MAFS 5030 - Quantitative Modeling of Derivative Securities

Solution to Homework Four

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1. When the dividends are taxed at the rate R, the differential of the portfolio value $\Pi = -c + \Delta S$ is given by

$$d\Pi = -\left[\frac{\partial c}{\partial t} + \frac{\sigma^2}{2}S^2\frac{\partial^2 c}{\partial S^2} + (1-R)q\Delta S\right]dt + \left(\Delta - \frac{\partial c}{\partial S}\right)dS.$$

Note that the capital gains on the change in value of c and S are not taxed. Again, we set $\Delta = \dfrac{\partial c}{\partial S}$ to eliminate the random term. Since interest incomes are taxed at the rate R, the deterministic rate of return from the money market account is $(1-R)r$. We set $d\Pi = (1-R)r\Pi\ dt$ to give

$$d\Pi = \left[-\frac{\partial c}{\partial t} - \frac{\sigma^2}{2}S^2\frac{\partial^2 c}{\partial S^2} + (1-R)qS\frac{\partial c}{\partial S}\right]dt = (1-R)r\left(-c + S\frac{\partial c}{\partial S}\right)dt.$$

This gives the governing equation for the call price function as follows

$$\frac{\partial c}{\partial t} + (1-R)(r-q)S\frac{\partial c}{\partial S} + \frac{\sigma^2}{2}S^2\frac{\partial^2 c}{\partial S^2} - (1-R)rc = 0.$$

The European call and put price formulas are given by

$$\begin{aligned} c(S,\tau) &= Se^{-(1-R)q\tau}N(d_1) - Xe^{-(1-R)r\tau}N(d_2) \\ p(S,\tau) &= Xe^{-(1-R)r\tau}N(-d_2) - Se^{-(1-R)q\tau}N(-d_1), \end{aligned}$$

where $\tau = T - t$ and

$$d_1 = \frac{\ln\frac{S}{X} + \left[(1-R)(r-q) + \frac{\sigma^2}{2}\right]\tau}{\sigma\sqrt{\tau}},\ \ d_2 = d_1 - \sigma\sqrt{\tau}.$$

2. An investor has two choices at time t: (i) holding cash amount of S until T, (ii) use the cash amount of S to buy the underlying asset and short one unit of forward (causes nothing to enter into the short position of the forward). The values of wealth at T for both strategies are $Se^{r(T-t)}$ and $F + \displaystyle\sum_{i=1}^{N} D_i e^{r(T-t_i)}$. These two strategies should have the same value, so we obtain

$$F(S,t) = Se^{r(T-t)} - \sum_{i=1}^{N} D_i e^{r(T-t_i)}.$$

For the call price function $c(S,t)$, the governing differential equation is

$$\frac{\partial c}{\partial t} + rS\frac{\partial c}{\partial S} + \frac{\sigma^2}{2}S^2\frac{\partial^2 c}{\partial S^2} - rc = 0,$$

where the risk neutral drift rate remains to be r due to the discrete nature of dividend payments. Consider the rules of differentials

$$\begin{aligned} \frac{\partial c}{\partial t}(S,t) &= \frac{\partial c}{\partial t} + \frac{\partial c}{\partial F}(F,t)\frac{\partial F}{\partial t} = \frac{\partial c}{\partial t}(F,t) - rSe^{r(T-t)}\frac{\partial c}{\partial F}(F,t) \\ \frac{\partial c}{\partial S}(S,t) &= \frac{\partial c}{\partial F}(F,t)\frac{\partial F}{\partial S} = e^{r(T-t)}\frac{\partial c}{\partial F}(F,t) \end{aligned}$$

so that

$$rS\frac{\partial c}{\partial S}(S,t) = rSe^{r(T-t)}\frac{\partial c}{\partial F}(F,t) \quad \text{and} \quad \frac{\sigma^2}{2}S^2\frac{\partial^2 c}{\partial S^2}(S,t) = \frac{\sigma^2}{2}\left[Se^{r(T-t)}\right]^2\frac{\partial^2 c}{\partial F^2}(F,t).$$

Putting all these relations together, we obtain

$$\frac{\partial c_F}{\partial t} + \frac{\sigma^2}{2}\left[F + \sum_{i=1}^{N} D_i e^{r(T-t_i)}\right]^2 \frac{\partial^2 c_F}{\partial F^2} - rc_F = 0.$$

Note that closed form solution to the above differential equation cannot be found.

