MAFS 5030 — Quantitative Modeling of Derivative Securities

Topic 1 – Introduction to Derivative Instruments

1.1 Basic instruments: bonds, forward contracts and futures

1.2 Exotic swap products

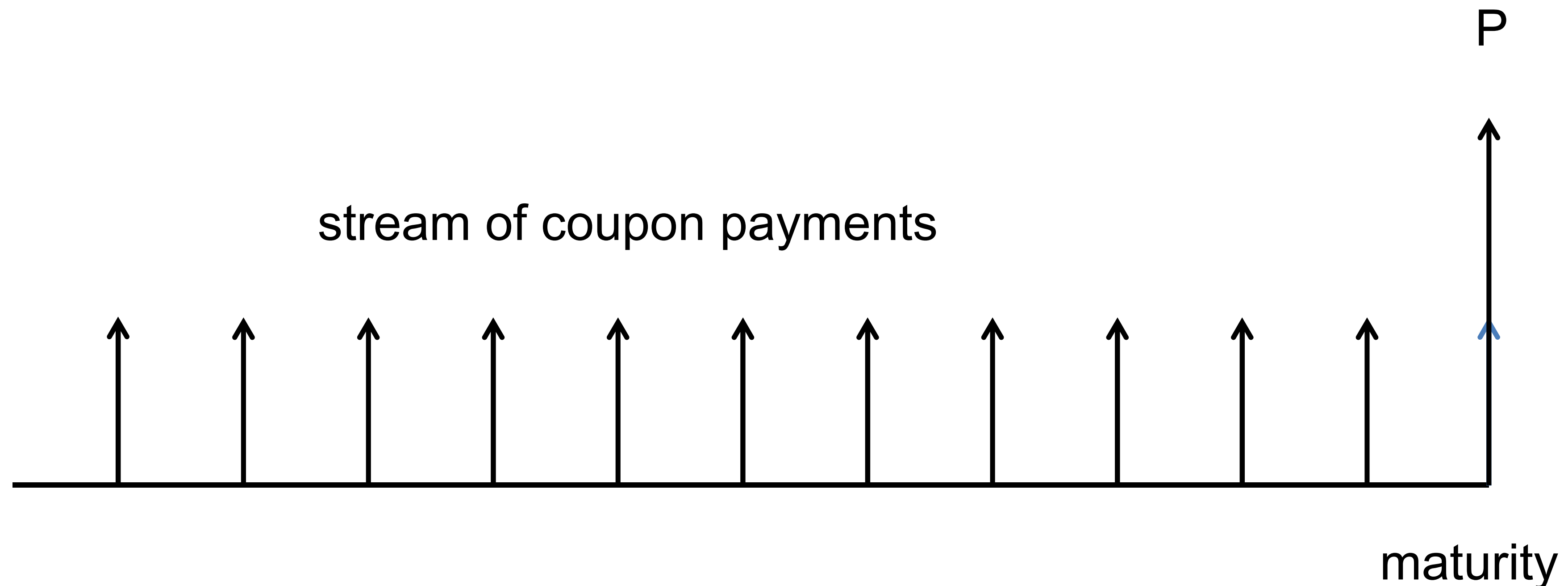
1.3 Options: rational boundaries of option values

1.4 American options: Optimal early exercise policies

1.1 Basic derivative instruments: bonds, forward contracts and futures

A bond is a *debt instrument* requiring the issuer to repay to the lender/investor the amount borrowed (par or face value) plus interests over a specified period of time.

Specify (i) the maturity date when the principal is repaid;
(ii) the coupon payments over the life of the bond



- The *coupon rate* offered by the bond issuer represents the *cost of raising capital*. It depends on the prevailing risk free interest rate and the creditworthiness of the bond issuer. It is also affected by the values of the embedded options in the bond, like the conversion right in a convertible bond.

- Assume that the bond issuer does not *default* or *redeem* the bond prior to maturity date, an investor holding this bond until maturity is assured of a *known* cash flow stream. This explains why bond products are also called the *Fixed Income Products/Derivatives*.

Pricing of a bond

Based on the current information of the term structure of interest rates (yield curve) and the embedded option provisions, find the fair price that the bond investor should pay at the current time so that the deal is fair to both counterparties. Also, potential default losses (probability of default, exposure and recovery value upon default) should be taken into account in computing the fair bond price.

Features in bond indenture

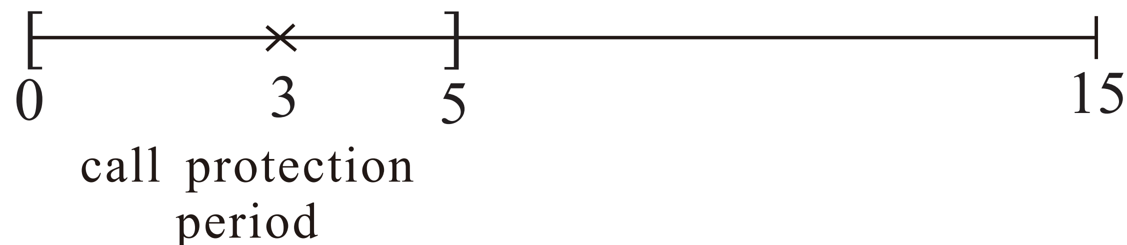
1. *Floating rate bond*

   The coupon rates are reset periodically according to some predetermined financial benchmark, like LIBOR + spread, where LIBOR is the LONDON INTER-BANK OFFERED RATE (phased out by June 2023). The new benchmark rate is SOFR (Secured Overnight Funding Rate).

2. Amortization feature – principal repaid over the life of the bond.

3. Callable feature (callable bonds)

   The issuer has the *right to buy back* the bond at a specified price. Usually this call price falls with time, and often there is an initial call protection period wherein the bond cannot be called.



4. Put provision – grants the bondholder the right to sell back to the issuer at par value on designated dates.

5. Convertible bond – gives the bondholder the right to *exchange the bond* for a *specified number of shares* of the issuer's firm.

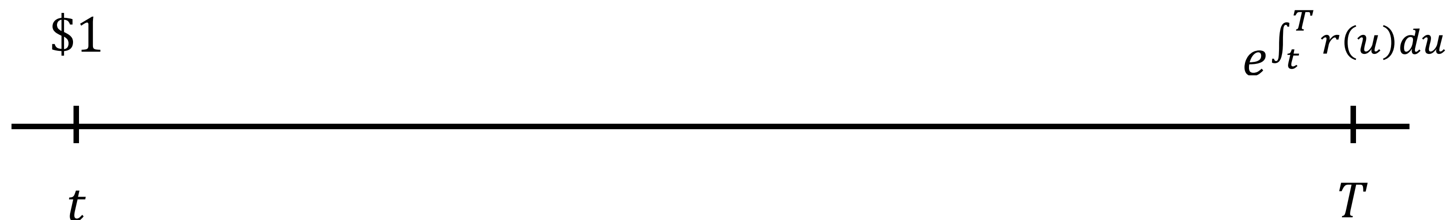
   - Bondholders can take advantage of the future growth of the issuer's company.
   - Issuer can raise capital at a lower cost.

6. Exchangeable bond – allows the bondholder to exchange the bond par for a specified number of common stocks of another corporation.

Short rate

Let $r(t)$ denote the short rate, which is in general stochastic. This is the interest rate that is applied over the next infinitesimal Δt time interval $(t, t + \Delta t]$. The short rate is a mathematical construction, not a market reality.

