Investment Science and Portfolio Analysis

Homework One

1. In a betting game with m possible outcomes, the return from a unit bet on i if outcome j occurs is given by

$$r_{ij} = \begin{cases} d_i & \text{if } j = i \\ -1 & \text{if } j \neq i \end{cases}, \quad j = 1, 2, \cdots, m,$$

   where $d_i > 0$, for all i. Assuming

$$\sum_{i=1}^{m} \frac{1}{1 + d_i} < 1,$$

   show that the betting strategy

$$\alpha_i = \frac{\frac{1}{1+d_i}}{1 - \sum_{i=1}^{m} \frac{1}{1+d_i}}, \quad i = 1, 2, \cdots, m,$$

   always yields a gain of exactly 1.

2. The random return vector of three securities achieves the following values: (4 2 3) and (2 4 3) with equal probabilities. Show that the optimal strategy based on the logarithm utility criterion is not unique. Find two such optimal strategies.

3. Consider a wheel with n sectors. If the wheel pointer lands on sector i, the payoff obtained is r_i for every unit bet on that sector. The chance of landing on sector i is $p_i, i = 1, 2, \cdots, n$. Let α_i be the fraction of one's capital bet on sector i. We require $\sum_{i=1}^{n} \alpha_i \leq 1$ and $\alpha_i \geq 0, i = 1, 2, \cdots, n$.

   (a) Assuming $\alpha_i > 0, i = 1, 2, \cdots, n$, show that the solution to α_k based on logarithm utility criterion must satisfy

$$\frac{p_k r_k}{r_k \alpha_k + 1 - \sum_{i=1}^{n} \alpha_i} - \sum_{j=1}^{n} \frac{p_j}{r_j \alpha_j + 1 - \sum_{i=1}^{n} \alpha_i} = 0, \quad k = 1, 2, \cdots, n.$$

   (b) Assuming $\sum_{i=1}^{n} \frac{1}{r_i} = 1$, show that a possible solution is $\alpha_i = p_i, i = 1, 2, \cdots, n$.

   (c) Suppose the sectors are ordered in such a way that $p_n r_n < p_i r_i$ for all $i = 1, 2, \cdots, n-1$, that is, sector n is the worst sector. Find a solution with $\alpha_n = 0$ and all other α_i's positive.

4. (The Dow Jones Average puzzle) The Dow Jones Industrial Average is an average of the prices of 30 industrial stocks with equal weights applied to all 30 stocks (but the sum of the weights is greater than 1). Occasionally (about twice per year) one of the 30 stocks splits (usually because its price has reached levels near $100 per share). When this happens all weights are adjusted upward by adding an amount ε to each of them, where ε is chosen so that the computed Dow Jones Average is continuous.

   Gavin Jones' father, Mr. D. Jones, uses the following investment strategy over 10-year period. At the beginning of the 10 years, Mr. Jones buys one share of each of the 30 stocks in the Dow Jones average. He puts the stock certificates in a drawer and does no more trading. If dividends arrive, he spends them. If additional certificates arrive due to stock splits, he tosses them in the drawer along with the others. At the end of 10 years, he cashes in all certificates. He then compares his overall return, based on the ratio of the final value to the original cost, with the hypothetical return defined as the ratio of the Dow Jones Average now to 10 years ago. He is surprised to see that there is a difference. Which return do you think will be larger? And why? (Ignore transactions costs, and assume that all 30 stocks remain in the average over the 10-year period.) [The difference, when actually measure, is close to 1% per year.]

5. Suppose there are n stocks. Each of them has a price that is governed by geometric Brownian motion, with $\mu_i = 15\%$ and $\sigma_i = 40\%$ for all i. However, these stocks are correlated, and for simplicity we assume that $\sigma_{ij} = 0.08$ for all $i \neq j$. What is the value of the expected growth rate and variance for a portfolio having equal portions invested in each of the stocks?

6. Suppose two people start with the same initial capital level X_0. Assume that person A invests using the log-optimal strategy while person B uses some other strategy. Denote the resulting capital streams by x_k^A and x_k^B, respectively, for periods $k = 1, 2, \cdots$. Show that

$$E\left[\frac{X_k^B}{X_k^A}\right] \leq 1 \quad \text{for all } k.$$

   This property argues in favor of using the log-optimal strategy.

7. Consider the class of power utility function

$$U(x) = \frac{x^\gamma}{\gamma} \quad \text{for} \quad \gamma \leq 1.$$

   Show that this class includes the logarithm utility. (Hint: consider $\gamma \to 0^+$). The log-optimal strategy has been shown to exhibit the property that the maximization of $E[U(X_k)]$ with a fixed-proportions strategy only requires the maximization of the expected utility of single-period investment as given by $E[U(X_1)]$. Check whether such property can be extended to the power utility functions.