Remark

Between two consecutive discrete dividend dates, the governing differential equation for the call price function remains the same. However, the transition density of the asset price will be affected by the discrete dividend payments since the asset price drops by the dividend amount on each ex-dividend date.

3. The forward start call option price is given by

$$\begin{aligned}
& e^{-r(T_2-t)}E_Q[(S_{T_2}-S_{T_1})^+|\mathcal{F}_t] \\
= {} & e^{-r(T_2-t)}E_Q[E_Q[(S_{T_2}-S_{T_1})^+|\mathcal{F}_{T_1}]|\mathcal{F}_t] \\
= {} & e^{-r(T_2-t)}E_Q\left[S_{T_1}e^{(r-q)(T_2-T_1)}N\left(\frac{r-q+\frac{\sigma^2}{2}}{\sigma}\sqrt{T_2-T_1}\right) - N\left(\frac{r-q-\frac{\sigma^2}{2}}{\sigma}\sqrt{T_2-T_1}\right)\middle|\mathcal{F}_t\right].
\end{aligned}$$

Note that $E_Q[S_{T_1}|\mathcal{F}_t] = e^{(r-q)(T_1-t)}S$ so that the call option price can be expressed as

$$\begin{aligned}
& e^{-q(T_1-t)}\left\{\left[Se^{-q(T_2-T_1)}N\left(\frac{r-q+\frac{\sigma^2}{2}}{\sigma}\sqrt{T_2-T_1}\right) - e^{-r(T_2-T_1)}N\left(\frac{r-q-\frac{\sigma^2}{2}}{\sigma}\sqrt{T_2-T_1}\right)\right]\right\} \\
= {} & e^{-q(T_1-t)}c(S,T_2-T_1;S),
\end{aligned}$$

where $c(S,T_2-T_1;S)$ is the value of an at-the-money call with time to expiry T_2-T_1.

4. The price formula of a European call on a continuous dividend paying asset is given by

$$c = Se^{-q\tau}N(\hat{d}_1) - Xe^{-r\tau}N_(\hat{d}_2),$$

where

$$\hat{d}_1 = \frac{\ln\frac{S}{X} + \left(r-q+\frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}, \quad \hat{d}_2 = \hat{d}_1 - \sigma\sqrt{\tau}.$$

The theta is found to be

$$\begin{aligned}
\Theta_c &= \frac{\partial c}{\partial t} = -\frac{\partial c}{\partial \tau} \\
&= qSe^{-q\tau}N(\hat{d}_1) - \frac{Se^{-q\tau-\frac{\hat{d}_1^2}{2}}}{\sqrt{2\pi}}\frac{\partial \hat{d}_1}{\partial \tau} - rXe^{-r\tau}N(\hat{d}_2) + \frac{Xe^{-r\tau-\frac{\hat{d}_2^2}{2}}}{\sqrt{2\pi}}\frac{\partial \hat{d}_2}{\partial \tau} \\
&= qSe^{-q\tau}N(\hat{d}_1) - \frac{1}{\sqrt{2\pi}}\frac{Se^{-q\tau-\frac{\hat{d}_1^2}{2}}\sigma}{2\sqrt{\tau}} - rXe^{-r\tau}N(\hat{d}_2) \\
&= rc + Se^{-q\tau}\left[(q-r)N(\hat{d}_1) - \frac{1}{\sqrt{2\pi}}\frac{e^{-\frac{\hat{d}_1^2}{2}}\sigma}{2\sqrt{\tau}}\right].
\end{aligned}$$

For an in-the-money call, $S > X$, when q is sufficiently high, the theta may become positive.

