MAFS 5030 - Quantitative Modeling of Derivative Securities

Solution to Homework Three

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1. (a) For $t > s$, consider

$$\begin{aligned} E[W_t^2 - W_s^2|\mathcal{F}_s] &= E\left[(W_t - W_s)^2 + 2W_s(W_t - W_s)\middle|\, \mathcal{F}_s\right] \\ &= E\left[(W_t - W_s)^2\middle|\, \mathcal{F}_s\right] + 2W_s E\left[W_t - W_s|\, \mathcal{F}_s\right] \\ &= t - s, \end{aligned}$$

We then have

$$\begin{aligned} E[W_t^2 - t|\mathcal{F}_s] &= E\left[W_t^2 - W_s^2 + W_s^2 - (t-s) - s\middle|\, \mathcal{F}_s\right] \\ &= (t-s) + W_s^2 - (t-s) - s = W_s^2 - s. \end{aligned}$$

Also, $E\left[\left|W_t^2 - t\right|\right] < \infty$ is observed. Therefore, $W_t^2 - t$ is a $(P, \{\mathcal{F}_t\}_{t\geq 0})$-martingale.

(b) From the lecture note p.8, we obtain

$$E\left[\exp(\sigma(W_t - W_s))|\mathcal{F}_s\right] = \frac{\sigma^2}{2}(t-s).$$

For $t > s$, consider

$$\begin{aligned} & E\left[\exp\left(\sigma W_t - \frac{\sigma^2}{2}t\right)\right] \\ = \;& E\left[\exp(\sigma(W_t - W_s)) - \frac{\sigma^2}{2}(t-s) + \exp(\sigma W_s) - \frac{\sigma^2}{2}s\middle|\, \mathcal{F}_s\right] \\ = \;& E\left[\exp(\sigma(W_t - W_s))|\, \mathcal{F}_s\right] - \frac{\sigma^2}{2}(t-s) + \exp(\sigma W_s) - \frac{\sigma^2}{2}s \\ = \;& \exp\left(\sigma W_s - \frac{\sigma^2}{2}s\right). \end{aligned}$$

Also, $E\left[\left|\exp(\sigma W_t) - \dfrac{\sigma^2}{2}t\right|\right] < \infty$ is observed. Therefore, $\exp(\sigma W_t) - \dfrac{\sigma^2}{2}t$ is a $(P, \{\mathcal{F}_t\}_{t\geq 0})$-martingale.

2. We have

$$E\left[\int_t^T [Z(u) - Z(t)]\, du\right] = \int_t^T E[Z(u) - Z(t)]\, du = 0$$

and

$$\begin{aligned}
& \text{var}\left(\sigma\int_t^T [Z(u)-Z(t)]\,du\right) \\
= \;& \sigma^2 E\left[\int_t^T [Z(u)-Z(t)]\,du\right]^2 \\
= \;& \sigma^2 E\left[\int_t^T\int_t^T [Z(u)-Z(t)][Z(v)-Z(t)]\,dudv\right] \\
= \;& \sigma^2 \int_t^T\int_t^T E[\{Z(u)-Z(t)\}\{Z(v)-Z(t)\}]\,dudv \\
= \;& \sigma^2 \int_t^T\int_t^T [\min(u,v)-t]\,dudv \\
= \;& \sigma^2\left[\int_t^T (u-t)\,du\int_u^T dv + \int_t^T (v-t)\,dv\int_v^T du\right] \\
= \;& \sigma^2(T-t)^3/3.
\end{aligned}$$

3. By virtue of the properties of normal distribution and the definition of a Brownian motion, we observe that $\sigma_1\,dZ_1(t)+\sigma_2\,dZ_2(t)$ is a Brownian motion with mean 0 and variance rate σ^2, where $\sigma^2=\sigma_1^2+\sigma_2^2+2\rho_{12}\sigma_1\sigma_2$. Define $Z(t)=\dfrac{\sigma_1 Z_1(t)+\sigma_2 Z_2(t)}{\sigma}$, which is seen to be a Brownian motion with zero mean and unit variance rate. Note that $dZ_1dZ_2=\rho_{12}\,dt$ in the mean square sense. For $f=S_1S_2$, it follows that

$$\begin{aligned}
df &= S_1\,dS_2+S_2\,dS_1+dS_1\,dS_2 \\
&= S_1S_2(\mu_2\,dt+\sigma_2\,dZ_2)+S_2S_1(\mu_1\,dt+\sigma_1\,dZ_1)+S_1S_2\sigma_1\sigma_2\,dZ_1\,dZ_2 \\
&= f(\mu_1+\mu_2+\rho_{12}\sigma_1\sigma_2)\,dt+f(\sigma_1\,dZ_1+\sigma_2\,dZ_2) \\
&= f\mu\,dt+f\sigma\,dZ.
\end{aligned}$$