Money market account: $M(t)$



An investor puts $1 at time t and let it earn interest at the rate $r(t)$ *continuously* over the period (t, T). The governing differential equation of $M(t)$ is given by

$$dM(t) = r(t)M(t)\,dt.$$

Here, $dM(t)$ represents the interest accrued over $(t, t + dt)$.

$$\int_t^T \frac{dM(u)}{M(u)} = \int_t^T r(u)\,du$$

so that

$$M(T) = M(t)e^{\int_t^T r(u)\,du}.$$

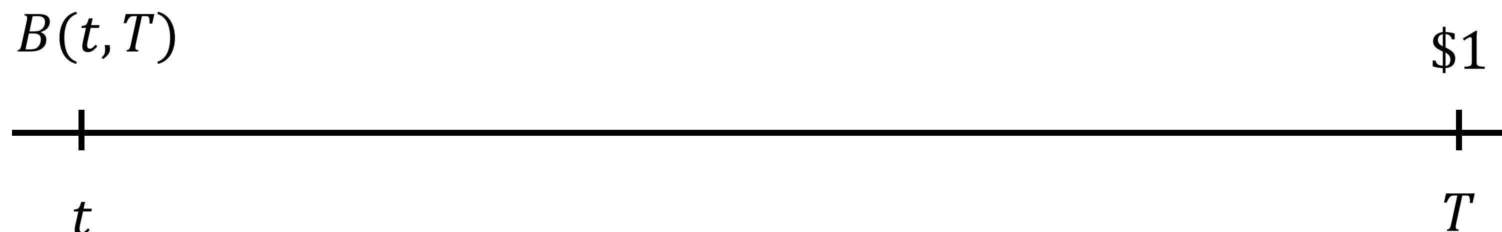
Here, $e^{\int_t^T r(u)\,du}$ is seen to be the growth factor of the money market account over $[t,T]$ based on continuous compounding. If r is constant, then

$$\text{growth factor } = e^{r\tau}, \quad \tau = T - t.$$

If $r(t)$ is stochastic, then $M(T)$ is also stochastic.

The reciprocal of the growth factor $e^{-\int_t^T r(u)\,du}$ is called the discount factor.

Discount bond price: $B(t,T)$



$\tau = T - t =$ time to bond's maturity

The price that an investor on the zero-coupon (discount) bond with unit par is willing to pay at time t if the bond promises to pay him back $1 at a later time (maturity date) T.

This fair value is called the discount bond price $B(t,T)$, which is given by the expectation of the discount factor based on the current information: $E_t\left[e^{-\int_t^T r(u)\,du}\right]$. If r is constant, then $B(t,T) = e^{-r\tau}, \tau = T - t$.

Forward contract 遠期合約

The buyer of the forward contract agrees to pay the delivery price K dollars at future time T to purchase a commodity whose value at time T is S_T. The pricing question is how to set K?

How about

$$E[\exp(-rT)(S_T - K)] = 0$$

so that $K = E[S_T]$?

This is the *expectation pricing* approach, which cannot enforce a price. When the expectation calculation $E[S_T]$ is performed, the distribution of the asset price process comes into play.

Objective of the buyer:

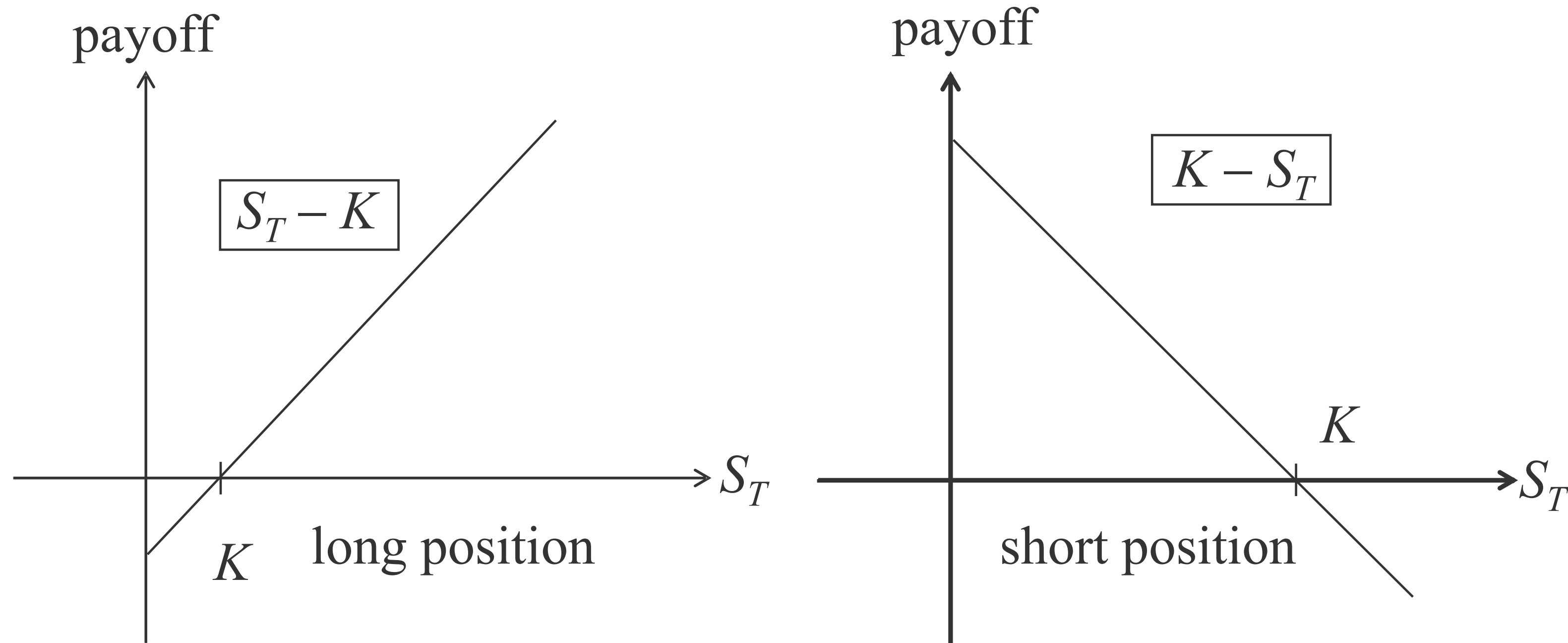
To hedge against the price fluctuation of the underlying commodity.

- Intension of a purchase to be decided earlier, actual transaction to be done later.
- The forward contract needs to specify the delivery price, amount, quality, delivery date, means of delivery, etc.

Potential default of either party (counterparty risk): writer or buyer.

Forward contract is the simplest form of a financial derivative. The exposure to the writer (short position) of the forward can be hedged by long holding one unit of the underlying asset. In this case, the hedge ratio is one.

Terminal payoff from a forward contract



K = delivery price, S_T = asset price at maturity

This is a *zero-sum game* between the writer (short position) and buyer (long position). For cash settlement of the terminal payoff, one can sell the underlying asset immediately to realize the cash value S_T upon maturity of the contract.

Replication enforces forward price

Is the forward price related to the expected price of the commodity on the delivery date? Provided that the underlying asset can be held for hedging by the writer, then

$$\begin{aligned} & \text{forward price} \\ = \; & \text{spot price} + \underbrace{\text{cost of fund} + \text{storage cost}}_{\text{cost of carry}} \end{aligned}$$

- Upfront cost (through borrowing) is required to acquire the underlying commodity at the spot price. Cost of fund is the interest costs accrued over the period of the forward contract.

- Cost of carry is the total cost incurred to acquire and hold the underlying asset, say, including the cost of fund and storage cost.

- Dividends paid to the holder of the asset are treated as negative contribution to the cost of carry.

Numerical example on arbitrage

- spot price of oil is US$19
- quoted 1-year forward price of oil is US$25
- 1-year US dollar interest rate is 5% pa
- storage cost of oil is 2% per annum, paid at maturity

Any arbitrage opportunity? *Yes*

Sell the forward and expect to receive US$25 one year later. Borrow $19 now to acquire oil, pay back $\$19(1+0.05) = \$19.95$ a year later. Also, one needs to spend $\$0.38 = \$19 \times 2\%$ as the storage cost.

$$
\begin{aligned}
& \text{total cost of replication (dollar value at maturity)} \\
= \; & \text{spot price} + \text{cost of fund} + \text{storage cost} \\
= \; & \$20.33 < \$25 \text{ to be received.}
\end{aligned}
$$

Close out all positions by delivering the oil to honor the forward. At maturity of the forward contract, guaranteed riskless profit $= \$4.67$.

How to perform replication when the underlying is a stock index?

- Long a basket of stocks whose composition is the same as the weights that calculate the stock index. This requires upfront cash on the seller of the index futures.

- Unlike the delivery of the physical asset in a forward contract on commodity, upon maturity of the index futures (traded in an exchange), the cash settlement of the index futures is realized by the cash received through liquidation of the basket of stocks.

However, since the settlement futures value is the average of the stock index taken at 5-minute time intervals on the settlement date of the index futures, liquidation has to be executed at every 5-minute time interval accordingly. There may be a potential slippage since the actual implementation of liquidation may take 30 seconds or more. Also, there are uncertainties with regard to the dividends received from the basket of stocks.