5. Recall $M(T) = e^{rT}$ with $M(0) = 1$. From the numeraire invariance theorem, we have

$$\frac{dQ^*}{dQ} = \frac{S_T}{S_0} \Bigg/ \frac{M(T)}{M(0)} = \frac{S_T}{S_0}e^{-rT} = e^{(r-\frac{\sigma^2}{2})T+\sigma Z_T}e^{-rT} = e^{-\frac{\sigma^2}{2}T+\sigma Z_T}.$$

Note that

$$\exp\left(-\frac{\sigma^2}{2}T + \sigma Z\right) = \exp\left(\int_0^T -(-\sigma)dZ - \frac{1}{2}\int_0^T (-\sigma)^2 ds\right),$$

we deduce that $Z_T^* = Z_T + \int_0^T -\sigma\, ds = Z_T - \sigma T$ is a Brownian process under Q^* by virtue of Girsanov's Theorem. We then obtain

$$\ln\frac{S_T}{S_0} = \left(r + \frac{\sigma^2}{2}\right)T + \sigma Z_T^*.$$

From the density function of a normal random variable , we deduce that the transition density function of S_T under Q is given by

$$\psi^*(S_T, T; S_0, 0) = \frac{1}{S_T\sigma\sqrt{2\pi T}}\exp\left(-\frac{\left[\ln\frac{S_T}{S_0} - \left(r + \frac{\sigma^2}{2}\right)T\right]^2}{2\sigma^2 T}\right).$$

Suppose we set $\ln\frac{S_T}{S_0} = y$, it follows that

$$\begin{aligned}
E_{Q^*}\left[\mathbf{1}_{\{S_T \geq X\}}\right] &= \int_X^\infty \psi^*(S_T, T; S_0, 0)\ dS_T \\
&= \int_{-\infty}^{\ln\frac{S_0}{X}} \frac{1}{\sigma\sqrt{2\pi T}}\exp\left(-\frac{\left[y + \left(r + \frac{\sigma^2}{2}\right)T\right]^2}{2\sigma^2 T}\right) dy \\
&= N\left(\frac{\ln\frac{S_0}{X} + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right).
\end{aligned}$$

Finally, we consider

$$\begin{aligned}
E_Q\left[S_T\mathbf{1}_{\{S_T \geq X\}}\right] &= E_{Q^*}\left[\frac{dQ}{dQ^*}S_T\mathbf{1}_{\{S_T \geq X\}}\right] \\
&= e^{rT}S_0 E_{Q^*}\left[\mathbf{1}_{\{S_T \geq X\}}\right] \\
&= e^{rT}S_0 N\left(\frac{\ln\frac{S_0}{X} + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right).
\end{aligned}$$

6. The time-t value of the contingent claim is given by

$$\begin{aligned}
& e^{-r(T-t)}E_Q[\min(S_{T_0}, S_T)|\mathcal{F}_t] \\
=\ & e^{-r(T-t)}E_Q[E_Q[\min(S_{T_0}, S_T)|\mathcal{F}_{T_0}]|\mathcal{F}_t] \\
=\ & e^{-r(T-t)}E_Q[E_Q[S_T - \max(S_T - S_{T_0}, 0)|\mathcal{F}_{T_0}]|\mathcal{F}_t] \\
=\ & e^{-r(T-t)}E_Q[e^{(r-q)(T-T_0)}S_{T_0}\left\{[1 - N(d_1)] + e^{-r(T-T_0)}N(d_2)\right\}|\mathcal{F}_t] \\
=\ & e^{-q(T-t)}S_t\left[1 - N(d_1) + e^{-r(T-T_0)}N(d_2)\right],
\end{aligned}$$

where

$$d_1 = \frac{r - q + \frac{\sigma^2}{2}}{\sigma}\sqrt{T - T_0}, \quad d_2 = d_1 - \sigma\sqrt{T - T_0}.$$

