MAFS 5030
Quantitative Modeling of Derivative Securities

Solution to Homework Two

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1. $\Leftarrow$ part: The trading strategy $\mathcal{H}$ with $V_0 < 0$ and $V_1(\omega) \geq 0, \forall \omega \in \Omega$, dominates the "zero holding of risky securities" strategy $\widehat{\mathcal{H}}$. This is because the terminal value under $\widehat{\mathcal{H}}$ remains at V_0, which is strictly negative. On the other hand, $V_1(\omega)$ resulting from $\mathcal{H}$ is always non-negative, so it is guaranteed to have a higher value than that resulting from $\widehat{\mathcal{H}}$.

   $\Rightarrow$ part: Existence of a dominant trading strategy implies that there exists a trading strategy $\mathcal{H} = (h_1 \quad \cdots \quad h_M)^T$ such that $V_0 = 0$ and $V_1(\omega) > 0, \forall \omega \in \Omega$. Let $G^*_{min} = \min_\omega G^*(\omega) = \min_\omega \sum_{m=1}^M h_m \Delta S^*_m$. Since $G^*(\omega) = V^*_1 - V^*_0 > 0$, we have $G^*_{min} > 0$. Consider the new trading strategy with

$$\begin{aligned}\widehat{h}_m &= h_m \quad \text{for } m = 1, \cdots, M,\\ \widehat{h}_0 &= -G^*_{min} - \sum_{m=1}^M h_m S^*_m(0).\end{aligned}$$

   Now, $\widehat{V}^*_0 = \widehat{h}_0 + \sum_{m=1}^M \widehat{h}_m S^*_m(0) = -G^*_{min} < 0$; while

$$\begin{aligned}\widehat{V}^*_1(\omega) &= \widehat{h}_0 + \sum_{m=1}^M \widehat{h}_m S^*_m(1;\omega)\\ &= -G^*_{min} + \sum_{m=1}^M h_m \Delta S^*_m(\omega) \geq 0,\end{aligned}$$

   by virtue of the definition of G^*_{min}. Thus, $\widehat{\mathcal{H}} = (\widehat{h}_1 \quad \cdots \quad \widehat{h}_M)^T$ is a trading strategy that gives $\widehat{V}_0 < 0, \widehat{V}_1(\omega) \geq 0, \forall \omega \in \Omega$.

2. For the given securities model, we have the discounted terminal payoff matrix: $S(1;\Omega) = \begin{pmatrix} 1.1 & 1.1 \\ 1.1 & 2.2 \\ 1.1 & 3.3 \end{pmatrix}$ and initial price vector $\boldsymbol{S}(0) = (1 \quad 4)$.

   (a) With $h_0 = 4, h_1 = -1$, we obtain

$$\begin{aligned}V_0 &= (1 \quad 4)\begin{pmatrix} 4 \\ -1 \end{pmatrix} = 0\\ V_1(\omega) &= S(1;\Omega)\begin{pmatrix} 4 \\ -1 \end{pmatrix} = \begin{pmatrix} 3.3 \\ 2.2 \\ 1.1 \end{pmatrix} > \mathbf{0}, \quad V^*_1(\omega) = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}.\end{aligned}$$

Thus $\begin{pmatrix} 4 \\ -1 \end{pmatrix}$ is a dominant trading strategy.

(b) $G^* = V_1^* - V_0 = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$.

(c) We shall use the result in Question 1. Now, $G^*_{min} = \min_\omega G^*(\omega) = 1$ so that $\widehat{h}_0 = -1 - (-1)(4) = 3$. Take $\widehat{\mathcal{H}} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$, then

$$\begin{aligned} \widehat{V}_0 &= (1 \quad 4)\begin{pmatrix} 3 \\ -1 \end{pmatrix} = -1 < 0 \\ \widehat{V}_1 &= S(1;\Omega)\begin{pmatrix} 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 2.2 \\ 1.1 \\ 0 \end{pmatrix} \geq \mathbf{0}. \end{aligned}$$

Thus $\widehat{\mathcal{H}}$ is a trading strategy that starts with negative wealth $\widehat{V}_0$ and ends with non-negative wealth $\widehat{V}_1$ for sure.