Value and price of a forward contract

Let $f(S,\tau)$ = value of forward, $F(S,\tau)$ = forward price,

$$\begin{aligned} \tau &= \text{time to expiration,} \\ S &= \text{spot price of the underlying asset.} \end{aligned}$$

Further, we let

$$B(\tau) = \text{value of an unit par discount bond with time to maturity } \tau$$

- If the interest rate r is constant and interests are compounded continuously, then $B(\tau) = e^{-r\tau}$.
- Assuming no dividend to be paid by the underlying asset and no storage cost.

We construct a "static" (fix the hedge ratio to be one throughout the life of the forward contract) replication of the forward contract by a portfolio of the underlying asset and bond.

Portfolio A: long one forward and a discount bond with par value K

Portfolio B: one unit of the underlying asset

Both portfolios become one unit of asset at maturity. Let $\Pi_A(t)$ denote the value of Portfolio A at time t. Note that $\Pi_A(T) = \Pi_B(T)$. By the "law of one price",* we must have $\Pi_A(t) = \Pi_B(t)$. The forward value is given by

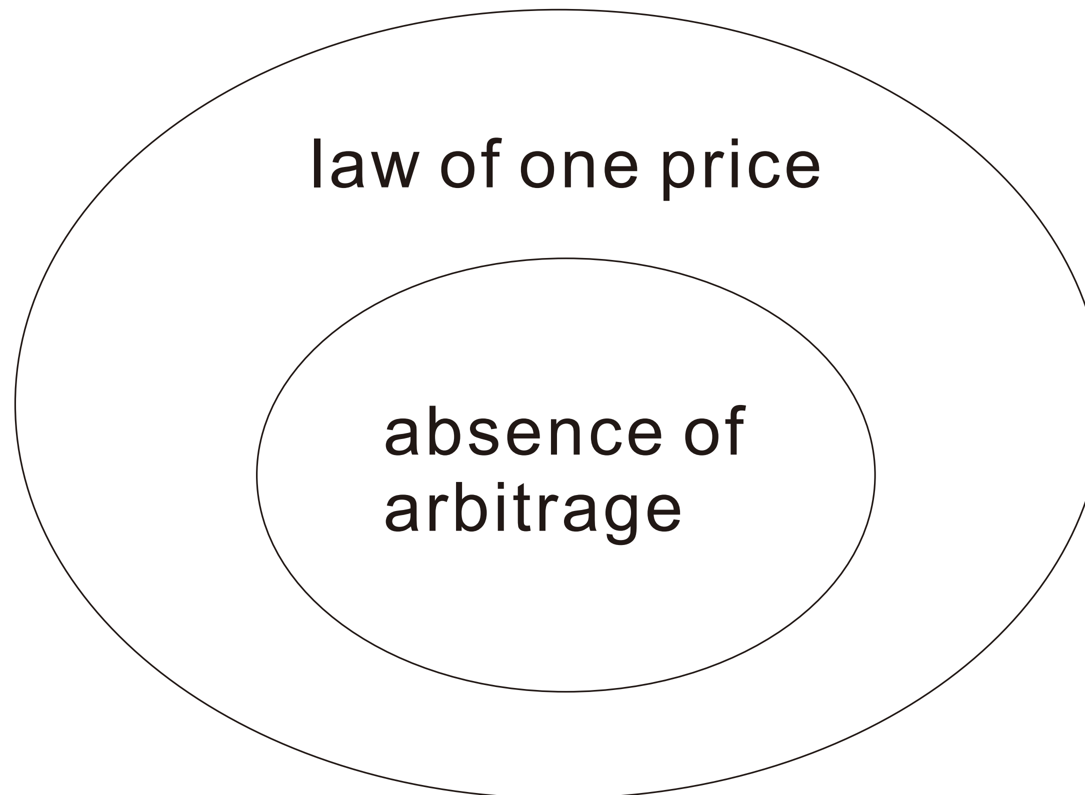
$$f = S - KB(\tau).$$

The forward price is defined to be the delivery price which makes $f = 0$, so $K = S/B(\tau)$. Hence, the forward price is given by

$$F(S, \tau) = S/B(\tau) = \text{ spot price } + \text{ cost of fund.}$$

*Suppose $\Pi_A(t) > \Pi_B(t)$, then an arbitrage can be taken by selling Portfolio A and buying Portfolio B. An upfront positive cash flow is resulted at time t but the portfolio values are offset at maturity T. Failure of law of one price implies the existence of an arbitrage opportunity.

Failure of the law of one price leads to existence of an arbitrage opportunity (an important financial economic concept to be discussed in Topic 2).



Can you find an example where law of one price is observed while there exists an arbitrage opportunity?

Forward price formula with discrete carrying costs

Suppose an asset has a holding cost of $c(k)$ per unit in period k, $k = 1, 2, \ldots, M$, and the asset can be sold short. Suppose the initial spot price is S. The theoretical forward price F is

$$F = \frac{S}{B(0,M)} + \sum_{k=1}^{M} \frac{c(k)}{E_0[d(k,M)]},$$

where the market expectation of the discount factor $d(k, M)$ at time zero is given by

$$E_0[d(k,M)] = B(0,M)/B(0,k).$$

Here, $B(0, k)$ denotes the market observable time-0 price of a discount bond that matures at time k.

The terms on the right hand side represent the future value at maturity date M of the total costs required for holding the underlying asset for hedging. As a fair deal, the buyer has to pay the delivery price F that is set equal to the sum of the future value at date M of all costs.

Discrete dividend paying asset

We may treat the carrying charges $c(k)$ as negative dividends $D(k)$, $k = 1, 2, \ldots, M$. Let D be present value of all dividends received from holding the asset during the life of the forward contract, where $D = \sum_{k=1}^{M} B(0,k)D(k)$, then

$$F = \frac{S}{B(0,M)} - \sum_{k=1}^{M} \frac{D(k)B(0,k)}{B(0,M)} = \frac{S-D}{B(0,M)}.$$

Remark

For a discount bond with unit par and maturity T, we may write the time-t price as $B_t(T)$ or $B_{t,T}$ or $B(\tau)$, where $\tau = T - t$. Here, $B(0,M)$ means $t = 0$ and $T = M$.

Alternative proof

We modify Portfolio B to contain one unit of the asset plus borrowing of D dollars. The loan of D dollars will be repaid by the dividends received by holding the asset. We then have

$$f + KB(\tau) = S - D$$

so that

$$f = S - [D + KB(\tau)].$$

Setting $f = 0$ to solve for K, we obtain $F = (S - D)/B(\tau)$.

The "net" asset value is reduced by the amount D due to the anticipation of the dividends. Unlike holding the asset, the holder of the forward will not receive the dividends. As a fair deal, he should pay a lower delivery price at forward's maturity.

Cost of carry

Additional costs to hold the commodities, like storage, insurance, deterioration, etc. These can be considered as negative dividends. Treating U as $-D$, we obtain

$$F = (S + U)e^{r\tau},$$

$U =$ present value of total cost incurred during the remaining life of the forward to hold the asset.

Suppose the costs are proportional and paid continuously, we have

$$F = Se^{(r+u)\tau},$$

where $u =$ cost per annum as a proportion of the spot price. Similar to interest amount, the holding cost paid over $(t, t + \mathrm{d}t)$ is $uS\ \mathrm{d}t$.

In general, $F = Se^{b\tau}$, where b is the cost of carry per annum. Let q denote the continuous dividend *yield* per annum paid by the asset. With both continuous holding cost and dividend yield, the cost of carry $b = r + u - q$.

Futures contracts 期貨合約

A futures contract is a legal agreement between a buyer (seller) and an established exchange or its clearing house in which the buyer (seller) agrees to take (make) delivery of a financial entity at a specified price at the end of a designated period of time. Usually the exchange specifies certain standardized features.

Mark to market the account

Pay or receive from the writer the change in the futures price through the margin account so that payment required on the maturity date is simply the spot price on that date.

Credit risk is limited to one-day performance period

Roles of the clearing house and margin account

- Minimize the **counterparty risk** through the margin account.
- Provide the **platform** for parties of a futures contract to unwind their position prior to the settlement date.