7. Recall
$$d\left(\frac{X_t}{S_t}\right) = \mu\ dt + \sigma_R dZ^Q_{R,t},$$
where $\mu = -\rho\sigma_X\sigma_S + \sigma_S^2$, $\sigma_R^2 = \sigma_X^2 - 2\rho\sigma_X\sigma_S + \sigma_S^2$ and
$$\sigma_R Z^Q_{R,t} = \sigma_X dZ^Q_{X,t} - \sigma_S dZ^Q_{S,t}.$$
Putting all these relations together, we obtain
$$\begin{aligned} d\left(\frac{X_t}{S_t}\right) &= (-\rho\sigma_X\sigma_S + \sigma_S^2)dt + \sigma_X dZ^Q_{X,t} - \sigma_S dZ^Q_{S,t} \\ &= \sigma_X(dZ^Q_{X,t} - \rho\sigma_S dt) - \sigma_S(dZ^Q_{S,t} - \sigma_S dt). \end{aligned}$$
Since $Z^Q_{X,t} - \rho\sigma_S t$ and $Z^Q_{S,t} - \sigma_S t$ are Q^*-Brownian, so the difference $\sigma_X(Z^Q_{X,t} - \rho\sigma_S t) - \sigma_S(Z^Q_{S,t} - \sigma_S t)$ is also Q^*-Brownian. Hence, X_t/S_t is a martingale under Q^* since the dynamics of $d\left(\frac{X_t}{S_t}\right)$ has zero drift.

8. Recall $V(S_1, S_2, \tau) = S_2 W(S_1, S_2, \tau)$ so that
$$\begin{aligned} &\frac{\partial V}{\partial \tau} = S_2\frac{\partial W}{\partial \tau}, \quad \frac{\partial V}{\partial S_1} = S_2\frac{\partial W}{\partial S_1}, \quad \frac{\partial V}{\partial S_2} = W + S_2\frac{\partial W}{\partial S_2}, \\ &\frac{\partial^2 V}{\partial S_1^2} = S_2\frac{\partial^2 W}{\partial S_1^2}, \quad \frac{\partial^2 V}{\partial S_1 \partial S_2} = \frac{\partial W}{\partial S_1} + S_2\frac{\partial^2 W}{\partial S_1\partial S_2}, \quad \frac{\partial^2 V}{\partial S_2^2} = 2\frac{\partial W}{\partial S_2} + S_2\frac{\partial^2 W}{\partial S_2^2}. \end{aligned}$$
The governing equation for $W = W(S_1, S_2, \tau)$ is given by
$$\begin{aligned} \frac{\partial W}{\partial \tau} &= \frac{\sigma_1^2}{2}S_1^2\frac{\partial^2 W}{\partial S_1^2} + \rho\sigma_1\sigma_2\left(S_1\frac{\partial W}{\partial S_1} + S_1S_2\frac{\partial^2 W}{\partial S_1\partial S_2}\right) + \frac{\sigma^2}{2}\left(2S_2\frac{\partial W}{\partial S_2} + S_2^2\frac{\partial^2 W}{\partial S_2^2}\right) \\ &\quad + (r - q_1)S_1\frac{\partial W}{\partial S_1} + (r - q_2)\left(W + S_2\frac{\partial W}{\partial S_2}\right) - rW. \end{aligned}$$
Next, we let $y_1 = \ln S_1$, $y_2 = \ln S_2$ and $y = \ln x = \ln S_1 - \ln S_2 = y_1 - y_2$.
Note that
$$\begin{aligned} &S_1\frac{\partial W}{\partial S_1} = \frac{\partial W}{\partial y_1}, \quad S_1^2\frac{\partial^2 W}{\partial S_1^2} + S_1\frac{\partial W}{\partial S_1} = \frac{\partial^2 V}{\partial y_1^2}, \\ &S_2\frac{\partial W}{\partial S_2} = \frac{\partial W}{\partial y_2}, \quad S_2^2\frac{\partial^2 W}{\partial S_2^2} + S_2\frac{\partial W}{\partial S_2} = \frac{\partial^2 W}{\partial y_2^2}, \quad S_1S_2\frac{\partial^2 W}{\partial S_1\partial S_2} = \frac{\partial^2 W}{\partial y_1\partial y_2}, \end{aligned}$$
so the governing equation for $W = W(y_1, y_2, \tau)$ can be expressed as
$$\begin{aligned} \frac{\partial W}{\partial \tau} &= \frac{\sigma_1^2}{2}\frac{\partial^2 W}{\partial y_1^2} + \rho\sigma_1\sigma_2\frac{\partial^2 W}{\partial y_1 y_2} + \frac{\sigma_2^2}{2}\frac{\partial^2 W}{\partial y_2^2} \\ &\quad + \left(r - q_1 - \frac{\sigma_1^2}{2} + \rho\sigma_1\sigma_2\right)\frac{\partial W}{\partial y_1} + \left(r - q_2 + \frac{\sigma_1^2}{2}\right)\frac{\partial W}{\partial y_2} - q_2 W. \end{aligned}$$
We define $W = W(y, \tau)$, where $y = y_1 - y_2$ and observe
$$\frac{\partial W}{\partial y_1} = \frac{\partial W}{\partial y} \quad \text{and} \quad \frac{\partial W}{\partial y_2} = -\frac{\partial W}{\partial y},$$