3. (a) If the law of one price does not hold, then there exist two trading strategies $\boldsymbol{h}$ and $\boldsymbol{h}'$ such that

$$S^*(1)\boldsymbol{h} = S^*(1)\boldsymbol{h}' \text{ but } \boldsymbol{S}(0)\boldsymbol{h} > \boldsymbol{S}(0)\boldsymbol{h}'.$$

For any payoff $\boldsymbol{x}$ in the asset span, it can be expressed as $\boldsymbol{x} = S^*(1)\widehat{\boldsymbol{h}}$ for some $\widehat{\boldsymbol{h}}$. Using the relation: $S^*(1)\boldsymbol{h} = S^*(1)\boldsymbol{h}'$, we have

$$\begin{aligned} \boldsymbol{x} &= S^*(1)\widehat{\boldsymbol{h}} + kS^*(1)\boldsymbol{h} - kS^*(1)\boldsymbol{h}' \\ &= S^*(1)[\widehat{\boldsymbol{h}} + k(\boldsymbol{h} - \boldsymbol{h}')], \quad \text{for any value of } k. \end{aligned}$$

The initial price of the portfolio that generates $\boldsymbol{x}$ is given by

$$\boldsymbol{S}(0)\widehat{\boldsymbol{h}} + k[\boldsymbol{S}(0)\boldsymbol{h} - \boldsymbol{S}(0)\boldsymbol{h}'], \quad \text{for any value of } k.$$

As $\boldsymbol{S}(0)\boldsymbol{h} - \boldsymbol{S}(0)\boldsymbol{h}' \neq 0$, the initial price of the portfolio with payoff $\boldsymbol{x}$ can assume any value.

(b) Uniqueness of the price of any security in the asset span is equivalent to satisfaction of law of one price. Consider the securities model

$$S^*(1) = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix} \quad \text{and} \quad \boldsymbol{S}(0) = \begin{pmatrix} 1 & \frac{4}{3} & \frac{2}{3} \end{pmatrix}.$$

The state prices $(\pi_1 \quad \pi_2 \quad \pi_3)$ can be found by solving

$$(\pi_1 \quad \pi_2 \quad \pi_3)\begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \frac{4}{3} & \frac{2}{3} \end{pmatrix}.$$

giving $(\pi_1 \quad \pi_2 \quad \pi_3) = \left(\frac{1}{3} \quad -\frac{1}{3} \quad 1\right)$. It can be shown that by taking the portfolio $\boldsymbol{h} = \begin{pmatrix} -6 \\ 2 \\ 5 \end{pmatrix}$, we have

$$V_0 = \begin{pmatrix} 1 & \frac{4}{3} & \frac{2}{3} \end{pmatrix} \begin{pmatrix} -6 \\ 2 \\ 5 \end{pmatrix} = 0$$

while

$$V_1^* = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} -6 \\ 2 \\ 5 \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \\ 1 \end{pmatrix} > \mathbf{0}.$$

This indicates that $\boldsymbol{h} = \begin{pmatrix} -6 \\ 2 \\ 5 \end{pmatrix}$ represents a dominant trading strategy (arbitrage opportunity as well). Indeed, V_0 and V_1^* are related by

$$V_0 = (\pi_1 \quad \pi_2 \quad \pi_3) V_1^* = \begin{pmatrix} \frac{1}{3} & -\frac{1}{3} & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 6 \\ 1 \end{pmatrix} = 0.$$

4. The state prices $(\pi_1 \quad \pi_2 \quad \pi_3)$ are found by solving

$$(1 \quad 3 \quad 2) = (\pi_1 \quad \pi_2 \quad \pi_3) \begin{pmatrix} 1 & 6 & 3 \\ 1 & 2 & 2 \\ 1 & 12 & 6 \end{pmatrix}.$$

The solution is found to be: $(\pi_1 \quad \pi_2 \quad \pi_3) = \left(\frac{2}{3} \quad \frac{1}{2} \quad -\frac{1}{6}\right)$. Positivity of the state prices is not observed so the securities model admits arbitrage opportunity. To find an arbitrage opportunity (for simplicity, we take $h_1 = 0$), we seek for $(h_0 \quad h_2)^T$ such that

$$V_0 = (1 \quad 2) \begin{pmatrix} h_0 \\ h_2 \end{pmatrix} = h_0 + 2h_2 = 0$$

while

$$V_1^*(\omega) = \begin{pmatrix} 1 & 3 \\ 1 & 2 \\ 1 & 6 \end{pmatrix} \begin{pmatrix} h_0 \\ h_2 \end{pmatrix} = \begin{pmatrix} h_0 + 3h_2 \\ h_0 + 2h_2 \\ h_0 + 6h_2 \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

with at least one strict inequality. A possible arbitrage portfolio is $(h_0 \quad h_2)^T = (-2 \quad 1)^T$. We short sell 2 units of the risk free asset, long hold one unit of the second risky asset and zero unit of the first risky asset (since $h_1 = 0$). The resulting discounted payoff of the portfolio is given by

$$V_1^*(\omega) = \begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix}.$$