Margin requirements

Initial margin – paid at inception as a deposit for the contract.

Maintenance margin – minimum level before the investor is required to deposit additional margin.

Example (Margin)

Suppose that Mr. Chan takes a long position of one contract in corn ($5,000$ kilograms) for March delivery at a price of \$2.10 (per kilogram). And suppose the broker requires margin of \$800 with a maintenance margin of \$600.

- The next day the price of this contract drops to \$2.07. This represents a loss of $0.03 \times 5,000 = \$150$. The broker will take this amount from the margin account, leaving a balance of \$650. The following day the price drops again to \$2.05. This represents an additional loss of \$100, which is again deducted from the margin account. As this point the margin account is \$550, which is below the maintenance level.

- The broker calls Mr. Chan and tells him that he must deposit at least \$250 in his margin account, or his position will be closed out.

Dynamic strategy that replicates the daily margin settlement in a futures contract

Consider an asset with price $\tilde{S}_T$ at time T. An investor who pays an amount $G_{t,T}$ that equals the futures price of the asset and together performs the dynamic strategy of long holding futures on successive dates is equivalent to pay the time-t spot price of a security which has a payoff

$$\frac{\widetilde{S}_T}{B_{t,t+1}\widetilde{B}_{t+1,t+2}\cdots\widetilde{B}_{T-1,T}}$$

at time T. Note that quantities with "tilde" at top indicate stochastic variables. Note that the payoff is seen to be some units of the underlying asset, where the number of units is $\dfrac{1}{B_{t,t+1}\widetilde{B}_{t+1,t+2}\cdots\widetilde{B}_{T-1,T}}$ (this is the same as the value of the money market account starting at $1 and accumulating over the period $[t,T]$).

The dynamic strategy is presented as follows.

- We start with long holds $\frac{1}{B_{t,t+1}}$ futures contracts at time t. On day τ, the investor long holds $\frac{1}{B_{t,t+1} \cdots B_{\tau,\tau+1}}$ futures contracts. To reflect the daily settlement feature, the gain/loss from the futures position on day τ earns/pays the overnight rate $\frac{1}{B_{\tau,\tau+1}}$. Recall that $B_{\tau,\tau+1}$ is known by day τ.

- Also, invest the dollar amount of $G_{t,T}$ in a one-day risk free bond and roll the cash position over on each day at the one-day rate. This is like "rolling over" in a money market account on a daily basis.

- In other words, paying upfront cash $G_{t,T}$ plus executing the dynamic strategy replicates the payoff of $\frac{1}{B_{t,t+1}\widetilde{B}_{t+1,t+2} \cdots \widetilde{B}_{T-1,T}}$ units of the underlying asset. The investment of $G_{t,T}$ is equivalent to the price paid to acquire this security.

As an illustrative example, take $t = 0$ and $T = 3$.

1. Take $1/B_{0,1}$ long futures at $t = 0$;

   $1/B_{0,1}B_{1,2}$ long futures at $t = 1$;

   $1/B_{0,1}B_{1,2}B_{2,3}$ long futures at $t = 2$.

   Note that $B_{1,2}$ and $B_{2,3}$ are market observable bond prices at $t = 1$ and $t = 2$, respectively. The holder holds extra $\dfrac{1}{B_{0,1}B_{1,2}} - \dfrac{1}{B_{0,1}}$ units of futures when the calendar time advances from $t = 0$ to $t = 1$. However, it costs nothing to change the holding of units of futures on successive dates.

2. Invest the dollar amount of $G_{0,3}$ in one-day risk free bond and roll over the net cash position

Time	Profits from futures	Bond position	Net position
0	—	$G_{0,3}$	$G_{0,3}$
1	$\frac{1}{B_{0,1}}(G_{1,3} - G_{0,3})$	$\frac{G_{0,3}}{B_{0,1}}$	$\frac{G_{1,3}}{B_{0,1}}$
2	$\frac{1}{B_{0,1}B_{1,2}}(G_{2,3} - G_{1,3})$	$\frac{G_{1,3}}{B_{0,1}B_{1,2}}$	$\frac{G_{2,3}}{B_{0,1}B_{1,2}}$
3	$\frac{1}{B_{0,1}B_{1,2}B_{2,3}}(G_{3,3} - G_{2,3})$	$\frac{G_{2,3}}{B_{0,1}B_{1,2}B_{2,3}}$	$\frac{G_{3,3}}{B_{0,1}B_{1,2}B_{2,3}} = \frac{S_3}{B_{0,1}B_{1,2}B_{2,3}}$

The net profit for one unit of the index futures with price $G_{0,3}$ at $t = 0$ after one day when the new index futures price becomes $G_{1,3}$ at $t = 1$ is $G_{1,3} - G_{0,3}$. Note that $G_{3,3} = S_3$.

- The key point is the combination of cash $\dfrac{G_{3,3}}{B_{0,1}B_{1,2}B_{2,3}}$ and $\dfrac{1}{B_{0,1}B_{1,2}B_{2,3}}$ units of futures. Upon exercising at time 3, they are converted into $\dfrac{1}{B_{0,1}B_{1,2}B_{2,3}}$ units of asset, each unit valued at S_3.

Pricing issues

We consider the discrete-time model and assume the existence of a risk neutral pricing measure Q. We have demonstrated the construction of the dynamic replication strategy. The fair time-t price of $\dfrac{\widetilde{S}_T}{B_{t,t+1}\cdots\widetilde{B}_{T-1,T}}$ is the cost of setting up the replicating portfolio, which is $G_{t,T}$. On the other hand, based on the risk neutral valuation principle, the time-t price of a security is given by the Q-expectation of the discounted terminal payoff at T, where

$$\begin{aligned} G_{t,T} &= E_Q\left[B_{t,t+1}\widetilde{B}_{t+1,t+2}\cdots\widetilde{B}_{T-1,T}\frac{\widetilde{S}_T}{B_{t,t+1}\widetilde{B}_{t,t+1}\widetilde{B}_{t+1,t+2}\cdots\widetilde{B}_{T-1,T}}\right] \\ &= E_Q[\widetilde{S}_T]. \end{aligned}$$

The result remains valid for the continuous-time counterpart. Under the continuous-time model, the risk neutral valuation principle gives the time-t price of a contingent claim as follows:

$$V_t = E_Q\left[e^{-\int_t^T r(u)\ du}V_T\right].$$

Difference in futures price G_t and forward price F_t

Difference in payment schedules may lead to difference in futures and forward prices since different interest rates are applied on intermediate payments. When the interest rates are deterministic, we have $G_{t,T} = F_{t,T}$. This is a sufficient condition for equality of the two prices. The necessary and sufficient condition is that the discount process and the terminal value of the underlying asset are uncorrelated under the risk neutral measure Q.

- When physical holding of the underlying index (say, snow fall amount in a ski resort) for hedging is infeasible, then the buyer sets

$$\text{forward price } = E_P[S_T],$$

where P is the subjective probability measure of the buyer.

- When the physical holding of the asset is subject to daily settlement through the margin requirement (dynamic rebalancing)

$$\text{futures price } = E_Q[S_T],$$

where Q is the risk neutral measure that uses the money market account as the numeraire. As a remark, $\frac{1}{B_{t,t+1}\tilde{B}_{t+1,t+2}\cdots\tilde{B}_{T-1,T}}$ is seen as the money market account rolling daily over the period $[t,T]$. The asset $\frac{\tilde{S}_T}{B_{t,t+1}\tilde{B}_{t+1,t+2}\cdots\tilde{B}_{T-1,T}}$ when normalized by the money market account becomes $\tilde{S}_T$.

Recall that the forward price (without daily settlement) is $F_t = \dfrac{S_t}{B(t,T)}$. By the risk neutral valuation principle, we have

$$B(t,T) = E_Q^t \left[e^{-\int_t^T r(u)du} \right] \text{ and } S_t = E_Q^t \left[e^{-\int_t^T r(u)\,du} S_T \right].$$

The difference between the futures price and forward price is

$$\begin{aligned} G_t - F_t &= E_Q[S_T] - \frac{S_t}{B(t,T)} \\ &= \frac{E_Q[S_T] E_Q \left[e^{-\int_t^T r(u)\,du} \right] - E_Q \left[e^{-\int_t^T r(u)\,du} S_T \right]}{B(t,T)} \\ &= -\frac{\text{cov}_Q \left[e^{-\int_t^T r(u)\,du}, S_T \right]}{B(t,T)}. \end{aligned}$$

The spread between G_t and F_t becomes zero when the discount process and the terminal value of the underlying asset are uncorrelated under the risk neutral measure Q. In the special case where the interest rates are deterministic, we have equality of G_t and F_t.