we obtain the following equation for $W = W(y, \tau)$:

$$\frac{\partial W}{\partial \tau} = \frac{\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2}{2}\frac{\partial^2 W}{\partial y^2} + \left(q_2 - q_1 - \frac{\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2}{2}\right)\frac{\partial W}{\partial y} - q_2 W.$$

In terms of $W = W(x, \tau)$, where $x = \ln y$, we have

$$\frac{\partial W}{\partial \tau} = \frac{\sigma^2}{2}x^2\frac{\partial^2 W}{\partial x^2} + (q_2 - q_1)x\frac{\partial W}{\partial x} - q_2 W,$$

where $\sigma^2 = \sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2$. The terminal payoff is $\max(x-1, 0)$. Corresponding to the usual call price formula, we set $X \equiv 1$, $r \equiv q_2$ and $q \equiv q_1$ and obtain the price formula presented in the question.

9. The digital quanto option pays one US dollar and it is in-the-money if one Singaporean dollar is more than α Hong Kong dollars, or equivalently, $F_{S\backslash H} > \alpha$. The digital quanto option value in Hong Kong is given by

$$e^{-r_U\tau}F_{H\backslash U}E^t_{Q^U}\left[\mathbf{1}_{\{F_{S\backslash H} > \alpha\}}\right] = e^{-r_U\tau}F_{H\backslash U}N(d),$$

where

$$d = \frac{\ln\frac{F_{S\backslash H}}{\alpha} + \left(\delta^U_{F_{S\backslash H}} - \frac{\sigma^2_{F_{S\backslash H}}}{2}\right)\tau}{\sigma_{F_{S\backslash H}}\sqrt{\tau}},$$

and

$$\delta^U_{F_{S\backslash H}} = \delta^S_{F_{S\backslash H}} - \rho\sigma_{F_{S\backslash H}}\sigma_{F_{U\backslash S}}, \quad \text{where } \delta^S_{F_{S\backslash H}} = r_{\text{SGD}} - r_{\text{HKD}}.$$

