MAFS 5030

Quantitative Modeling of Derivative Securities

Solution to Homework One

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1. The seller of the forward hedges the risk by borrowing $\alpha S(0)$ dollars at time 0 to buy α units of the underlying stock and selling out q units of the stock at each period in order to pay for the carrying charge. After M periods, the number of units of the stock remaining is $\alpha - qM$. The goal is to have one unit of the stock available for delivery at maturity M. This gives

$$\alpha - qM = 1 \quad \text{or} \quad \alpha = 1 + qM.$$

Let F be the forward price. At maturity M, the seller of the forward receives F but he has to pay back $(1 + qM)S(0)/B(0, M)$ to the borrower. By netting the cash flows to be zero at maturity, we obtain

$$F = \frac{(1 + qM)S(0)}{B(0, M)}.$$

Alternative approach: Recall the forward price formula:

$$F = \frac{S(0)}{B(0, M)} + \sum_{k=1}^{M} \frac{c(k)}{B(0, M)/B(0, k)},$$

where $c(k)$ is the carrying charge at time k. Here, $c(k) = qS(k)$. The price of the forward maturity at k is $S(0)/B(0, k)$. The forward price formula is modified to become

$$F = \frac{S(0)}{B(0, M)} + \sum_{k=1}^{M} \frac{qS(0)/B(0, k)}{B(0, M)/B(0, k)} = \frac{(1 + qM)S(0)}{B(0, M)}.$$

2. The fixed rate receiver takes the following bonds portfolio:

   (i) short holding of the T_2-maturity discount bond with par $N[1 + K_{\text{gen}}(T_2 - T_1)]$;

   (ii) long holding of the T_1-maturity discount bond with par N.

The dollar amount N collected at T_1 is deposited in a money market account that earns the floating LIBOR $L(T_1, T_2)$. The interest earned can be used to pay the floating leg payment $NL[T_1, T_2](T_2 - T_1)$. In return, the fixed rate receiver receives $NK_{\text{gen}}(T_2 - T_1)$. Together with cash N left, he is able to honor the par payment of $N[1 + K_{\text{gen}}(T_2 - T_1)]$ of the short position of the T_2-maturity discount bond.

The net cost of acquiring the long and short positions of the two bonds at the current time
$= N[1 + K_{\text{gen}}(T_2 - T_1)B_t(T_2) - B_t(T_1)]$

$> N[1 + K(T_2 - T_1)B_t(T_2) - B_t(T_1)] = 0.$
Here, the fixed rate receiver has upfront positive gain and net offsetting position at T_2. This represents an arbitrage opportunity.

3. Consider the fixed-leg and floating-leg payments of a swap of unit notional. Suppose the current time is indexed by 0 (i.e. $t = 0$), and $t = 1$ means one year away from now.

   Fixed-leg payments

   At $t = \frac{1}{4}$, in-flow of $\frac{10\%}{2} = 0.05$ of interest.

   At $t = \frac{3}{4}$, in-flow of $\frac{10\%}{2} = 0.05$ of interest.

   Floating-leg payments

   At $t = \frac{1}{4}$, in-flow of $L_{\frac{1}{2}}\left(-\frac{1}{4}\right)\left(\frac{3}{4} - \frac{1}{4}\right) = \frac{1}{2}L_{\frac{1}{2}}\left(-\frac{1}{4}\right)$.

   Here, $L_{\frac{1}{2}}\left(-\frac{1}{4}\right)$ denotes the half-year LIBOR set at an earlier time $t = -\frac{1}{4}$.

   At $t = \frac{3}{4}$, receives $L_{\frac{1}{2}}\left(\frac{1}{4}\right)\left(\frac{3}{4} - \frac{1}{4}\right) = \frac{1}{2}L_{\frac{1}{2}}\left(\frac{1}{4}\right)$.