Currency forward

The underlying is the exchange rate X, which is the domestic currency price of one unit of foreign currency.

$$\begin{aligned} r_d &= \text{constant domestic interest rate} \\ r_f &= \text{constant foreign interest rate} \end{aligned}$$

Portfolio A: Hold one currency forward with delivery price K and a domestic bond of par K maturing on the delivery date of forward.

Portfolio B: Hold a foreign bond of unit par maturing on the delivery date of forward.

Holding of the domestic and foreign bonds allow the bonds to earn the interest rate in the respective currency.

Remark

The underlying asset of a foreign currency forward is one unit of foreign currency (in the form of market account) that pays dividend yield r_f.

Let $\Pi_A(t)$ and $\Pi_A(T)$ denote the value of Portfolio A at time t and T, respectively. On the delivery date, the holder of the currency forward has to pay K domestic dollars to buy one unit of foreign currency. Hence, $\Pi_A(T) = \Pi_B(T)$, where T is the delivery date.

Using the law of one price, $\Pi_A(t) = \Pi_B(t)$ must be observed at the current time t, $t < T$.

Interest rate parity relation

Under the assumption of constant interest rates, we have

$$B_d(\tau) = e^{-r_d\tau}, B_f(\tau) = e^{-r_f\tau},$$

where $\tau = T - t$ is the time to expiry. Let f be the time-t value of the currency forward in domestic currency,

$$f + KB_d(\tau) = XB_f(\tau),$$

where $XB_f(\tau)$ is the value of the foreign bond in domestic currency.

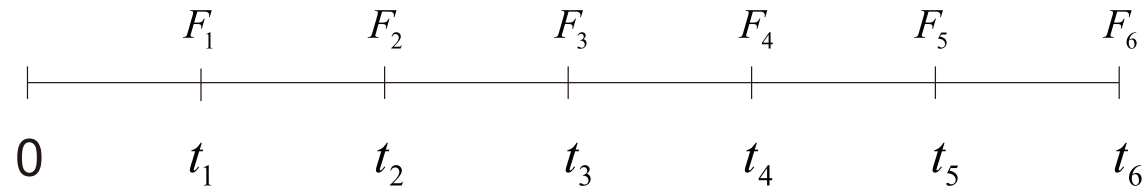
By setting $f = 0$ at initiation of the forward contract, the forward price of the currency forward is

$$K = \frac{XB_f(\tau)}{B_d(\tau)} = Xe^{(r_d - r_f)\tau}.$$

We may recognize r_d as the cost of fund and r_f as the dividend yield. This result is the well known *Interest Rate Parity Relation*.

Flexible notional currency forward

Consider a 6-month forward contract. The exchange rate over each one-month period is preset to assume some constant value.



The holder can exercise certain percentage of the notional at any preset time point during the life of the forward, but she has to exercise the whole notional by the maturity date of the currency forward. Without such flexibility, we observe $F_j = Xe^{(r_d - r_f)t_j}$, $j = 1, 2, \ldots, 6$, so that $F_j = F_1 e^{(r_d - r_f)(t_j - t_1)}$, $j = 2, 3, \ldots, 6$. Market conventions set the same relation under this flexible forward.

How to set F_1 so that it is fair to both parties? We expect $F_1 > Xe^{(r_d - r_f)t_1}$ since the buyer needs to pay more domestic currency to buy one unit of foreign currency. The extra amount represents the premium of this flexibility.

How to determine the optimal policy of the percentage of the notional to be exercised at each time instant? Since the optimal policy is independent of the notional, we argue that the holder chooses either no action or exercise the full notional (partial exercise is non-optimal). This is called the bang-bang strategy. If otherwise, for any optimal proportion of exercise, $\alpha < 1$, the remaining $1 - \alpha$ is also subject to an optimal exercise of α portion, and so forth. It ends up with the exercise of the full notional.

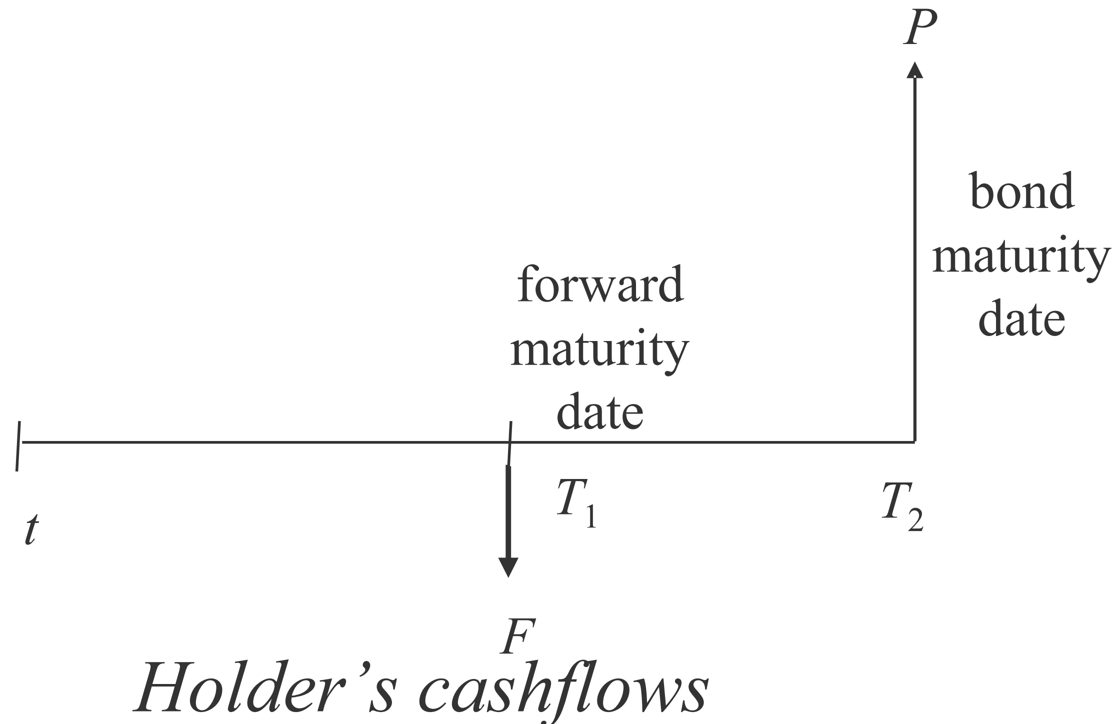
Remark

Suppose differential transaction costs are charged according to the notional amount, say 0.1% on the first \$10,000 and 0.08% on the next \$10,000,etc., then the property of independence of the notional fails. However, if the transaction cost is proportional to the notional, then the independence property remains.

As part of the pricing issue, besides F_1, one has to determine the threshold exchange rate above which it is optimal to exercise the whole notional.

Bond forward

The underlying asset is a zero-coupon bond of maturity T_2 with the settlement date T_1, where $t < T_1 < T_2$.



Holder's cashflows

The holder pays the delivery price F of the bond forward on the forward maturity date T_1 to receive a bond with par value P on the maturity date T_2, where $T_2 > T_1$.