10. We write $P(t, T)$ as the time-t value of the unit par discount bond maturing at time T. Consider the function defined by

$$g(S_T, T) = P(t, T)(S_T - K)^+,$$

and assume that S_t follows the Ito process under a risk neutral measure Q

$$\frac{dS_t}{S_t} = (r_t - q_t)dt + \sigma_t(S_t, t)dW_t.$$

By Ito's lemma, the differential of g is given by

$$dg = \left[\frac{\partial g}{\partial T} + (r_T - q_T)S_T\frac{\partial g}{\partial S_T} + \frac{\sigma_T^2}{2}S_T^2\frac{\partial^2 g}{\partial S_T^2}\right]d\tau + \sigma_T S_T\frac{\partial g}{\partial S_T}dW_T.$$

Recall the following identities:

$$\begin{aligned}
&\frac{\partial}{\partial S}(S-K)^+ = \mathbf{1}_{\{S>K\}},\ \frac{\partial}{\partial S}\mathbf{1}_{\{S>K\}} = \delta(S-K),\\
&\frac{\partial}{\partial K}(S-K)^+ = -\mathbf{1}_{\{S>K\}},\ \frac{\partial}{\partial K}\mathbf{1}_{\{S>K\}} = -\delta(S-K),\\
&\frac{\partial c}{\partial K} = -P(t,T)E[\mathbf{1}_{\{S>K\}}], \frac{\partial^2 c}{\partial K^2} = P(t,T)E[\delta(S-K)].
\end{aligned}$$

We then have

$$\begin{aligned}
dg &= P(t,T)[-r_T(S_T-K)^+ + (r_T - q_T)S_T\mathbf{1}_{\{S_T>K\}} + \frac{\sigma_T^2}{2}S_T^2\delta(S_T-K)]dT\\
&\quad + P(t,T)\sigma_T S_T\mathbf{1}_{\{S_T>K\}}dW_T.
\end{aligned}$$

By substituting all the necessary relations and observe $E[dW_T]=0$, we obtain

$$\begin{aligned} dc &= E[dg] \\ &= P(t,T)E\left[r_T K\mathbf{1}_{\{S_T>K\}} - q_T S_T \mathbf{1}_{\{S_T>K\}} + \frac{\sigma_T^2}{2}S_T^2\delta(S_T-K)\right]dT. \end{aligned}$$

Furthermore, we observe

$$P(t,T)E\left[S_T\mathbf{1}_{\{S_T>K\}}\right] = c + KP(t,T)E\left[\mathbf{1}_{\{S_T>K\}}\right]$$

so that

$$\begin{aligned} \frac{\partial c}{\partial T} &= KP(t,T)r_T E[\mathbf{1}_{\{S_T>K\}}] - q_T\left\{c + KP(t,T)E\left[\mathbf{1}_{\{S_T>K\}}\right]\right\} \\ &\quad +P(t,T)E\left[\frac{\sigma_T^2}{2}S_T^2\delta(S_T-K)\right] \\ &= -K(r_T-q_T)\frac{\partial c}{\partial K} - q_T c + P(t,T)E\left[\frac{\sigma_T^2}{2}S_T^2\delta(S_T-K)\right]. \end{aligned}$$

The last term can be rewritten as

$$\begin{aligned} & P(t,T)E\left[\frac{\sigma_T^2}{2}S_T^2\,\middle|\, S_T=K\right]E[\delta(S_T-K)] \\ = \;& P(t,T)E\left[\frac{\sigma_T^2}{2}\,\middle|\, S_T=K\right]K^2\frac{\partial^2 c}{\partial K^2}. \end{aligned}$$

Lastly, we obtain

$$\frac{\partial c}{\partial T} = -K(r_T-q_T)\frac{\partial c}{\partial K} - K^2E\left[\frac{\sigma_T^2}{2}\,\middle|\, S_T=K\right]\frac{\partial^2 c}{\partial K^2} - q_T c.$$