   Discount bond prices observed at $t = 0$: $B\left(0, \frac{1}{4}\right) = 0.972, B\left(0, \frac{3}{4}\right) = 0.918$. The 3-month maturity floating rate bond that is entitled to receive $1 + \frac{1}{2}L_{\frac{1}{2}}\left(-\frac{1}{4}\right)$ at time $\frac{1}{4}$ is now priced at 0.992. Note that one dollar at time $\frac{1}{4}$ is worth $B\left(0, \frac{1}{4}\right)$ at the current time. This gives the present value of $\frac{1}{2}L_{\frac{1}{2}}\left(-\frac{1}{4}\right)$ to be

   $$0.992 - B\left(0, \frac{1}{4}\right) = 0.992 - 0.972 = 0.02.$$

   Also, by paying $B\left(0, \frac{1}{4}\right)$ at $t = 0$ to acquire the 3-month maturity discount bond, we can generate the cash flow of $\$1 + \frac{1}{2}L_{\frac{1}{2}}\left(\frac{1}{4}\right)$ at $t = \frac{3}{4}$. This is done by putting $1 collected at $t = \frac{1}{4}$ and depositing in a bank account to earn $L_{\frac{1}{2}}\left(\frac{1}{4}\right)$ for 6-month period. Note that the implied present value of $1 at $t = \frac{3}{4}$ is $B\left(0, \frac{3}{4}\right)$, so the implied present value of $\frac{1}{2}L_{\frac{1}{2}}\left(\frac{1}{4}\right)$ is

   $$B\left(0, \frac{1}{4}\right) - B\left(0, \frac{3}{4}\right) = 0.972 - 0.918 = 0.054.$$

Therefore, the present value of the two floating-leg payments at $t = \frac{1}{4}$ and $t = \frac{3}{4}$ is \$0.02 + \$0.054 = \$0.074 per unit notional. The present value of the fixed-leg payments is

$$0.05\left[B\left(0, \frac{1}{4}\right) + B\left(0, \frac{3}{4}\right)\right] = (\$0.05)(0.972 + 0.918) = \$0.0945$$

per unit notional.

The value of the swap to the fixed-rate payer with notional one million

$$\begin{aligned} &= \$1,000,000 \times (0.074 - 0.0945) \\ &= -\$20,500. \end{aligned}$$

4. Let the initiation date of the swaption be time zero, let T_S and T denote the maturity date of the swaption and the underlying interest rate swap, respectively, where $0 < T_S < T$. Recall that the fixed rate K has been structured in the swaption contract, which should be set to be the time-0 expectation of the swap rate. During the time period $[0, T_S]$, the market swap rate fluctuates due to the stochastic dynamics of the interest rate. Therefore, one may visualize the market swap rate to be floating.

   On the swaption maturity date T_S, the fair value of the fixed leg payments to the fixed rate payer is given by

$$N(\text{market swap rate at } T_S - K)A(T_S),$$

   where N is the notional and $A(T_S)$ is the value of the annuity based on the tenor of the underlying interest rate swap. It becomes in-the-money when the market swap rate at T_S is higher than K since the fixed rate payer is happy to pay K that is below the market swap rate.

5. We can rewrite the caplet payoff as

$$\begin{aligned} &(1 + K_R) \max\left(\frac{R_T(T, T+s) - K_R}{[1 + R_T(T, T+s)](1 + K_R)}, 0\right) \\ = \; &(1 + K_R) \max\left(\frac{1}{1 + K_R} - \frac{1}{1 + R_T(T, T+s)}, 0\right) \\ = \; &(1 + K_R) \max\left(\frac{1}{1 + K_R} - P_T(T+s), 0\right). \end{aligned}$$

   It is the same as the payoff of $1 + K_R$ units of put option on an s-period bond with strike price $\frac{1}{1 + K_R}$.

6. The pricing argument for determining the asset spread $s^A(t)$ prevails at any time t, so the in-progress asset swap spread $s^A(t)$ is given by

$$s^A(t) = \frac{C(t) - \overline{C}(t)}{A(t)},$$

where $A(t)$ is the time-t value of annuity associated with the remaining tenor of the interest rate swap. The net gain in the asset swap spread is $s^A(0) - s^A(t)$ since $s^A(0)$ has been fixed at initiation. This is translated into net change in value of $A(t)[s^A(0) - s^A(t)]$ of the asset swap.