Bond forward price in terms of traded bond prices

Let $B_t(T)$ denote the traded price of unit par discount bond at current time t with maturity date T.

$$\begin{aligned} & \text{Present value of the two net future cashflows} \\ = \; & -FB_t(T_1) + PB_t(T_2). \end{aligned}$$

To determine the forward price F, we set the above value zero and obtain

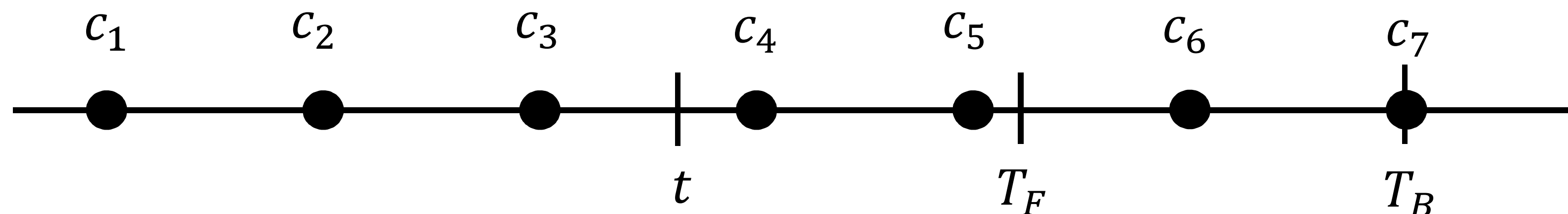
$$F = PB_t(T_2)/B_t(T_1).$$

Here, $PB_t(T_2)$ can be visualized as the spot price of the discount bond. The forward price is given in terms of the known market bond prices observed at time t with maturity dates T_1 and T_2.

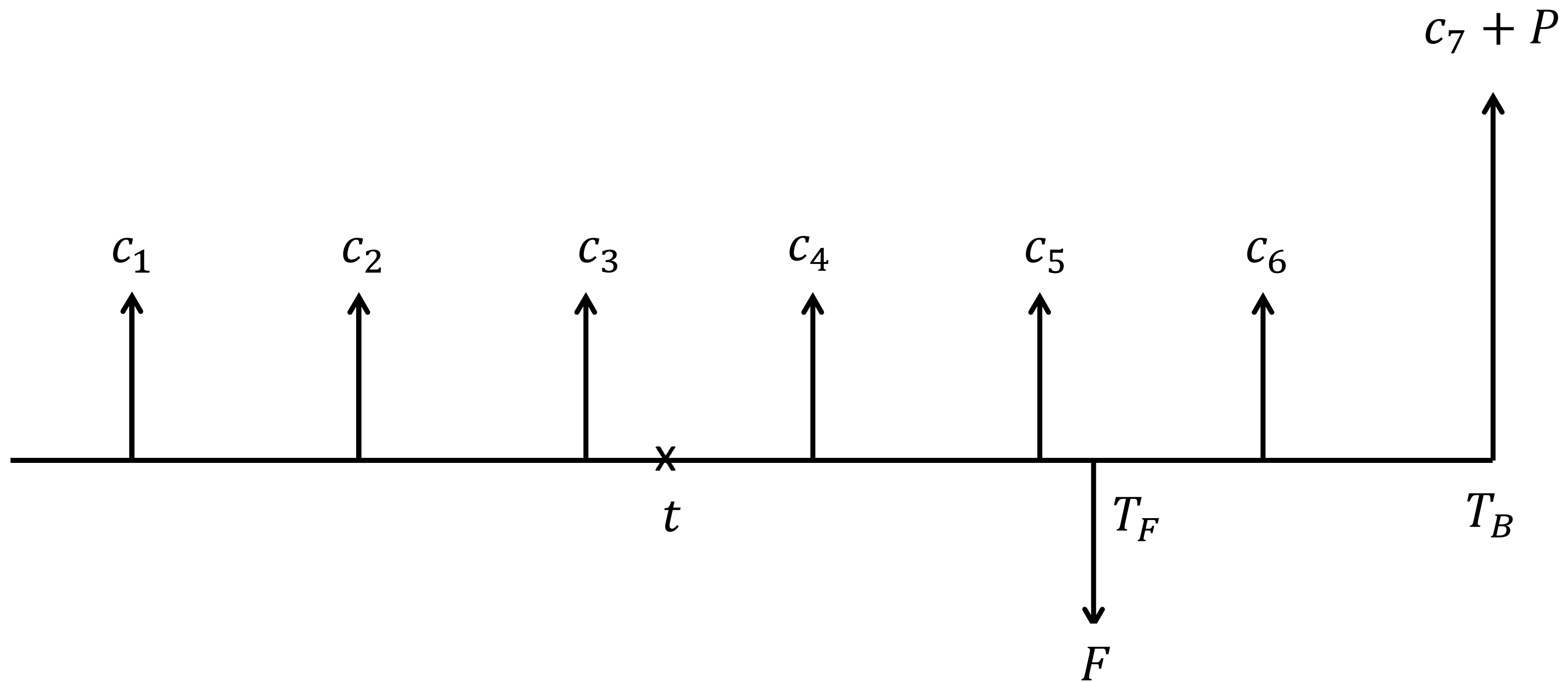
Forward on a coupon-paying bond

The underlying is a coupon-paying bond with maturity date T_B.

Note that the bond is a traded security whose value changes with respect to time.



Let T_F be the delivery date of the bond forward, where $T_F < T_B$. Let t_i be the coupon payment date of the bond on which deterministic coupon c_i is paid. Let t be the current time, where $t < T_F < T_B$. Some of the coupons have been paid at earlier times. Let F be the forward price, the amount paid by the forward contract holder at time T_F to buy the bond.



At T_B, the bondholder receives par plus the last coupon.

Based on the forward price formula: $F = \frac{S-D}{B(\tau)}$, we deduce that

$$F = \frac{\text{spot price of bond}}{B_t(T_F)} - \frac{c_4 B_t(t_4)}{B_t(T_F)} - \frac{c_5 B_t(t_5)}{B_t(T_F)}.$$

Let P be the par value of the bond. After receiving the bond at T_F, the bond forward holder is entitled to receive c_6, c_7 and P once he has received the underlying bond. By considering the cash flows after T_F, he pays F at T_F and receives c_6 at t_6, $c_7 + P$ at T_B.

$$\begin{aligned} & \text{Present value of cash flows at time } t \\ = \;\; & -F B_t(T_F) + c_6 B_t(t_6) + c_7 B_t(T_B) + P B_t(T_B). \end{aligned}$$

Hence, the bond forward price is given by

$$F = \frac{c_6 B_t(t_6) + c_7 B_t(T_B) + P B_t(T_B)}{B_t(T_F)}.$$

Remark

Equating the two expressions gives the spot price of the bond in terms of the cash flows.

Example — Bond forward

- A 10-year bond is currently selling for $920.

- Currently, hold a forward contract on this bond that has a delivery date in 1 year and a delivery price of $940.

- The bond pays coupons of $80 every 6 months, with one due 6 months from now and another just before maturity of the forward.

- The current interest rates for 6 months and 1 year (compounded semi-annually) are 7% and 8%, respectively (annual rates compounded every 6 months).

- The bond forward is in-progress, so its value is not zero. This is unlike the case at initiation where the value of the forward is zero. What is the current value of the forward?

Let $d(0,k)$ denote the discount factor over the $(0,k)$ semi-annual period. We have $d(0,2) = \frac{1}{(1.04)^2}$ and $d(0,1) = \frac{1}{1.035}$. Consider the future value of the cash flows associated with holding the bond one year later and payment of F_0 under the forward contract. The current forward price of the bond

$$\begin{aligned} F_0 &= \frac{\text{spot price}}{d(0,2)} - \frac{c(1)d(0,1)}{d(0,2)} - \frac{c(2)d(0,2)}{d(0,2)} \\ &= 920(1.04)^2 - \frac{80(1.04)^2}{1.035} - \frac{80(1.04)^2}{(1.04)^2} = 831.47. \end{aligned}$$

Note that the future value at time 2 of coupon amount $c(1)$ is given by

$$c(1) \ / \ \text{expected discount factor over } (1,2)$$

The expected discount factor over $(1,2)$ is $d(0,2)/d(0,1)$. The difference in the forward prices is discounted to the present value. The current value of the forward contract $= \frac{831.47-940}{(1.04)^2} = -100.34$.

Implied forward interest rate

The forward price of a forward on a discount bond should be related to the implied forward interest rate $R(t; T_1, T_2)$. The implied forward rate is the interest rate over the future time period $[T_1, T_2]$ as implied by time-t discount bond prices. The bond forward buyer pays F at T_1 and receives P at T_2 and she is expected to earn $R(t; T_1, T_2)$ over $[T_1, T_2]$, so

$$F[1 + R(t; T_1, T_2)(T_2 - T_1)] = P.$$

Together with

$$F = PB_t(T_2)/B_t(T_1),$$

we obtain

$$R(t; T_1, T_2) = \frac{1}{T_2 - T_1}\left[\frac{B_t(T_1)}{B_t(T_2)} - 1\right].$$

Forward rate agreement (FRA)

The FRA is an agreement between two counterparties to exchange floating and fixed interest payments on the future settlement date T_2.