11. The Black-Scholes call price can be written as

$$c_{BS}(S_0,K,\Sigma(K,T),T) = F_T[N(d_1) - e^y N(d_2)],$$

where

$$\begin{aligned} & y = \ln\frac{K}{F_T},\;\; w = \Sigma(K,T)^2T, \\ & d_1 = \frac{\ln\frac{S_0}{K} + \int_0^T r_t - q_t\,dt + \frac{w}{2}}{\sqrt{w}} = \frac{-y}{\sqrt{w}} + \frac{\sqrt{w}}{2},\;\; d_2 = d_1 - \sqrt{w}. \end{aligned}$$

Note that

$$n(d_1) = \frac{1}{\sqrt{2\pi}}e^{-\frac{(d_2+\sqrt{w})^2}{2}} = n(d_2)e^{-d_2\sqrt{w}-\frac{w}{2}} = n(d_2)e^y,$$

so that

$$\frac{\partial c_{BS}}{\partial w} = \frac{F_T}{2}e^y n(d_2)/\sqrt{w}.$$

The other derivatives of c_{BS} with respect to w and y are found to be

$$
\begin{aligned}
\frac{\partial^2 c_{BS}}{\partial w^2} &= \frac{F_T}{2}\left[e^y n(d_2)/\sqrt{w}\right]\left(-d_2\frac{\partial d_2}{\partial w} - \frac{1}{2\sqrt{w}}\right) \\
&= \frac{\partial c_{BS}}{\partial w}\left[\left(\frac{y}{\sqrt{w}} + \frac{\sqrt{w}}{2}\right)\left(\frac{y}{2w\sqrt{w}} - \frac{1}{4\sqrt{w}}\right) - \frac{1}{2\sqrt{w}}\right] \\
&= \frac{\partial c_{BS}}{\partial w}\left(-\frac{1}{8} - \frac{1}{2w} + \frac{y^2}{2w^2}\right), \\
\frac{\partial^2 c_{BS}}{\partial w \partial y} &= \frac{F_T}{2}\frac{1}{\sqrt{w}}\frac{\partial}{\partial y}\left[e^y n(d_2)\right] \\
&= \frac{F_T}{2}\frac{1}{\sqrt{w}}\left[e^y n(d_2) - e^y n(d_2) d_2 \frac{\partial d_2}{\partial y}\right] \\
&= \frac{\partial c_{BS}}{\partial w}\left(\frac{1}{2} - \frac{y}{w}\right), \\
\frac{\partial c_{BS}}{\partial y} &= -F_T e^y N(d_2) \\
\frac{\partial^2 c_{BS}}{\partial y^2} &= \frac{\partial c_{BS}}{\partial y} + 2\frac{\partial c_{BS}}{\partial w}.
\end{aligned}
$$

If we write $c(S_0, K, T) = c_{BS}(S_0, F_T e^y, w(0), T)$, we obtain

$$
\frac{\partial c}{\partial y} = a(w, y) + b(w, y)\ell(y)
$$

where $a(w, y) = \dfrac{\partial c_{BS}}{\partial y}$, $b(w, y) = \dfrac{\partial c_{BS}}{\partial w}$, $\ell(y) = \dfrac{\partial w}{\partial y}$. The other derivatives of c are found to be

$$
\begin{aligned}
\frac{\partial^2 c}{\partial y^2} &= \frac{\partial a}{\partial y} + \frac{\partial a}{\partial w}\frac{\partial w}{\partial y} + b(w, y)\frac{\partial \ell}{\partial y} + \left(\frac{\partial b}{\partial y} + \frac{\partial b}{\partial w}\frac{\partial w}{\partial y}\right)\ell(y) \\
&= \frac{\partial^2 c_{BS}}{\partial y^2} + 2\frac{\partial^2 c_{BS}}{\partial y \partial w}\frac{\partial w}{\partial y} + \frac{\partial c_{BS}}{\partial w}\frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 c_{BS}}{\partial w^2}\left(\frac{\partial w}{\partial y}\right)^2, \\
\frac{\partial c}{\partial T} &= \frac{\partial c_{BS}}{\partial T} + \frac{\partial c_{BS}}{\partial w}\frac{\partial w}{\partial T} = \mu_T c_{BS} + \frac{\partial c_{BS}}{\partial w}\frac{\partial w}{\partial T}.
\end{aligned}
$$

