable to ignore $\text{cov}(r_M, e_i)$ if the market represents a well-diversified portfolio.

We write the beta of the asset as

$$\begin{aligned}\beta_i &= \frac{\text{cov}(r_M, r_i)}{\sigma_M^2} \\ &= b_{i1} \underbrace{\frac{\text{cov}(r_M, f_1)}{\sigma_M^2}}_{\beta_{f_1}} + b_{i2} \underbrace{\frac{\text{cov}(r_M, f_2)}{\sigma_M^2}}_{\beta_{f_2}}.\end{aligned}$$

The factor betas β_{f_1} and β_{f_2} do not depend on the particular asset.

The weight of these factor betas in the overall asset beta is equal to the factor loadings. In this framework, different assets have different betas corresponding to different loadings.

Portfolio management

- Employing a multi-index model allows the creation of an index fund that more closely matches the desired index such that the index has been matched in terms of all important sources of return movement.
  - allow exclusion (tobacco stocks) or inclusion (business relationship) of certain types of stocks.
- Allows one to closely match an index while purposely taking positions with respect to certain types of risk different from the positions contained in the index
  - pension fund: overseers want a portfolio that will perform especially well when the rate of inflation increases; has the same response to all factors affecting the index fund, except the inflation risk factor.

Solve a quadratic programming problem that minimizes the residual risks.

Example 2

Assume that a two factor model is appropriate, and there are an infinite number of assets in the economy. The cross-sectional a relationship between expected return and factor betas indicates the price of factor 1 is 0.15, and the price of factor 2 is -0.2. You have estimated factor betas for stocks X and Y as follows:

	β_1	β_2
Stock X	1.4	0.4
Stock Y	0.9	0.2

Also, the expected return on an asset having zero betas (with respect to both factors) is 0.05. What are the approximate equilibrium returns on each of the two stocks?

Solution

The expected return of an asset based on a two-factor model is given by

$$\overline{r}_i = \lambda_0 + \lambda_1 \beta_{i1} + \lambda_2 \beta_{i2}.$$

Here, $\lambda_0 =$ zero-beta return $= 0.05$

$$\lambda_1 = 0.15 \quad \text{and} \quad \lambda_2 = -0.2.$$

Now,

$$\begin{aligned} \overline{r}_1 &= \lambda_0 + \lambda_1 \beta_{11} + \lambda_2 \beta_{12} = 0.05 + 0.15 \times 1.4 - 0.2 \times 0.4 = 01.8; \\ \overline{r}_2 &= \lambda_0 + \lambda_1 \beta_{21} + \lambda_2 \beta_{22} = 0.05 + 0.15 \times 0.9 - 0.2 \times 0.2 = 0.145. \end{aligned}$$

Example 3

Assume that a three-factor model is appropriate, and there are an infinite number of assets. The expected return on a portfolio with zero beta values is 5 percent. You are interested in an equally weighted portfolio of two stocks, A and B. The factor prices are indicated in the accompanying table, along with the factor betas for A and B. Compute the approximate expected return of the portfolio.

Factor i	β_{iA}	β_{iB}	Factor Prices
1	0.3	0.5	0.07
2	0.2	0.6	0.09
3	1.0	0.7	0.02

Solution:

By APT, the expected return of a portfolio is given by

$$\bar{r}_p = \lambda_0 + \lambda_1 \beta_{P1} + \lambda_2 \beta_{P2} + \lambda_3 \beta_{P3}.$$

Here, $\lambda_0 = 5\%, \beta_{P1} = \frac{1}{2}(\beta_{1A} + \beta_{1B}) = \frac{1}{2}(0.3 + 0.5) = 0.4,$

$$\begin{aligned}\beta_{P2} &= \frac{1}{2}(\beta_{2A} + \beta_{2B}) = \frac{1}{2}(0.2 + 0.6) = 0.4,\\ \beta_{P3} &= \frac{1}{2}(\beta_{3A} + \beta_{3B}) = \frac{1}{2}(1.0 + 0.7) = 0.85.\end{aligned}$$

Given $\lambda_1 = 0.07, \lambda_2 = 0.09, \lambda_3 = 0.02$, so

$$\bar{r}_P = 5\% + 0.07 \times 0.4 + 0.09 \times 0.4 + 0.02 \times 0.85 = 13.1\%.$$

Example 4

Stocks 1 and 2 are affected by three factors, as shown here. Factor 2 and 3 are unique to each stock. Expected values of each are $E(F_1) = 3.0\%, E(F_2) = 0.0\%$, and $E(F_3) = 0.0\%$. Neither stock pays a dividend, and they are now selling at prices $P_1 = \$40$ and $P_2 = \$10$. You expect their prices in a year to be $E(P_1) = \$45$ and $E(P_2) = \$10.70$.

$$\begin{aligned}\widetilde{R}_1 &= 6.0(\widetilde{F}_1) + 0.3(\widetilde{F}_2) + 0.0(\widetilde{F}_3)\\ \widetilde{R}_2 &= 1.5(\widetilde{F}_1) + 0.0(\widetilde{F}_2) + 0.4(\widetilde{F}_3)\end{aligned}$$

a. What do factors 2 and 3 reflect? In the context of a broadly diversified portfolio, should the weights 0.3 and 0.4 be positive, as they are shown?

b. Neglecting F_2 and F_3, create a riskless arbitrage.

c. Relate the return equations to the CAPM.

Solution

a. Factors 2 and 3 appear to be firm specific factors in that they affect only a single stock. Across a large number of stocks, these factors will net to zero.

b. By APT, the expected returns of Stock 1 and Stock 2 are

$$\overline{r}_1 = 6\overline{F}_1 = 18\%, \overline{r}_2 = 1.5\overline{F}_2 = 4,5\%.$$

The "market" expected returns are

$$\overline{r}_{1,market} = \frac{45}{40} - 1.0 = 12.5\%; \quad \overline{r}_{2,market} = \frac{10.7}{10} - 1 = 7\%.$$

The arbitrage strategy is to short 1/4 unit of Stock 1 and buy one unit of Stock 2. The portfolio is riskless but the expected return $= \frac{-1}{4} \times 12.5\% + 7\% = 4.875\%$.

c. Here, $\overline{r}_i = 0 + \beta_i \overline{F}_1$; while CAPM gives $\overline{r}_i = r_f + \beta_i(\overline{r}_M - r_f)$. These correspond to $r_f = 0$ and $\overline{F}_1 = \overline{r}_M$.