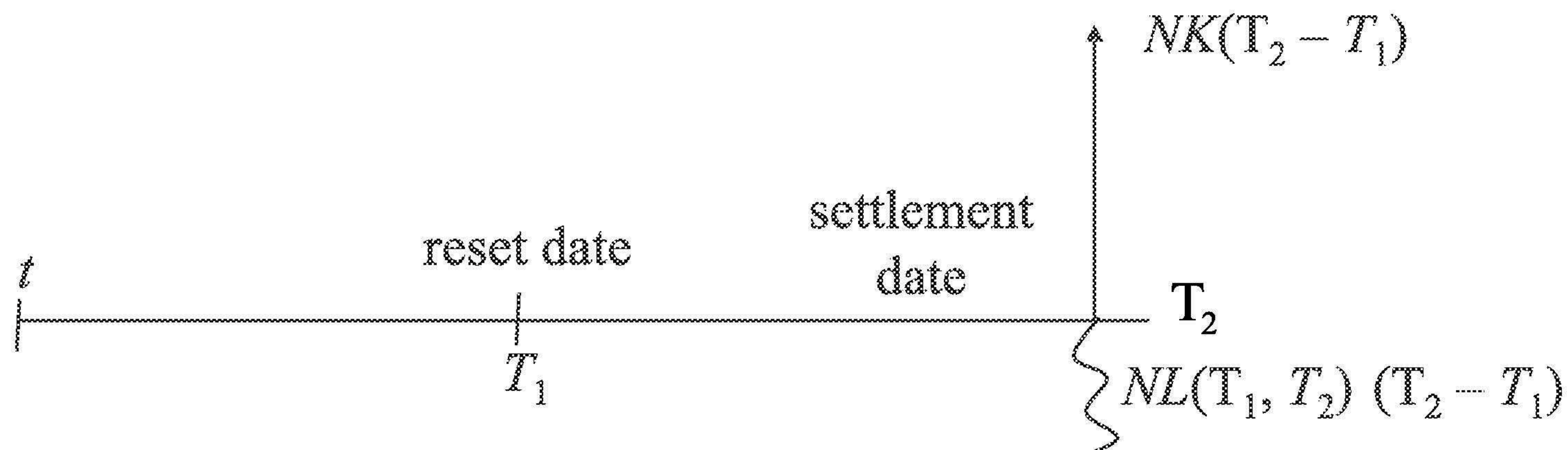
- The floating rate will be the LIBOR $L[T_1, T_2]$ as observed on the future reset date T_1.

Question

Should the fixed rate be set equal to the implied forward interest rate over the same period (determined based on traded bond prices as observed today)?

Determination of the forward price of LIBOR

$$
\begin{aligned}
L[T_1, T_2] &= \text{LIBOR to be observed at future time } T_1 \\
&\quad \text{for the accrual period } [T_1, T_2] \\
K &= \text{fixed rate} \\
N &= \text{notional of the FRA}
\end{aligned}
$$



Cash flows of the fixed rate receiver

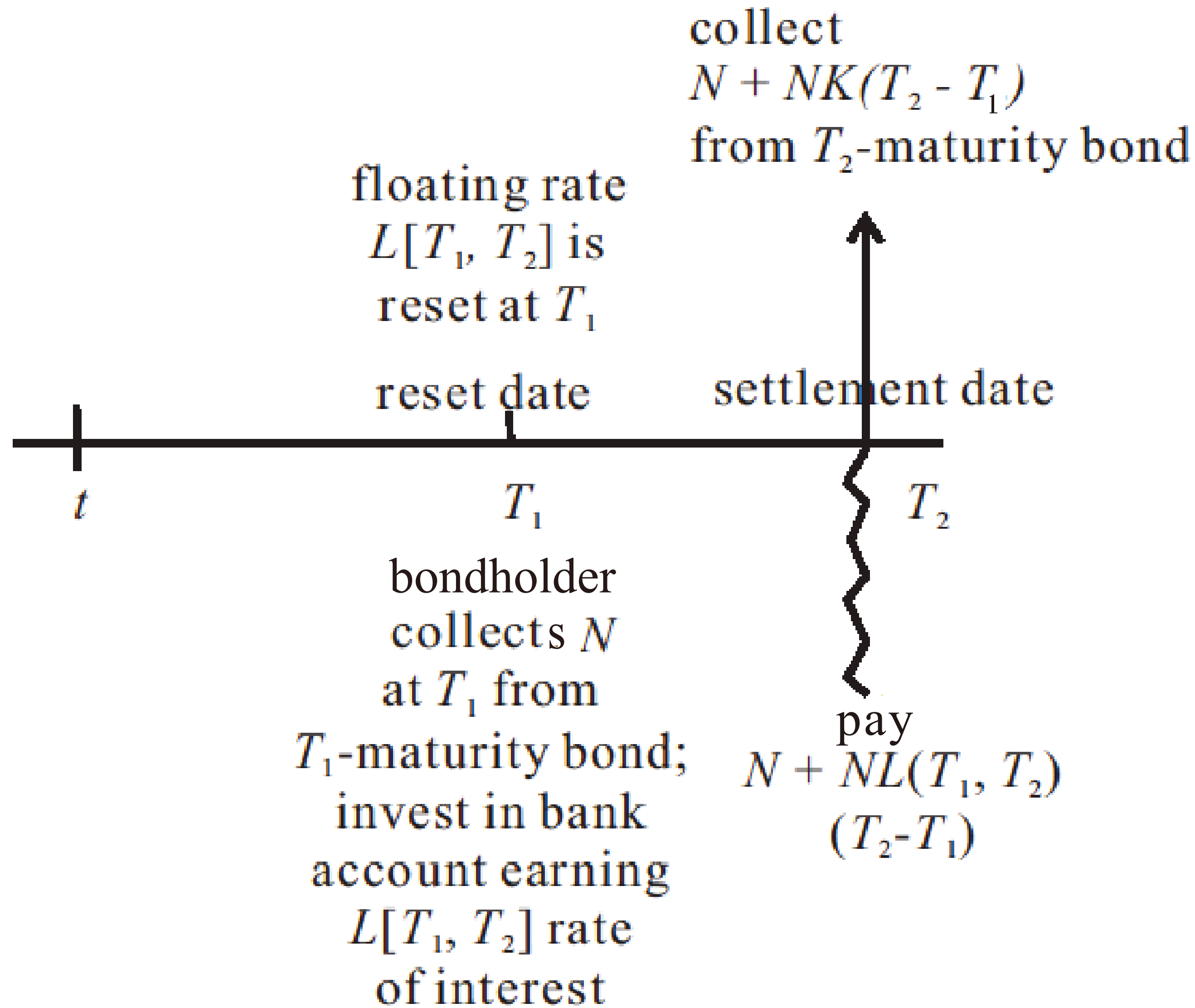
Replication argument

We add N to both the cash flows of the fixed rate receiver (see the diagram on the next page). These cash flows can be replicated by the following portfolio of bonds:

(i) long holding of the T_2-maturity discount bond with par $N[1 + K(T_2 - T_1)]$;

(ii) short holding of the T_1-maturity discount bond with par N.