7. Consider the first strategy of entering into the off-market forward swap:

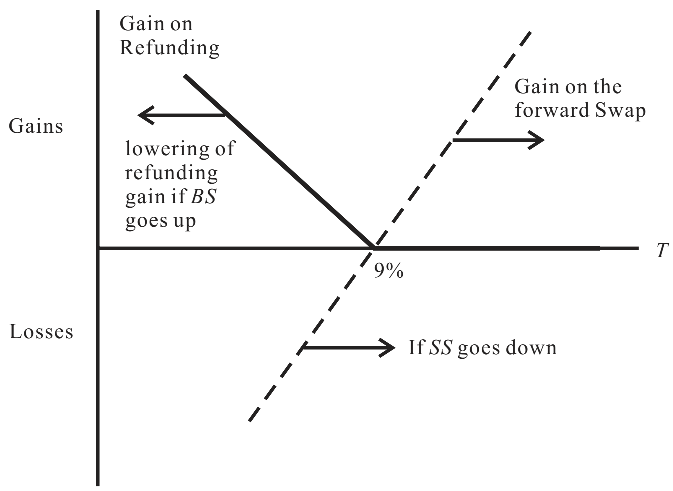
   - Gain on refunding (per settlement period): embedded callable right

$$\begin{cases} [10 \text{ percent } -(T+BS)] & \text{if } T+BS < 10 \text{ percent,} \\ 0 & \text{if } T+BS \geq 10 \text{ percent.} \end{cases}$$

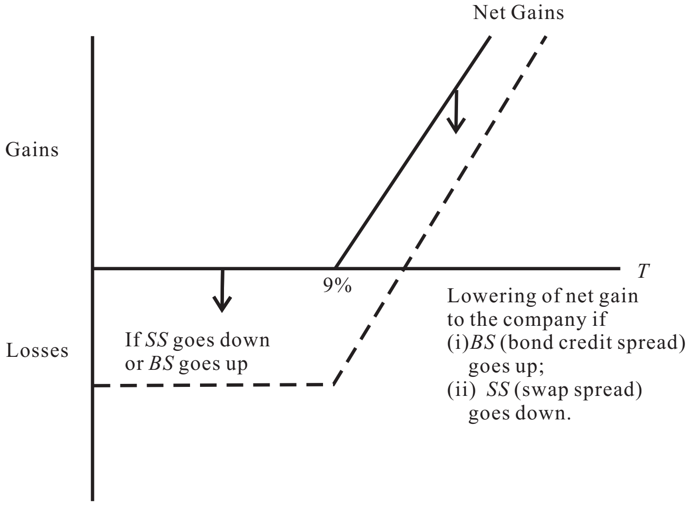
   - Gain (or loss) on the swap forward (per settlement period):

$$\begin{cases} -[9.50\text{percent} - (T+SS)] & \text{if } T+SS < 9.50\text{percent,} \\ [(T+SS) - 9.50 \text{ percent}] & \text{if } T+SS \geq 9.50\text{percent.} \end{cases}$$

   Assuming that $BS = 1.00$ percent and $SS = 0.50$ percent, these gains and losses two years later are:



   - Refunding payoff resembles a put payoff on T
   - Forward swap payoff resembles a forward payoff on T



Since the company stands to gain if rates rise, it has not fully monetized the embedded callable right. This is because a symmetric payoff instrument (a forward swap) is used to hedge an asymmetric payoff (option).

Next, consider the second strategy of buying a payer swaption expiring in two years with a strike rate of 9.5%.

Initial cash flow: Pay \$1.10 million as the cost of the swaption (the swaption is out-of-the-money)

