MAFS 5030
Quantitative Modeling of Derivative Securities

Homework Three

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1. Let $\{W_t\}_{t\geq 0}$ generate the filtration $\{\mathcal{F}_t\}_{t\geq 0}$. If $\{W_t\}_{t\geq 0}$ is a standard Brownian motion under the probability measure P, then

   (a) $W_t^2 - t$ is a $(P, \{\mathcal{F}_t\}_{t\geq 0})$-martingale;

   (b) $\exp\left(\sigma W_t - \dfrac{\sigma^2}{2}t\right)$ is a $(P, \{\mathcal{F}_t\}_{t\geq 0})$-martingale, called an exponential martingale.

2. Show that
$$\sigma \int_t^T [Z(u) - Z(t)]\, du$$
has zero mean and variance $\sigma^2(T-t)^3/3$.

   Hint: Consider
$$\begin{aligned}
&\text{var}\left(\int_t^T [Z(u) - Z(t)]\, du\right) \\
&= E\left[\int_t^T \int_t^T [Z(u) - Z(t)][Z(v) - Z(t)]\, dudv\right] \\
&= \int_t^T \int_t^T E[\{Z(u) - Z(t)\}\{Z(v) - Z(t)\}]\, dudv \\
&= \int_t^T \int_t^T [\min(u, v) - t]\, dudv.
\end{aligned}$$

3. Suppose the stochastic variables S_1 and S_2 follow the Geometric Brownian motions where
$$\frac{dS_i}{S_i} = \mu_i\, dt + \sigma_i\, dZ_i, \qquad i = 1, 2.$$
Let ρ_{12} denote the correlation coefficient between dZ_1 and dZ_2, where $dZ_1 dZ_2 = \rho dt$. Let $f = S_1 S_2$, show that f also follows the Geometric Brownian motion of the form
$$\frac{df}{f} = \mu\, dt + \sigma\, dZ_f$$
where $\mu = \mu_1 + \mu_2 + \rho_{12}\sigma_1\sigma_2$ and $\sigma^2 = \sigma_1^2 + \sigma_2^2 + 2\rho_{12}\sigma_1\sigma_2$. Similarly, we let $g = \dfrac{S_1}{S_2}$. Show that
$$\frac{dg}{g} = \widetilde{\mu}\, dt + \widetilde{\sigma}\, dZ_g$$
where $\widetilde{\mu} = \mu_1 - \mu_2 - \rho_{12}\sigma_1\sigma_2 + \sigma_2^2$ and $\widetilde{\sigma}^2 = \sigma_1^2 + \sigma_2^2 - 2\rho_{12}\sigma_1\sigma_2$.

Hint: Note that

$$\frac{d\left(\frac{1}{S_2}\right)}{\frac{1}{S_2}} = -\mu_2\, dt + \sigma_2^2\, dt - \sigma_2\, dZ_2.$$

Treat S_1/S_2 as the product of S_1 and $1/S_2$ and use the result obtained for the product of Geometric Brownian motions.

4. Define the discrete random variable X by

$$X(\omega) = \begin{cases} 2 & \text{if } \omega = \omega_1 \\ 3 & \text{if } \omega = \omega_2 \\ 4 & \text{if } \omega = \omega_3 \end{cases},$$

where the sample space $\Omega = \{\omega_1, \omega_2, \omega_3\}, P[\omega_1] = P[\omega_2] = P[\omega_3] = 1/3$. Find a new probability measure $\widetilde{P}$ such that the mean becomes $E_{\widetilde{P}}[X] = 3.5$ while the variance remains unchanged. Is $\widetilde{P}$ unique?

5. Given that S_t is a Geometric Brownian motion which follows

$$\frac{dS_t}{S_t} = \mu\ dt + \sigma\ dZ_t$$

where Z_t is P-Brownian motion. Find another measure $\widetilde{P}$ by specifying the Radon-Nikodym derivative $\dfrac{d\widetilde{P}}{dP}$ such that S_t is governed by

$$\frac{dS_t}{S_t} = \mu'\ dt + \sigma\ d\widetilde{Z}_t$$

under the measure $\widetilde{P}$, where $\widetilde{Z}_t$ is $\widetilde{P}$-Brownian motion and μ' is the new drift rate.