5. Let $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ be two discounted terminal payoff vectors in the asset span $\mathcal{S}$. This would imply that there exist $\boldsymbol{h}_1, \boldsymbol{h}_2$ such that $\boldsymbol{x}_i = S^*(1)\boldsymbol{h}_i$ for $i = 1, 2$. By the law of one price, the pricing functional is given by $F(\boldsymbol{x}_i) = \boldsymbol{S}(0)\boldsymbol{h}_i$ for $i = 1, 2$. For any scalars α_1 and α_2, we consider

$$\begin{aligned} \alpha_1 F(\boldsymbol{x}_1) + \alpha_2 F(\boldsymbol{x}_2) &= \alpha_1 \boldsymbol{S}(0)\boldsymbol{h}_1 + \alpha_2 \boldsymbol{S}(0)\boldsymbol{h}_2 \\ &= \boldsymbol{S}(0)(\alpha_1 \boldsymbol{h}_1 + \alpha_2 \boldsymbol{h}_2) \end{aligned}$$

while

$$\begin{aligned} S^*(1)(\alpha_1 \boldsymbol{h}_1 + \alpha_2 \boldsymbol{h}_2) &= \alpha_1 S^*(1)\boldsymbol{h}_1 + \alpha_2 S^*(1)\boldsymbol{h}_2 \\ &= \alpha_1 \boldsymbol{x}_1 + \alpha_2 \boldsymbol{x}_2 \in \mathcal{S}. \end{aligned}$$

Knowing that $\alpha_1 \boldsymbol{x}_1 + \alpha_2 \boldsymbol{x}_2 \in \mathcal{S}$, $F(\alpha_1 \boldsymbol{x}_1 + \alpha_2 \boldsymbol{x}_2)$ is given by $\boldsymbol{S}(0)(\alpha_1 \boldsymbol{h}_1 + \alpha_2 \boldsymbol{h}_2)$ as deduced from the relation: $\alpha_1 \boldsymbol{x}_1 + \alpha_2 \boldsymbol{x}_2 = S^*(1)(\alpha_1 \boldsymbol{h}_1 + \alpha_2 \boldsymbol{h}_2)$. We then have

$$F(\alpha_1 \boldsymbol{x}_1 + \alpha_2 \boldsymbol{x}_2) = \boldsymbol{S}(0)(\alpha_1 \boldsymbol{h}_1 + \alpha_2 \boldsymbol{h}_2) = \alpha_1 F(\boldsymbol{x}_1) + \alpha_2 F(\boldsymbol{x}_2).$$

This proves linearity of the pricing functional.

6. Consider $\widehat{S}^*(1) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 3 \end{pmatrix}$ and $\widehat{\boldsymbol{S}}(0) = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$.

Since the three rows of $\widehat{S}^*(1)$ are independent, so that the row space of $\widehat{S}^*(1)$ spans the whole $\mathbb{R}^3$. Hence, $\widehat{\boldsymbol{S}}(0)$ is sure to lie in the row space of $\widehat{S}^*(1)$. Therefore, we can conclude that the law of one price holds for the given securities model. However, we observe that $(-1 \quad 1 \quad 1)^T$ dominates the trading strategy $(0 \quad 0 \quad 0)^T$ as $V_0 = \boldsymbol{S}(0)(-1 \quad 1 \quad 1)^T = 0$ and

$$V_1^* = \widehat{S}^*(1)\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} > \mathbf{0}.$$

7. Let $\boldsymbol{q} = (q(\omega_1) \quad q(\omega_2) \quad q(\omega_3))$. Since the initial bet is one dollar, we have to solve

$$\boldsymbol{q}S(1;\Omega) = (1 \quad 1 \quad 1),$$

giving

$$q(\omega_i) = \frac{1}{d_i + 1} > 0 \quad \text{for} \quad i = 1, 2, 3. \tag{1}$$

We also have to observe $\sum_{i=1}^{3} q(\omega_i) = 1$, that is,

$$\sum_{i=1}^{3} \frac{1}{d_i + 1} = 1. \tag{2}$$

Eqs. (1) and (2) state the required conditions for the existence of a risk neutral probability measure for the betting game. An example would be $d_1 = 1, d_2 = 3$ and $d_3 = 3$. The betting game pays out \$2 if ω_1 occurs and \$4 if either ω_2 or ω_3 occurs.