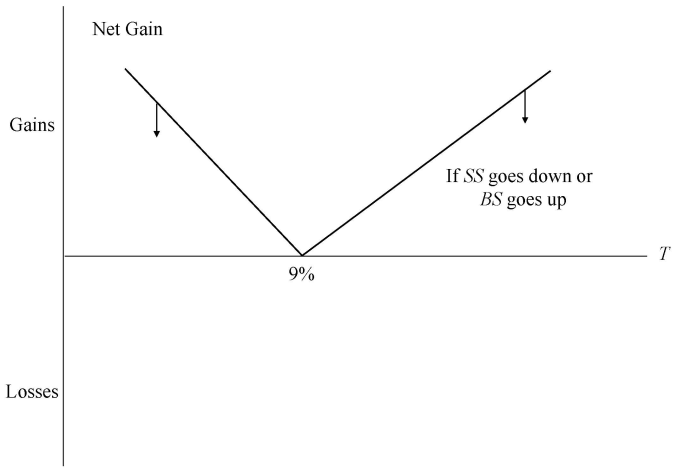
- Gain on refunding (per settlement period):

$$\begin{cases} 10 \text{ percent } -(T+BS) & \text{if } T+BS < 10 \text{ percent,} \\ 0 & \text{if } T+BS \geq 10 \text{ percent.} \end{cases}$$

- Gain (or loss) on unwinding the swap (per settlement period):

$$\begin{cases} (T+SS) - 9.50 \text{ percent} & \text{if } T+SS > 9.50 \text{ percent,} \\ 0 & \text{if } T+SS \leq 9.50 \text{ percent..} \end{cases}$$

With $BS = 1.00$ percent and $SS = 0.50$ percent, these gains and losses are:



This strategy is too conservative. The company will benefit from Treasury rates being either higher or lower than 9%. However, the treasurer had to spend \$1.1 million to lock in this straddle.

Lastly, consider the third strategy of selling a receiver swaption at a strike rate of 9.5% expiring in two years.

Initial cash flow: Receive \$2.50 million (in-the-money swaption)

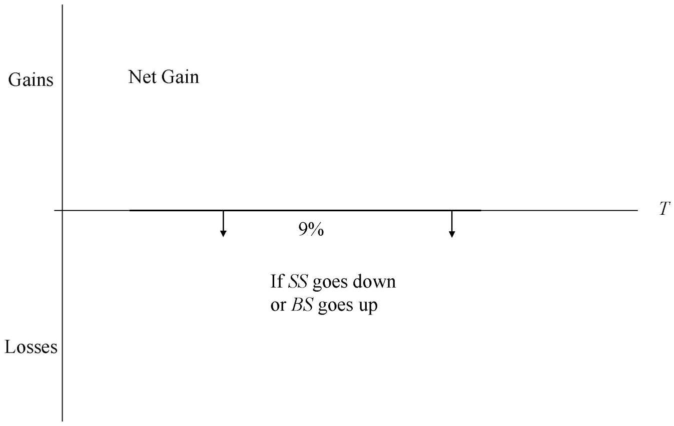
- Gain on refunding (per settlement period):

$$\begin{cases} [10 \text{ percent } - (T + BS)] & \text{if } T + BS < 10 \text{ percent,} \\ 0 & \text{if } T + BS \geq 10 \text{ percent.} \end{cases}$$

- Loss on unwinding the swap (per settlement period):

$$\begin{cases} [9.50 \text{ percent } -(T + SS)] & \text{if } T + SS < 9.50 \text{ percent,} \\ 0 & \text{if } T + SS \geq 9.50 \text{ percent.} \end{cases}$$

With $BS = 1.00$ percent and $SS = 0.50$ percent, these gains and losses are:



By selling the receiver swaption, the company has been able to simulate the sale of the embedded call feature of the bond, thus fully monetizing the callable right. The only remaining uncertainty is the basis risk associated with unanticipated changes in swap and bond spreads.

8. Payoff streams of a total return swap (TRS) to the total return receiver B (the payoffs to the total return payer A are the converse of these).

Time	Defaultable bond	TRS payments		
		Funding	Returns	Marking to market
$t = 0$	$-\overline{C}(0)$	0	0	0
$t = T_i$	$\overline{c}$	$-\overline{C}(0)(L_{i-1} + s^{TRS})$	$+\overline{c}$	$+\overline{C}(T_i) - \overline{C}(T_{i-1})$
$t = T_N$	$\overline{C}(T_N) + \overline{c}$	$-\overline{C}(0)(L_{N-1} + s^{TRS})$	$+\overline{c}$	$+\overline{C}(T_N) - \overline{C}(T_{N-1})$
Default	Recovery	$-\overline{C}(0)(L_{i-1} + s^{TRS})$	0	$-(\overline{C}(T_{i-1}) -$ Recovery$)$

The TRS is unwound upon default of the underlying bond. Day count fractions are set to one, $\delta_i = 1$, for convenience.

