MAFS 5030

Quantitative Modeling of Derivative Securities

Homework One

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1. A forward contract is written at time 0 and there are M periods until delivery. The carrying charge in period k for one unit of stock is $qS(k)$ to be paid at time k, where q is a proportional constant. Show that the forward price is

$$F = \frac{(1 + qM)S(0)}{B(0, M)},$$

where $B(0, M)$ is the time-0 price of the discount bond maturity at M.

2. Suppose the fixed rate payer of the Forward Rate Agreement is too generous to agree to pay a fixed rate K_{gen} that is higher than the theoretical fixed rate K, where

$$K = \frac{1}{T_2 - T_1}\left[\frac{B_t(T_1)}{B_t(T_2)} - 1\right].$$

Here, k is visualized as the forward price of LIBOR $L[T_1, T_2]$. How can the counterparty, the fixed rate receiver, take arbitrage?

3. Consider an interest rate swap of notional principal \$1 million and remaining life of 9 months, the terms of the swap specify that the floating six-month LIBOR is exchanged for the fixed rate of 10% per annum (quoted with semi-annual compounding). The market prices of unit par zero coupon bonds with maturity dates 3 months and 9 months from now are \$0.972 and \$0.918, respectively, while the market price of unit par floating rate bond with maturity date 3 months from now is \$0.992. Find the value of the interest rate swap to the fixed-rate payer, assuming no default risk of the swap counterparty.

   Hint: The discount factors over 3 months and 9 months from now are implied by the respective zero coupon bond prices, and from which the implied 6-month LIBOR that is reset 3 months from now can be determined. The 6-month LIBOR payment to be paid 3 months from now has been fixed, whose value should be reflected in the price of the floating rate bond maturing 3 months from now.

4. The value of the fixed leg payments in the underlying interest rate swap in a swaption is visualized to have a floating value on the maturity date of the swaption. Explain why.

5. This question considers equivalence of two instruments: option on bond and interest rate caplet. We let $P_t(T)$ denote the time-t price of a bond maturing at $T > t$. The forward rate quoted at time t for a loan commencing at time T with repayment at time $T+s$ is

$$R_t(T, T+s) = \frac{P_t(T)}{P_t(T+s)} - 1.$$

(i) Options on bond $P_t(T+s)$
For T-maturity call and put options on an s-period zero-coupon bond, suppose the strike price is K, their payments at maturity are

$$c(P_T(T+s), T; K) = \max(P_T(T+s) - K, 0)$$
$$p(P_T(T+s), T; K) = \max(K - P_T(T+s), 0).$$

(ii) Interest rate caplet
An interest rate caplet makes a payment if the interest rate over $[T, T+s]$ is above the strike rate K_R. While the caplet payment is made at time $T+s$, the value of the payoff seen at time T is given by

$$\text{caplet}(P_T(T+s), T) = \frac{1}{1 + R_T(T, T+s)} \max(R_T(T, T+s) - K_R, 0).$$

The discount factor $\dfrac{1}{1 + R_T(T, T+s)}$ is applied over the time interval $[T, T+s]$ and will be known at T. Indeed, $R_T(T, T+s) = \dfrac{1}{P_T(T+s)} - 1$ so that $\dfrac{1}{1 + R_T(T, T+s)} = P_T(T+s)$. On the other hand, the payoff to a forward rate agreement at time $T+s$ is $R_T(T, T+s) - R_t(T, T+s)$ [see the analogy between a caplet and a forward rate agreement if we set $K_R = R_t(T, T+s)$].

Show that an interest rate caplet is equivalent to a bond put option.

6. Consider the asset swap package, where asset swap spread $s^A(0)$ at initiation is given by

$$s^A(0) = \frac{C(0) - \overline{C}(0)}{A(0)}.$$

Find the in-progress net change of value of the asset swap at a later time $t > 0$ prior to default of the underlying bond [in terms of $C(t)$, $\overline{C}(t)$ and $A(t)$].

7. Two years ago, a corporation issued 7-year bonds with a fixed coupon rate of 10% payable semiannually on Feb 15 and Aug 15 of each year. The debt was structured to be callable (at par) offer a 4-year deferment period and was issued at par value of \$100 million. The bonds are now trading in the market at a price of 106, reflecting the general decline in market interest rates and the corporation's recent upgrade in its credit quality. The corporate treasurer believes that the current interest rate cycle has bottomed. If the bonds were callable today, the firm would realize a considerable savings in annual interest expense. However, the bonds are still in their call protection period. Also, the treasurer's fear is that the market rate might rise considerably prior to the call date two years later.

