

MAFS 5030

Quantitative Modeling of Derivative Securities

Homework Two

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1. Show that a dominant trading strategy exists if and only if there exists a trading strategy satisfying $V_0 < 0$ and $V_1(\omega) \geq 0$ for all $\omega \in \Omega$.

Hint: Consider the dominant trading strategy $\mathcal{H} = (h_0 \ h_1 \ \cdots \ h_M)^T$ satisfying $V_0 = 0$ and $V_1(\omega) > 0$ for all $\omega \in \Omega$. Take $G_{\min}^* = \min_{\omega} G^*(\omega) > 0$ and define a new trading strategy with $\hat{h}_m = h_m, m = 1, \dots, M$

$$\text{and } \hat{h}_0 = -G_{\min}^* - \sum_{m=1}^M h_m S_m^*(0).$$

2. Consider a portfolio with one risky security and the riskfree security. Suppose the price of the risky asset at time 0 is 4 and the possible values of the $t = 1$ price are 1.1, 2.2 and 3.3 (3 possible states of the world at the end of a single trading period). Let the riskfree interest rate r be 0.1 and take the price of the riskfree security at $t = 0$ to be unity.
- (a) Show that the trading strategy: $h_0 = 4$ and $h_1 = -1$ is a dominant trading strategy that starts with zero wealth and ends with positive wealth with certainty.
 - (b) Find the discounted gain G^* over the single trading period.
 - (c) Find a trading strategy that starts with negative wealth and ends with non-negative wealth with certainty.
3. (a) Show that if the law of one price does not hold, then every payoff in the asset span can be bought at any price.
- (b) Suppose the price of any security in the asset span is unique, does there exist any arbitrage opportunity? Give a proof or quote a counterexample.
4. Given the discounted terminal payoff matrix

$$\hat{S}^*(1) = \begin{pmatrix} 1 & 6 & 3 \\ 1 & 2 & 2 \\ 1 & 12 & 6 \end{pmatrix},$$

and the current discounted price vector $\mathbf{S}(0) = (1 \ 3 \ 2)$, find the state price of the Arrow security with discounted payoff $\mathbf{e}_k, k = 1, 2, 3$. Do we observe positivity of the state prices? Does the securities model admit any arbitrage opportunities? If so, find one such example.

5. Define the pricing functional $F(\mathbf{x})$ on the asset span \mathcal{S} by $F(\mathbf{x}) = \{y : y = \mathbf{S}(0)\mathbf{h} \text{ for some } \mathbf{h} \text{ such that } \mathbf{x} = S^*(1)\mathbf{h}, \text{ where } \mathbf{x} \in \mathcal{S}\}$. Show that if the law of one price holds, then F is a *linear* functional.
6. Construct a securities model with 2 risky securities and the riskfree security and 3 possible states of world such that the law of one price holds but there are dominant trading strategies.

7. Suppose a betting game has 3 possible outcomes. If a gambler bets on outcome i , then he receives a net gain of d_i dollars for one dollar betted, $i = 1, 2, 3$. The payoff matrix thus takes the form (discounting is immaterial in a betting game)

$$S(1; \Omega) = \begin{pmatrix} d_1 + 1 & 0 & 0 \\ 0 & d_2 + 1 & 0 \\ 0 & 0 & d_3 + 1 \end{pmatrix}$$

Find the condition on d_i such that a risk neutral probability measure exists for the above betting game (visualized as an investment model).

8. Consider the following securities model with discounted terminal payoff of the securities given by the payoff matrix

$$\widehat{S}^*(1; \Omega) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 5 \\ 1 & 5 & 6 & 7 \end{pmatrix},$$

where the first column gives the discounted payoff of the riskfree security. Let the initial price vector $\widehat{\mathbf{S}}^*(0)$ be $(1 \ 3 \ 5 \ 9)$. Does the law of one price hold for this securities model? Show that the contingent claim with discounted payoff $\begin{pmatrix} 6 \\ 8 \\ 12 \end{pmatrix}$ is attainable and find the set of all possible trading securities that generate the payoff. Can we find the price at $t = 0$ of this contingent claim?