The source of value difference lies in the marking-to-market of the TRS at the intermediate intervals.

Final payoff of the strategy

B sells the bond in the market for $\overline{C}(T_N)$, and has to pay back his debt which costs him $\overline{C}(0)$. (The LIBOR coupon payment is already cancelled with the TRS.) This yields:

$$\overline{C}(T_N) - \overline{C}(0),$$

which is the amount that B receives at time T_N by following strategy (a). This is the netting of intermediate interest and coupon payments.

We decompose this total price difference between $t = 0$ and $t = T_N$ into the smaller and incremental differences that occur between the individual times T_i:

$$\overline{C}(T_N) - \overline{C}(0) = \sum_{i=1}^{N} \overline{C}(T_i) - \overline{C}(T_{i-1}).$$

This representation allows us to distribute the final payoff of the strategy over the intermediate time intervals and to compare them to the payout of the TRS position (b).

- Each time interval $[T_{i-1}, T_i]$ contributes an amount of

  $$\overline{C}(T_i) - \overline{C}(T_{i-1})$$

  to the final payoff, and this amount is directly observable at time T_i.
- This payoff contribution can be converted into a payoff that occurs at time T_i by discounting it back from T_N to T_i, giving

  $$[\overline{C}(T_i) - \overline{C}(T_{i-1})]B(T_i, T_N).$$

Conversely, if we paid B the amount given in the above equation at each T_i, and assume that B reinvested this money at the default-free interest rate until T_N, then B would have exactly the same final payoff as in strategy (a).

From the TRS position in strategy (b), B has a slightly different payoff:

$$\overline{C}(T_i) - \overline{C}(T_{i-1})$$

at all times $T_i > T_0$ net of his funding expenses.

Time value of intermediate payments

- The difference (b) − (a) is:

  $$[\overline{C}(T_i) - \overline{C}(T_{i-1})][1 - B(T_i, T_N)] = \Delta\overline{C}(T_i)[1 - B(T_i, T_N)].$$

  The above gives the excess payoff at time T_i of the TRS position over the outright purchase of the bond.
- This term will be positive if the change in value of the underlying bond $\Delta\overline{C}(T_i)$ is positive. It will be negative if the change in value of the underlying bond is negative, and zero if $\Delta\overline{C}(T_i)$ is zero.

- If the underlying asset is a bond, the likely sign of its change in value $\Delta\overline{C}(T_i)$ can be inferred from the deviation of its initial value $\overline{C}(0)$ from par. For example, if $\overline{C}(0)$ is above par, the price changes will have to be negative on average.
- The most extreme example of this kind would be a TRS on a default-free zero-coupon bond with maturity T_N.
- If we assume constant interest rates of R, this bond will always increase in value because it was issued at such a deep discount.
- A direct investment in the bond will only realise this increase in value at maturity of the bond, while the TRS receiver effectively receives prepayments. He can reinvest these prepayments and earn an additional return.

Bonds that initially trade at a discount to par should command a positive TRS spread s^{TRS}, while bonds that trade above par should have a negative TRS spread s^{TRS}.

9. Consider a portfolio consisting of a call of strike X_1 and a discount bond with par X_1, both have the same date of maturity. The terminal payoff of the portfolio is $\max(S_T, X_1)$. We compare this portfolio with another similar portfolio, except that X_1 is replaced by X_2, where $X_2 > X_1$. Since

$$\max(S_T, X_1) \leq \max(S_T, X_2),$$

so the second portfolio dominates over the first portfolio. By no arbitrage principle, the present value of the second portfolio is greater than or equal to that of the first portfolio. We then have

$$\begin{aligned} & \text{value of the first portfolio} = c(S, \tau; X_1) + B(\tau)X_1 \\ \leq \; & c(S, \tau; X_2) + B(\tau)X_2 = \text{value of the second portfolio} \end{aligned}$$

so

$$c(S, \tau; X_1) - c(S, \tau; X_2) \leq B(\tau)(X_2 - X_1)$$

$$B(\tau) \geq -\frac{c(S, \tau; X_2) - c(S, \tau; X_1)}{X_2 - X_1}.$$

By taking the limit $X_1 \to X_2$, we then obtain

$$B(\tau) \geq -\frac{\partial c}{\partial X}(S, \tau; X_2) \quad \text{or} \quad \frac{\partial c}{\partial X}(S, \tau; X) \geq -B(\tau).$$