Let T be the 3-year Treasury yield that prevails in two years later. Let BS be the company specific bond credit spread so that $T + BS$ is the refunding rate of the corporation. The prevailing 3-year swap fixed rate is given by $T+SS$, where SS stands for the swap spread.

Consider the following 3 strategies:

1. Strategy I. Enter into the off-market forward swap as the fixed rate payer

   Agreeing to pay 9.5% (rather than the at-market rate of 8.55%) for a three-year swap, two years forward. Initial cash flow: Receive \$2.25 million since the the fixed rate is above the at-market rate. Assume that the corporation's refunding spread remains at its current 100 bps level and the 3-year swap spread over Treasuries remains at 50 bps.

2. Strategy II. Buy payer swaption expiring in two years with a strike rate of 9.5%

   Initial cash flow: Pay \$1.10 million as the cost of the swaption (the swaption is out-of-the-money).

3. Strategy III. Sell a receiver swaption at a strike rate of 9.5% expiring in two years

   Initial cash flow: Receive \$2.50 million (in-the-money swaption).

In the first and third strategies, the corporation receives upfront cash inflow. In the second strategy, cash out-flow is required. Examine the gains and losses under various scenarios of the above three strategies and comment on the choices of these strategies. Note the difference between the off-market forward swap and swaption. In both instruments, the underlying is an interest rate swap.

8. This problem examines the differences in cash flows between entering a total return swap and an out-right purchase

   (a) An outright purchase of the C-bond at $t = 0$ with a sale at $t = T_N$. B finances this position with debt that is rolled over at LIBOR, maturing at T_N.

   (b) A total return receiver B in a TRS with the asset holder A.

   1. B receives the coupon payments of the underlying security at the same time in both positions.

   2. The debt service payments in strategy (a) and the LIBOR part of the funding payment in the TRS (strategy (b)) coincide, too.

9. Suppose the strike prices X_1 and X_2 satisfy $X_1 < X_2$, show that for European calls on a non-dividend paying asset, the difference in call values satisfies

$$B(\tau)(X_2 - X_1) \geq c(S, \tau; X_1) - c(S, \tau; X_2) \geq 0,$$

where $B(\tau)$ is the value of a pure discount bond with par value of unity and time to maturity τ. Furthermore, deduce that

$$-B(\tau) \leq \frac{\partial c}{\partial X}(S, \tau; X) \leq 0.$$

In other words, suppose the call price can be expressed as a differentiable function of the strike price, then the derivative must be non-positive and no greater in absolute value than the price of a pure discount bond of the same maturity. Do the above results also hold for European calls on a dividend paying asset?

10. Show that the put prices (European and American) are convex functions of the asset price, that is,

$$p(\lambda S_1 + (1-\lambda)S_2, X) \le \lambda p(S_1, X) + (1-\lambda)p(S_2, X), \quad 0 \le \lambda \le 1,$$

where S_1 and S_2 denote the asset prices and X denotes the strike price.

Hint: Let $S_1 = h_1 X$ and $S_2 = h_2 X$. Note that the put price function is homogeneous of degree one in the asset price and strike price. This means if we multiply the asset price and strike price by the same scalar multiplier, the new put price will be the same scalar multiplier times the old put price. The above inequality can be expressed as

$$\begin{aligned} &[\lambda h_1 + (1-\lambda)h_2]p\left(X, \frac{X}{\lambda h_1 + (1-\lambda)h_2}\right) \\ &\le \lambda h_1 p\left(X, \frac{X}{h_1}\right) + (1-\lambda)h_2 p\left(X, \frac{X}{h_2}\right). \end{aligned}$$

Use the property that the put prices are convex functions of the strike price.

11. Suppose the strike price is growing at the riskless interest rate, show that the price of an American put option is the same as that of its European counterpart. Find the value of an American put option when (i) strike price $X = 0$, (ii) asset price $S = 0$.

12. Suppose we have an option to exchange one asset A for another asset B. Let the underlying asset A have price S_t and the strike asset B have the price Q_t. Let $F^P_{t,T}(S)$ denote the time-t price of a prepaid forward on the underlying asset, paying S_T at time T; and similar definition for $F^P_{t,T}(Q)$. For European exchange options, the time-T call and put payoffs are

$$c(S_T, Q_T, T) = \max(S_T - Q_T, 0) \text{ and } p(S_T, Q_T, T) = \max(Q_T - S_T, 0).$$

respectively. Show the following put-call parity relation at $t < T$:

$$c(S_t, Q_t, t) - p(S_t, Q_t, t) = F^P_{t,T}(S) - F^P_{t,T}(Q).$$