Recall the Dupire equation

$$
\frac{\partial c}{\partial T} = \frac{v_L}{2}\left(\frac{\partial^2 c}{\partial y^2} - \frac{\partial c}{\partial y}\right) + \mu_T c.
$$

Substituting all the above relations, we obtain

$$
\begin{aligned}
\frac{\partial c_{BS}}{\partial w}\frac{\partial w}{\partial T} &= \frac{v_L}{2}\left[\frac{\partial^2 c_{BS}}{\partial y^2} + 2\frac{\partial^2 c_{BS}}{\partial y \partial w}\frac{\partial w}{\partial y} + \frac{\partial c_{BS}}{\partial w}\frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 c_{BS}}{\partial w^2}\left(\frac{\partial w}{\partial y}\right)^2 - \frac{\partial c_{BS}}{\partial y} + \frac{\partial c_{BS}}{\partial w}\frac{\partial w}{\partial y}\right] \\
&= \frac{v_L}{2}\frac{\partial c_{BS}}{\partial w}\left[2 + 2\left(\frac{1}{2} - \frac{y}{w}\right)\frac{\partial w}{\partial y} + \left(-\frac{1}{8} - \frac{1}{2w} + \frac{y^2}{2w}\right)\left(\frac{\partial w}{\partial y}\right)^2 + \frac{\partial^2 w}{\partial y^2} - \frac{\partial w}{\partial y}\right].
\end{aligned}
$$

Solving for v_L, we obtain the first identity in the question. For the second identity, we use an alternative derivation approach to arrive at

$$
v_L = \frac{\frac{\partial w}{\partial T} + \mu_T K \frac{\partial w}{\partial K}}{\frac{K^2}{2}\left[\frac{2}{K^2} + \frac{2}{K}\left(\frac{1}{2} - \frac{y}{w}\right)\frac{\partial w}{\partial K} + \left(-\frac{1}{8} - \frac{1}{2w} + \frac{y^2}{2w}\right)\left(\frac{\partial w}{\partial K}\right)^2 + \frac{\partial^2 w}{\partial K^2}\right]}
$$

and observe the following relations:

$$\frac{\partial w}{\partial T} = 2\Sigma T \frac{\partial \Sigma}{\partial T} + \Sigma^2, \quad \frac{\partial w}{\partial K} = 2\Sigma T \frac{\partial \Sigma}{\partial K} \text{ and } \quad \frac{\partial^2 w}{\partial K^2} = 2T \left[\left(\frac{\partial \Sigma}{\partial K} \right)^2 + \Sigma \frac{\partial^2 \Sigma}{\partial K^2} \right]$$

so that the numerator and denominator can be expressed as

$$\Sigma^2 + 2\Sigma T \left(\frac{\partial \Sigma}{\partial T} + \mu_T K \frac{\partial \Sigma}{\partial K} \right)$$

and

$$1 + 2K\Sigma T \left(\frac{1}{2} - \frac{y}{w} \right) \frac{\partial \Sigma}{\partial K} + 2K^2 \Sigma^2 T^2 \left(-\frac{1}{8} - \frac{1}{2w} + \frac{y^2}{2w} \right) \left(\frac{\partial \Sigma}{\partial K} \right)^2 + K^2 T \left[\left(\frac{\partial \Sigma}{\partial K} \right)^2 + \Sigma \frac{\partial^2 \Sigma}{\partial K^2} \right],$$

respectively. After some simplification, we obtain the second identity.