8. Note that the last two columns are seen to be

$$\begin{pmatrix} 3 \\ 4 \\ 6 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 4 \\ 5 \\ 7 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix}.$$

The rank of $\widehat{S}^*(1;\Omega)$ is 2. We also observe that

$$\begin{aligned} \boldsymbol{S}_2^*(1;\Omega) &= \boldsymbol{S}_0^*(1;\Omega) + \boldsymbol{S}_1^*(1;\Omega) \text{ while } S_2(0) \neq S_0(0) + S_1(0); \\ \boldsymbol{S}_3^*(1;\Omega) &= \boldsymbol{S}_0^*(1;\Omega) + \boldsymbol{S}_2^*(1;\Omega) \text{ while } S_3(0) \neq S_0(0) + S_2(0). \end{aligned}$$

Hence, the law of one price does not hold. In fact, $\widehat{\boldsymbol{S}}(0) = (1 \quad 3 \quad 5 \quad 9)$ does not lie in the row space of $\widehat{S}^*(1;\Omega)$. This is equivalent to saying that solution to the linear system

$$\widehat{\boldsymbol{S}}(0) = \boldsymbol{q}\widehat{S}^*(1;\Omega)$$

does not exist.

Next, we check whether $\begin{pmatrix} 6 \\ 8 \\ 12 \end{pmatrix}$ is attainable by asking whether solution to the following linear system

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 5 \\ 1 & 5 & 6 & 7 \end{pmatrix} \begin{pmatrix} h_0 \\ h_1 \\ h_2 \\ h_3 \end{pmatrix} = \begin{pmatrix} 6 \\ 8 \\ 12 \end{pmatrix}$$

exists. The Gaussian elimination procedure gives

$$\begin{aligned} \left(\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 6 \\ 1 & 3 & 4 & 5 & 8 \\ 1 & 5 & 6 & 7 & 12 \end{array}\right) &\to \left(\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 6 \\ 0 & 1 & 1 & 1 & 2 \\ 0 & 3 & 3 & 3 & 6 \end{array}\right) \\ &\to \left(\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 6 \\ 0 & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \\ &\to \left(\begin{array}{cccc|c} 1 & 0 & 1 & 2 & 2 \\ 0 & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right). \end{aligned}$$

The set of all possible trading strategies that generate the payoff is seen to be

$$\begin{pmatrix} h_0 \\ h_1 \\ h_2 \\ h_3 \end{pmatrix} = \begin{pmatrix} 2 - h_2 - 2h_3 \\ 2 - h_2 - h_3 \\ h_2 \\ h_3 \end{pmatrix} \quad \text{for any values of } h_2, h_3 \in \mathbb{R}.$$

Thus, $\begin{pmatrix} 6 \\ 8 \\ 2 \end{pmatrix}$ lies in the asset span. For example, we take $h_2 = h_3 = 1$ so that $h_1 = 0$ and $h_0 = -1$, giving the following replicating strategy:

$$\begin{pmatrix} 6 \\ 8 \\ 12 \end{pmatrix} = -\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 3 \\ 4 \\ 6 \end{pmatrix} + \begin{pmatrix} 4 \\ 5 \\ 7 \end{pmatrix}.$$

Note that $\widehat{\boldsymbol{S}}(0)\begin{pmatrix} h_0 \\ h_1 \\ h_2 \\ h_3 \end{pmatrix} = 8 + h_2 + 4h_3$. The cost of the replicating portfolio is dependent on h_2 and h_3. This verifies that the law of one price does not hold in this securities model. There are infinitely many possible prices for this contingent claim.