9. Let Π_u and Π_d denote the state price corresponding to the state of the asset value going up and going down, respectively. The state prices can also be interpreted as state contingent discount rates. Assuming absence of arbitrage opportunities, all securities (including the money market account, asset and call option) would have returns with the same state contingent discount rates Π_u and Π_d . Hence, the respective relations for the money market account, asset price and call option value with Π_u and Π_d are given by

$$\begin{aligned} 1 &= \Pi_u R + \Pi_d R \\ S &= \Pi_u uS + \Pi_d dS \\ c &= \Pi_u c_u + \Pi_d c_d. \end{aligned}$$

By solving for Π_u and Π_d from the first two equations and substituting the solutions into the third equation, show that the binomial call price formula over one period is given by

$$c = \frac{pc_u + (1-p)c_d}{R} \quad \text{where} \quad p = \frac{R-d}{u-d}.$$

10. Consider the single-period securities model:

$$\mathbf{S}(0) = (1 \ 2 \ 3) \quad \text{and} \quad \widehat{S}(1; \Omega) = \begin{pmatrix} 1 & 4 & 5 \\ 1 & 2 & 3 \\ 1 & 1 & 2 \end{pmatrix}.$$

Find the range of reasonable initial price of the non-attainable contingent claim $Y^* = (3 \ 4 \ 5)^T$ such that arbitrage opportunities do not exist.

11. Suppose the one-period securities model is endowed with

$$S(0) = (1 \ 10) \quad \text{and} \quad S^*(1) = \begin{pmatrix} 1 & 8 \\ 1 & 10 \\ 1 & 12 \end{pmatrix}.$$

Show that the contingent claim $Y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$ is attainable if and only if

$$y_1 - 2y_2 + y_3 = 0.$$

12. Suppose the universe of securities does not contain the riskfree security, all Arrow securities lie in the asset span and all state prices are strictly positive, show that the single-period securities model admits no arbitrage opportunity.
13. Consider the sample space $\Omega = \{-3, -2, -1, 1, 2, 3\}$ and the algebra $\mathcal{F} = \{\emptyset, \{-3, -2\}, \{-1, 1\}, \{2, 3\}, \{-3, -2, -1, 1\}, \{-3, -2, 2, 3\}, \{-1, 1, 2, 3\}, \Omega\}$. For each of the following random variables, determine whether it is \mathcal{F} -measurable:
 - (i) $X(\omega) = \omega^2$, (ii) $X(\omega) = \max(\omega, 2)$.
 Find a random variable that is \mathcal{F} -measurable.
14. Let X, X_1, \dots, X_n be random variables defined on (Ω, \mathcal{F}, P) . Prove the following properties on conditional expectation:
 - (a) $E[XI_B] = E[I_B E[X|\mathcal{F}]]$ for all $B \in \mathcal{F}$,
 - (b) $E[\max(X_1, \dots, X_n)|\mathcal{F}] \geq \max(E[X_1|\mathcal{F}], \dots, E[X_n|\mathcal{F}])$.
15. Let $X = \{X_t; t = 0, 1, \dots, T\}$ be a stochastic process adapted to the filtration $\mathbb{F} = \{\mathcal{F}_t; t = 0, 1, \dots, T\}$. Does the property: $E[X_{t+1} - X_t|\mathcal{F}_t] = 0, t = 0, 1, \dots, T - 1$ imply that X is a martingale?
16. Consider the binomial experiment with probability of success $p, 0 < p < 1$. We let N_k denote the number of successes after k independent trials. Define the discrete process Y_k by $N_k - kp$, the excess number of successes above the mean kp . Show that Y_k is a martingale.
17. Consider the two-period securities model in the lecture note of Topic 2, p.101. Suppose the riskless interest rate r violates the restriction $r < 0.2$, say, $r = 0.3$. Construct an arbitrage opportunity associated with the securities model.