Also, the call price function is a decreasing function of the strike price, so it is obvious that

$$\frac{\partial c}{\partial X}(S, \tau; X) \leq 0.$$

The above result holds for European options on a dividend paying asset since the holder of a European option is not entitled to receive the dividends. The terminal payoffs of the two portfolios remain the same even the underlying asset is dividend paying.

10. The put price function is homogeneous of degree one, that is,

$$p(\lambda S; \lambda X) = \lambda p(S; X).$$

Let $h_1 > h_2$ so that $h_1 \geq \lambda h_1 + (1-\lambda)h_2$ and $\lambda h_1 + (1-\lambda)h_2 \geq h_2, \forall \lambda \in [0,1]$. Let $\mu = \dfrac{\lambda h_1}{\lambda h_1 + (1-\lambda)h_2}$, and observe $\dfrac{X}{h_1} \leq \dfrac{X}{\lambda h_1 + (1-\lambda)h_2} \leq \dfrac{X}{h_2}$, we obtain from the convexity properties of the option price functions with respect to the strike price:

$$p\left(X; \frac{X}{\lambda h_1 + (1-\lambda)h_2}\right) \leq \mu p\left(X; \frac{X}{h_1}\right) + (1-\mu)p\left(X; \frac{X}{h_2}\right)$$

$$\Leftrightarrow [\lambda h_1 + (1-\lambda)h_2] p\left(X; \frac{X}{\lambda h_1 + (1-\lambda)h_2}\right) \leq \lambda h_1 p\left(X; \frac{X}{h_1}\right) + (1-\lambda)h_2 p\left(X; \frac{X}{h_2}\right).$$

Using the homogeneous property of the price function, we obtain

$$p(\lambda h_1 X + (1-\lambda)h_2 X; X) \leq \lambda p(h_1 X; X) + (1-\lambda)p(h_2 X; X).$$

Lastly, by setting $S_1 = h_1 X$ and $S_2 = h_2 X$, we deduce that

$$p(\lambda S_1 + (1-\lambda)S_2; X) \leq \lambda p(S_1; X) + (1-\lambda)p(S_2; X), \text{ where } \lambda \in [0,1].$$

11. When the strike price is growing at the riskless interest rate, there will be no gain on the time value of the strike price upon early exercise of the American put. In this case, the American early exercise right is rendered worthless, so the price of the American put is the same as that of its European counterpart.

    (i) When $X = 0$, the American put becomes worthless since the exercise of the American put always gives zero value.

    (ii) Once $S = 0$, the asset price stays at zero value forever. There will be no loss in dividends from the asset and insurance value associated with the holding of the American put. The American put should be exercised immediately to receive the strike price X. Hence, the value of the American put is equal to X.

12. Consider the following payoff table:

Transactions	time t	time T	
		$S_T \leq Q_T$	$S_T > Q_T$
buy call	$-c(S_t, Q_t, t)$	0	$S_T - Q_T$
sell put	$p(S_t, Q_t, t)$	$S_T - Q_T$	0
sell forward on A	$F^P_{t,T}(S)$	$-S_T$	$-S_T$
buy forward on B	$-F^P_{t,T}(Q)$	Q_T	Q_T
total	$-c(S_t, Q_t, t)$ $+ p(S_t, Q_t, t)$ $+ F^P_{t,T}(S) - F^P_{t,T}(Q)$	0	0

Since the time-T portfolio value is always zero under all scenarios, by virtue of the law of one price, the time-t portfolio value must also be zero. If otherwise, then arbitrage opportunity arises. We then have the put-call parity relation.