Alternatively, we may consider

$$\begin{aligned}
S_1(t) &= S_1(0)\exp\left(\left(\mu_1-\frac{\sigma_1^2}{2}\right)t+\sigma_1 Z_1(t)\right) \\
S_2(t) &= S_2(0)\exp\left(\left(\mu_2-\frac{\sigma_2^2}{2}\right)t+\sigma_2 Z_2(t)\right),
\end{aligned}$$

so that

$$\begin{aligned}
f=S_1S_2 &= S_1(0)S_2(0)\exp\left(\left(\mu_1+\mu_2-\frac{\sigma_1^2}{2}-\frac{\sigma_2^2}{2}\right)t+\sigma_1Z_1(t)+\sigma_2Z_2(t)\right) \\
&= S_1(0)S_2(0)\exp\left((\mu_1+\mu_2+\rho_{12}\sigma_1\sigma_2)t-\frac{\sigma_1^2+\sigma_2^2+2\rho_{12}\sigma_1\sigma_2}{2}\,t+\sigma Z(t)\right).
\end{aligned}$$

From $S_2(t)=S_2(0)\exp\left(\left(\mu_2-\dfrac{\sigma_2^2}{2}\right)t+\sigma_2Z_2(t)\right)$, we deduce

$$S_2^{-1}=\frac{1}{S_2(0)}\exp\left[\left(-\mu_2+\frac{\sigma_2^2}{2}\right)t-\sigma_2Z_2(t)\right].$$

The corresponding dynamic equation is seen to be

$$\frac{dS_2^{-1}}{S_2^{-1}} = \left(-\mu_2 + \sigma_2^2\right)\, dt - \sigma_2\, dZ_2.$$

From the first result, for $g = S_1/S_2$, it follows that

$$dg = g\tilde{\mu}\, dt + g\tilde{\sigma}\, d\widehat{Z},$$

where $\tilde{\mu} = \mu_1 - \mu_2 - \rho_{12}\sigma_1\sigma_2 + \sigma_2^2$, $\tilde{\sigma}^2 = \sigma_1^2 + \sigma_2^2 - 2\rho_{12}\sigma_1\sigma_2$ and

$$\widehat{Z}(t) = \frac{\sigma_1 Z_1(t) - \sigma_2 Z_2(t)}{\widetilde{\sigma}}.$$

4. First, we have $\text{var}_P(X) = 2/3$. By solving the following equations:

$$E_{\widetilde{P}}[X] = 2\widetilde{P}[\omega_1] + 3\widetilde{P}[\omega_2] + 4\widetilde{P}[\omega_3] = 3.5,$$
$$\text{var}_{\widetilde{P}}(X) = (2-3.5)^2\widetilde{P}[\omega_1] + (3-3.5)^2\widetilde{P}[\omega_2] + (4-3.5)^2\widetilde{P}[\omega_3] = 2/3,$$

and

$$\widetilde{P}[\omega_1] + \widetilde{P}[\omega_2] + \widetilde{P}[\omega_3] = 1,$$

We obtain the unique solution: $\widetilde{P}[\omega_1] = 5/24, \widetilde{P}[\omega_2] = 1/12, \widetilde{P}[\omega_3] = 17/24$. Since $\widetilde{P}[\omega_i], i = 1, 2, 3$ are all positive, we do obtain the required probability measure $\widetilde{P}$ and it is unique.

5. Let $\gamma = \dfrac{\mu - \mu'}{\sigma}$ and consider the Radon-Nikodym derivative:

$$\frac{d\widetilde{P}}{dP} = \rho(t)$$

where

$$\rho(t) = \exp\left(\int_0^t -\gamma(s)\ dZ(s) - \frac{1}{2}\int_0^t \gamma(s)^2\ ds\right).$$

Under the measure $\widetilde{P}$, the stochastic process

$$\widetilde{Z}_t = Z_t + \int_0^t \gamma(s)\, ds$$

is $\widetilde{P}$-Brownian by the Girsanov Theorem. It is seen that when we set $\gamma = \dfrac{\mu - \mu'}{\sigma}$, then

$$\mu'\, dt + \sigma\, d\widetilde{Z}_t = \mu'\, dt + \sigma(dZ_t + \gamma\, dt) = \mu\, dt + \sigma\, dZ_t.$$

Therefore, S_t is governed by

$$\frac{dS_t}{S_t} = \mu'\, dt + \sigma\, d\widetilde{Z}_t$$

under the measure $\widetilde{P}$.