6. Show that $\exp\left(-\mu Z_P(t) - \dfrac{\mu^2 t}{2}\right)$ is a martingale under P, where $Z_P(t)$ is P-Brownian.

7. Consider a forward contract on an underlying commodity, find the portfolio consisting of the underlying commodity and bond (bond's maturity coincides with forward's maturity) that replicates the forward contract. Show that the hedge ratio $\triangle$ is always equal to one. Give the financial argument to justify why the hedge ratio is one. Let $B(t,T)$ denote the price at current time t of the unit-par zero-coupon bond maturing at time T and S denote the price of commodity at time t. Show that the forward price $F(S,\tau)$ is given by

$$F(S,\tau) = S/B(t,T), \quad \tau = T - t.$$

8. Consider a portfolio containing Δ units of asset and M dollars of riskless asset in the form of money market account. The portfolio is dynamically adjusted so as to replicate an option. Let S and $V(S,t)$ denote the value of the underlying asset and the option, respectively. Let r denote the riskless interest rate and Π denote the value of the *self-financing* replicating portfolio. When the self-financing trading strategy is adopted, explain why

$$\Pi = \Delta S + M \quad \text{and} \quad d\Pi = \Delta\ dS + rM\ dt,$$

where r is the riskless interest rate. Here, the differential term $S\ d\Delta$ does not enter into $d\Pi$. Assume that the asset price dynamics follows the Geometric Brownian motion:

$$\frac{dS}{S} = \rho\ dt + \sigma\ dZ.$$

Replication means that the option value and the value of the replicating portfolio should match at all times, show that the number of units of asset held must be given by

$$\Delta = \frac{\partial V}{\partial S}.$$

How to proceed further in order to obtain the Black-Scholes equation for V?

9. When a European option is currently out-of-the-money, show that a higher volatility of the asset price or a longer time to expiry makes it more likely for the option to expire in-the-money. What would be the impact on the value of delta? Do we have the same effect or opposite effect when the option is currently in-the-money?

10. Show that when the European call price is a convex function of the asset price, the elasticity of the call price is always greater than or equal to one. Can you think of a situation where the European put's elasticity has absolute value less than one, that is, the European put option is less riskier than the underlying asset?

11. The greeks of the value of a derivative security are defined by

$$\Theta = \frac{\partial f}{\partial t}, \quad \triangle = \frac{\partial}{\partial S}, \quad \Gamma = \frac{\partial^2 f}{\partial S^2}.$$

   (a) Find the relation between Θ and Γ for a delta-neutral portfolio where $\triangle = 0$.
   (b) Explain by financial argument why the theta value tends asymptotically to $-rXe^{-r\tau}$ from below when the asset value is sufficiently high.

12. Consider a European capped call option whose terminal payoff function is given by

$$c_M(S,0;X,M) = \min(\max(S - X, 0), M),$$

where X is the strike price and M is the cap. Show that the value of the European capped call is given by

$$c_M(S,\tau;X,M) = c(S,\tau;X) - c(S,\tau;X+M),$$

where $c(S,\tau;X+M)$ is the value of a European vanilla call with strike price $X+M$.

13. Consider the value of a European call option written by an issuer who holds only up to α (< 1) units of the underlying asset. As a result, the terminal payoff of this call is then given by

$$S_T - X \quad \text{if} \quad \alpha S_T \geq S_T - X \geq 0 \quad \text{and} \quad \alpha S_T \quad \text{if} \quad S_T - X > \alpha S_T,$$

and zero otherwise. Show that the value of this European call option is given by

$$c_L(S, \tau; X, \alpha) = c(S, \tau; X) - (1 - \alpha) c\left(S, \tau; \frac{X}{1 - \alpha}\right), \quad \alpha < 1,$$

where $c\left(S, \tau; \frac{X}{1 - \alpha}\right)$ is the value of a European vanilla call with strike price $\frac{X}{1 - \alpha}$. Show that the delta of this European call is always less than α.