9. From $\begin{cases} 1 = \Pi_u R + \Pi_d R \\ S = \Pi_u uS + \Pi_d \, dS \end{cases}$, the state prices Π_u and Π_d can be expressed in terms of u, d and R:

$$\Pi_u = \frac{R-d}{u-d}\frac{1}{R} \quad \text{and} \quad \Pi_d = \frac{u-R}{u-d}\frac{1}{R}.$$

The call value under the binomial model is given by

$$c = \Pi_u c_u + \Pi_d c_d = \frac{\frac{R-d}{u-d}c_u + \frac{u-R}{u-d}c_d}{R} = \frac{pc_u + (1-p)c_d}{R},$$

where $p = \dfrac{R-d}{u-d}$.

10. We test whether a risk neutral measure $\boldsymbol{Q} = (Q_1 \quad Q_2 \quad Q_3)$ exists for the given securities model. This is done by solving

$$(Q_1 \quad Q_2 \quad Q_3)\begin{pmatrix} 1 & 4 & 5 \\ 1 & 2 & 3 \\ 1 & 1 & 2 \end{pmatrix} = (1 \quad 2 \quad 3).$$

We obtain the set of risk neutral measures R as characterized by $(Q_1 \quad Q_2 \quad Q_3) = (\lambda \quad 1-3\lambda \quad 2\lambda)$, $0 < \lambda < \frac{1}{3}$. For $Y^* = \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix}$, we have $E_Q[Y^*] = (\lambda \quad 1 - 3\lambda \quad 2\lambda)\begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix} = 4 + \lambda$. We deduce that

$$\begin{aligned} V_+ &= \sup\{E_Q[Y^*] : Q \in R\} = 4 + \frac{1}{3} = \frac{13}{3} \\ V_- &= \inf\{E_Q[Y^*] : Q \in R\} = 4. \end{aligned}$$

Hence, in order to avoid arbitrage, the range of reasonable initial price is $\left[4, \quad 4\frac{1}{3}\right]$.

11. For the securities model, it is easy to check that the set of risk neutral measures is characterized by

$$(Q_1 \quad Q_2 \quad Q_3) = (\alpha \quad 1-2\alpha \quad \alpha),\ 0 < \alpha < \frac{1}{2}.$$

Consider $E_Q[Y^*] = \frac{1}{S_0(1)}(Q_1 \quad Q_2 \quad Q_3)\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \frac{\alpha(y_1 - 2y_2 + y_3) + y_2}{S_0(1)}$, which is independent of α if and only if $y_1 - 2y_2 + y_3 = 0$. Since attainability of a contingent claim is equivalent to uniqueness of risk neutral price, so the necessary and sufficient condition for Y to be attainable is $y_1 - 2y_2 + y_3 = 0$.

12. Consider a portfolio with discounted terminal payoff $V^*(1; \omega_k) \geq 0$, $k = 1, 2, \ldots, K$, and strict inequality for at least one state, since the state prices $s_k, k = 1, 2, \ldots, K$ exist and they are all positive, we have

$$V(0) = \sum_{k=1}^{K} s_k V^*(1; \omega_k) > 0.$$

Therefore, it is impossible to have $V(0) = 0$ while $V^*(1; \omega_k) \geq 0$, $k = 1, 2, \ldots, K$, and strict inequality for at least one state. Therefore, there is no arbitrage opportunity.

Remark

Consider the securities model:

$$\boldsymbol{S}(0) = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} & \frac{1}{4} \end{pmatrix} \text{ and } S^*(1) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

All Arrow securities $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ lie in the asset span and the state prices are $\begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{4} \end{pmatrix}$. The state prices are all positive and their sum is *not equal* to 1. They are not risk neutral measures.

Suppose the riskfree security is now included, then the securities model becomes

$$\widehat{S}(0) = \begin{pmatrix} 1 & \frac{1}{3} & \frac{2}{3} & \frac{1}{4} \end{pmatrix} \text{ and } \widehat{S}^*(1) = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}.$$

In this case, even the law of one price is not satisfied since the portfolio that contains one unit of the last two securities replicates that of the riskfree security but the cost of the replicating portfolio is $\frac{2}{3} + \frac{1}{4} + \frac{11}{12} < 1$. The state prices cannot be specified under failure of law of one price.