6. For $s < t$, we consider

$$\begin{aligned} & E_P\left[\exp\left(-\mu Z_P(t) - \frac{\mu^2 t}{2}\right)\Big|\, \mathcal{F}_s\right] \\ = \; & E_P\left[\exp\left(-\mu Z_P(s) - \frac{\mu^2 s}{2}\right)\exp\Big(-\mu \underbrace{(Z_P(t) - Z_P(s))}_{\substack{\text{normal with} \\ \text{variance } t-s}} - \frac{\mu^2 s}{2}(t-s)\Big)\Big|\, \mathcal{F}_s\right] \end{aligned}$$

Recall that the expectation of a normal random variable with variance $\mu^2(t-s)$ is $\exp\left(\frac{\mu^2}{2}(t-s)\right)$.

We then have

$$\begin{aligned} & E_P\left[\exp\left(-\mu Z_P(t) - \frac{\mu^2 t}{2}\right)\Big|\, \mathcal{F}_s\right] \\ = \; & \exp\left(-\mu Z_P(s) - \frac{\mu^2 s}{2}\right)\exp\left(\frac{\mu^2}{2}(t-s)\right)\exp\left(-\frac{\mu^2}{2}(t-s)\right) \\ = \; & \exp\left(-\mu Z_P(s) - \frac{\mu^2 s}{2}\right). \end{aligned}$$

7. Let K be the delivery price of the commodity forward, then the value of the forward contract is given by

$$f = S - Ke^{-r\tau} = S - B(\tau; K);$$

where $B(\tau; K)$ denotes the price of a T-maturity bond with par value K at time to expiry τ, hence the hedge ratio Δ is always one. The forward contract is an agreement where the holder agrees to buy the commodity at the delivery time T for the delivery price K. It can be replicated by holding one unit of the commodity and shorting one unit of a bond with par value K, implying the hedge ratio is one. Setting $f = 0$, we get $K = Se^{r\tau}$. It follows that the forward price $F(S, \tau)$ is given by

$$F(S, \tau) = Se^{r\tau} = S/B(t, T), \;\; \tau = T - t.$$

8. When the self-financing trading strategy is adopted, the purchase of additional units of asset is financed by the sale of the riskless asset, hence $Sd\Delta + dM = 0$ and it follows that

$$\begin{aligned} d\Pi &= \Delta dS + Sd\Delta + rMdt + dM \\ &= \Delta dS + rMdt. \end{aligned}$$

By substituting $dS = \rho S\, dt + \sigma S\, dZ$ into the above equation, we obtain

$$d\Pi = (\rho S\Delta + rM)\, dt + \sigma S\Delta\, dZ.$$

On the other hand, Ito's Lemma gives

$$\begin{aligned} dV &= \frac{\partial V}{\partial t}\, dt + \frac{\partial V}{\partial S}\, dS + \frac{\sigma^2}{2}S^2\frac{\partial^2 V}{\partial S^2}\, dt \\ &= \left(\frac{\partial V}{\partial t} + \rho S\frac{\partial V}{\partial S} + \frac{\sigma^2}{2}S^2\frac{\partial^2 V}{\partial S^2}\right) dt + \sigma S\frac{\partial V}{\partial S}\, dZ. \end{aligned}$$

Since Π is a replicating portfolio, in order to match $d\Pi = dV$, it is necessary to choose

$$\Delta = \frac{\partial V}{\partial S}.$$

This leads to

$$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} - rM = 0.$$

Note that $M = V - \Delta S = V - S\frac{\partial V}{\partial S}$, the Black-Scholes equation for V is then given by

$$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0.$$

9. Consider $\Delta_c = N(d_1)$ and $\Delta_p = -N(-d_1)$, where

$$d_1 = \frac{\ln\frac{S}{X} + \left(r + \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}.$$

We have

$$\frac{\partial d_1}{\partial \sigma} = \frac{\sqrt{\tau}}{2} - \frac{\ln\frac{S}{X} + r\tau}{\sigma^2\sqrt{\tau}}, \quad \frac{\partial d_1}{\partial \tau} = \frac{-\ln\frac{S}{X} + \left(r + \frac{\sigma^2}{2}\right)\tau}{2\sigma\tau^{\frac{3}{2}}}.$$

Financial interpretation: When the option is sufficiently out-of-the-money currently, a higher volatility of the asset price or a longer time to expiry implies a greater value of delta. The option becomes more likely for the option to expire in-the-money. When the option is currently in-the-money, $\ln\dfrac{S}{X}$ changes in sign; so higher σ and longer τ lead to a small value of delta.