13. Note that $\mathcal{F}$ is generated by the partition $\mathcal{P} = \{\{-3, -2\}, \{-1, 1\}, \{2, 3\}\}$.

   (i) Since $\{2, 3\} \in \mathcal{P}$ and $X(2) = 4 \neq X(3) = 9$, X is not $\mathcal{F}$-measurable.

   (ii) Since $\{2, 3\} \in \mathcal{P}$ and $X(2) = 2 \neq X(3) = 3$, X is not $\mathcal{F}$-measurable.

Define the random variable $X(\omega) = \max(\omega, 3)$. Now, $X(\omega) = 3$ for all $\omega \in \Omega$, hence X is $\mathcal{F}$-measurable.

14. (a) Suppose $\mathcal{F}$ is generated by a partition $\mathcal{P}$. It suffices to show that this property is valid for every $B \in \mathcal{P}$. Consider

$$\begin{aligned} E[I_B E[X|\mathcal{F}]] &= \sum_{\omega \in B} E[X|B]P(\omega) \\ &= E[X|B]P(B) \\ &= \sum_{\omega \in B} X(\omega)(P(\omega)/P(B))P(B) \\ &= \sum_{\omega \in B} X(\omega)P(\omega) \\ &= E[XI_B]. \end{aligned}$$

(b) Recall $E[X|\mathcal{F}] = \sum_{j=1}^{J} E[X|B_j]\mathbf{1}_{B_j}$, and consider

$$\begin{aligned} & E[\max(X_1, \cdots, X_n)|\mathcal{F}] \\ = & \sum_{j=1}^{J} E[\max(X_1, \cdots, X_n)|B_j]\mathbf{1}_{B_j} \\ = & \sum_{j=1}^{J} \frac{1}{P(B_j)} \sum_{k=1}^{K_j} \max(X_1(\omega_{k,j}), \cdots, X_n(\omega_{k,j}))P(\omega_{k,j})\mathbf{1}_{B_j} \end{aligned}$$

while

$$\begin{aligned} & \max(E[X_1|\mathcal{F}], \cdots, E[X_n|\mathcal{F}]) \\ = & \max\left(\sum_{j=1}^{J} \frac{1}{P(B_j)} \sum_{k=1}^{K_j} X_1(\omega_{k,j})P(\omega_{k,j})\mathbf{1}_{B_j}, \cdots, \sum_{j=1}^{J} \frac{1}{P(B_j)} \sum_{k=1}^{K_j} X_n(\omega_{k,j})P(\omega_{k,j})\mathbf{1}_{B_j} \right). \end{aligned}$$

It is obvious that the maximum value among the various sums of $X_\ell(\omega_{k,j})$, $\ell = 1, \cdots, n$, cannot be greater than the value obtained by taking the maximum value among $X_1(\omega_{k,j})$, $\cdots$, $X_n(\omega_{k,j})$ and performing the summation afterward. Hence, we obtain the desired result.

15. The property: $E[X_{t+1} - X_t|\mathcal{F}_t] = 0, t = 0, 1, \ldots, T-1$, does imply that X is a martingale because

$$\begin{aligned} E[X_T|\mathcal{F}_t] &= \sum_{s=t}^{T-1} E[X_{s+1} - X_s|\mathcal{F}_t] + E[X_t|\mathcal{F}_t] \\ &= \sum_{s=t}^{T-1} E[E[X_{s+1} - X_s|\mathcal{F}_s]|\mathcal{F}_t] + X_t \\ &= X_t, \quad t = 0, 1, \ldots, T-1. \end{aligned}$$

16. Note that $N_{k+1} - N_k$ is independent of $\mathcal{F}_k$ since the successive binomial trials are independent and $p =$ probability of success in the $(k+1)^{\text{th}}$ trial, we have

$$\begin{aligned} E[N_{k+1} - (k+1)p - (N_k - kp)|\mathcal{F}_k] &= E[N_{k+1} - N_k - p|\mathcal{F}_k] \\ &= E[N_{k+1} - N_k - p] \\ &= 0. \end{aligned}$$

Then from Problem 15, we deduce that Y_k is a martingale.

17. Consider a portfolio consisting of 4 units of money market account with interest rate $r = 0.3$ and shorting one unit of asset, then we have

$$V(0) = 4 - S(0) = 0, V(2;\omega_i) = 4(1+r)^2 - S(2;\omega_i) \geq 0.76, \quad i = 1,2,3,4.$$

Hence, this is an arbitrage opportunity.