10. The convexity of the European call price with respect to the asset price implies

$$\frac{c(S,\tau;X) - c(S',\tau;X)}{S - S'} \geq \frac{c(S,\tau;X)}{S}.$$

Let $S' \to S$, we obtain $\frac{\partial c}{\partial S} \geq \frac{c}{S}$, hence $e_c \geq 1$.

For a European put option, we have

$$e_p = \left(\frac{\partial p}{\partial S}\right)\left(\frac{S}{p}\right) = \frac{-SN(-d_1)}{Xe^{-r\tau}N(-d_2) - SN(-d_1)}.$$

In order to have $|e_p| < 1$, this is equivalent to observe

$$2SN(-d_1) < Xe^{-r\tau}N(-d_2).$$

This occurs when S is sufficiently low or the put is sufficiently in-the-money.

11. (a) By Ito's Lemma and observing $\Delta = 0$, it follows that

$$\begin{aligned} df &= \left(\frac{\partial f}{\partial t} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 f}{\partial S^2}\right)dt + \frac{\partial f}{\partial S}dS \\ &= \left(\Theta + \frac{\sigma^2 S^2}{2}\Gamma\right)dt. \end{aligned}$$

By virtue of no arbitrage, we set $df = rf\,dt$ and obtain

$$\Theta + \frac{\sigma^2 S^2}{2}\Gamma = rf.$$

(b) When the asset value is sufficiently high, the call will always be exercised with terminal payoff $S_T - X$. Therefore, the call price tends asymptotically to $S - Xe^{-r\tau}$. By differentiating $S - Xe^{-r\tau}$ with respect to τ, it is seen that the theta tends asymptotically to $-rXe^{-\tau}$ from below.

12. Using the risk neutral valuation principle, the value of the European call option is given by

$$\begin{aligned} c_M(S,\tau;X,M) &= e^{-r\tau}E_Q[c_M(S_T,0;X,M)] \\ &= e^{-r\tau}E_Q[\max(S_T - X, 0)] + e^{-r\tau}E_Q[\max(S_T - X - M, 0)] \\ &= c(S,\tau;X) - c(S,\tau;X+M). \end{aligned}$$

Alternatively, we can derive the price relation without the assumption of existence of Q. The capped European call and the portfolio of long one unit of European call with strike X and short one unit of European call with strike $X+M$ have the same payoff at T under all scenarios of S_T, so their current prices have the same relation. This relation is model free, that is, it is independent of the underlying assumption of the underlying asset price distribution.

13. The terminal payoff function of this call is given by

$$c_L(S,\tau;X,\alpha) = \min(\max(S_T - X, 0), \alpha S_T).$$

Using the risk neutral valuation principle, it follows that

$$\begin{aligned} c_L(S,\tau;X,\alpha) &= e^{-r\tau}E_Q[c_L(S_T,0;X,\alpha S_T)] \\ &= e^{-r\tau}E_Q[\max(S_T - X, 0) + \alpha S_T - \max(\max(S_T - X, 0), \alpha S_T)] \\ &= e^{-r\tau}E_Q[\max(S_T - X, 0)] - (1-\alpha)e^{-r\tau}E_Q\left[\max\left(S_T - \frac{X}{1-\alpha}, 0\right)\right] \\ &= c(S,\tau;X) - (1-\alpha)c\left(S,\tau;\frac{X}{1-\alpha}\right). \end{aligned}$$

Note that the delta of c_L is an increasing function of S. This is because

$$\Delta_{c_L} = N(d_1(X)) - (1-\alpha)N\left(d_1\left(\frac{X}{1-\alpha}\right)\right)$$

and

$$\frac{\partial \Delta_{c_L}}{\partial S} = n(d_1(X))\frac{1}{S\sigma\sqrt{\tau}} - (1-\alpha)n\left(d_1\left(\frac{X}{1-\alpha}\right)\right)\frac{1}{S\sigma\sqrt{\tau}} > 0.$$

As $S \to \infty$, c_L becomes α units of forward with zero forward price, so $\Delta_{c_L} \to \alpha$ as $S \to \infty$. Here, Δ_{c_L} is always less than α and tends to α from below when the asset price becomes exceedingly large.