Cash flow of the fixed rate receiver



$$
\begin{aligned}
& \text{Value of the portfolio of bonds that replicate} \\
& \text{the cash flow of the fixed rate receiver at the current time} \\
= \; & \text{net cost of acquiring the long and short positions of the two bonds} \\
& \text{at the current time} \\
= \; & N\{[1 + K(T_2 - T_1)]B_t(T_2) - B_t(T_1)\}.
\end{aligned}
$$

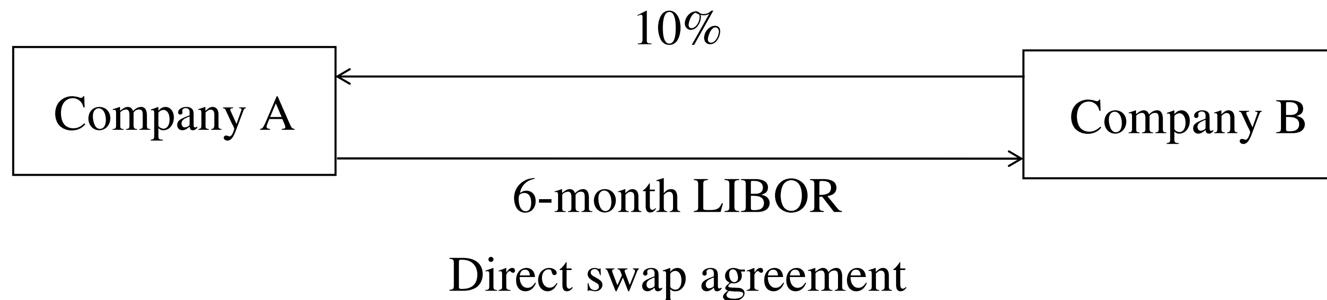
We find K such that the above value is zero.

$$
K = \underbrace{\frac{1}{T_2 - T_1}\left[\frac{B_t(T_1)}{B_t(T_2)} - 1\right]}_{\text{forward rate over } [T_1, T_2]}.
$$

The fair fixed rate K is visualized as the forward price of the LIBOR $L[T_1, T_2]$ over the time period $[T_1, T_2]$.

1.2 Swap products

Interest rate swaps



Direct swap agreement

The two parties agree to exchange periodic interest payments.

- The interest payments exchanged are calculated based on some predetermined dollar principal, called the notional amount.

- One party is the fixed-rate payer and the other party is the floating-rate payer. The floating interest rate is based on some reference rate (the most common index is the LONDON INTERBANK OFFERED RATE, LIBOR).

Example

Notional amount = \$50 million
fixed rate = 10%
floating rate = 6-month LIBOR

Tenor = 3 years, semi-annual payments

6-month period	Cash flows		
	Net (float − fix)	floating rate bond	fixed rate bond
0	0	−50	50
1	$\text{LIBOR}_1/2 \times 50 - 2.5$	$\text{LIBOR}_1/2 \times 50$	−2.5
2	$\text{LIBOR}_2/2 \times 50 - 2.5$	$\text{LIBOR}_2/2 \times 50$	−2.5
3	$\text{LIBOR}_3/2 \times 50 - 2.5$	$\text{LIBOR}_3/2 \times 50$	−2.5
4	$\text{LIBOR}_4/2 \times 50 - 2.5$	$\text{LIBOR}_4/2 \times 50$	−2.5
5	$\text{LIBOR}_5/2 \times 50 - 2.5$	$\text{LIBOR}_5/2 \times 50$	−2.5
6	$\text{LIBOR}_6/2 \times 50 - 2.5$	$\text{LIBOR}_6/2 \times 50$	−2.5

- One interest rate swap contract can effectively establish a payoff equivalent to a package of forward contracts.

A swap can be interpreted as a package of cash market instruments – a portfolio of forward rate agreements.

- Buy \$50 million par of a 3-year floating rate bond that pays 6-month LIBOR semi-annually.
- Finance the purchase by borrowing \$50 million for 3 years at 10% interest rate paid semi-annually.

The fixed-rate payer has a cash market position equivalent to a long position in a floating-rate bond and a short position in a fixed rate bond (borrowing through issuance of a fixed rate bond). The floating rate bond is a par bond (the present value is equal to the par value) since it pays LIBOR exactly. Suppose the interest rate swap costs nothing to either party, then the present value of the fixed rate bond is also par. Both bonds have values that are equal to the par value.

Application to asset/liability management

- Holding a 5-year term commercial loans of $50 million with a fixed interest rate of 10%, that is, interest of $2.5 million received semi-annually and par received at the end of 5 years.

- To fund its loan portfolio, the bank issues 6-month certificates of deposit with floating interest rate of LIBOR + 40 bps (100 bps = 1%). This may be a response to the market demand from investors for floating rate certificates of deposit.

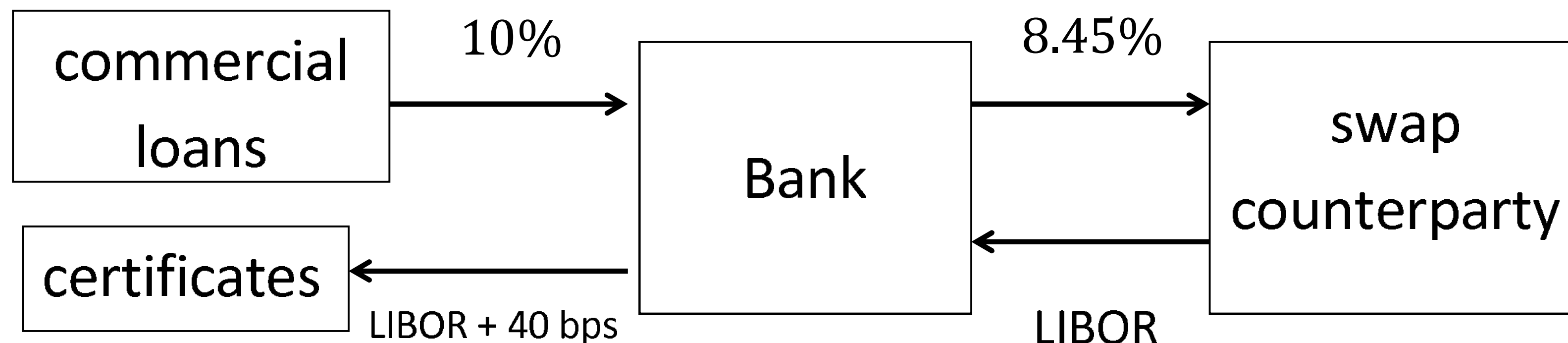
Risk	6-month LIBOR may be 9.6% or greater.
Possible strategy	Swap the fixed rate income into a floating rate cash stream to hedge against uncertainty in floating rate.

Choice of swap for the bank

- Every six months, the bank will pay 8.45% (annualized rate).
- Every six month, the bank will receive LIBOR.

Outcome

To be received	10.00%+ 6-month LIBOR
To be paid	8.45% + 0.4%+ 6-month LIBOR
spread income	1.15% or 115 basis points



The bank faces potential default risk of loans.

Life insurance company's position

- Has committed to pay a 9% rate for the next 5 years on a guaranteed investment contract (GIC) of amount \$50 million.
- Can invest \$50 million in an attractive 5-year floating-rate instrument with floating interest rate of 6-month LIBOR +160 bps.

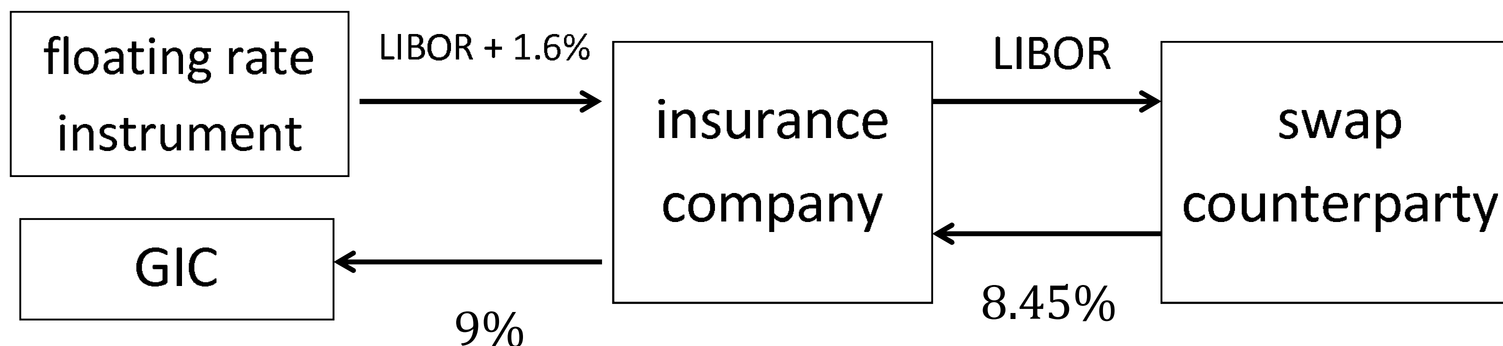
Risk	6-month LIBOR may fall to 7.4%.
Possible strategy	Swap the floating rate income into a fixed rate cash stream.

Choice of swap for the insurance company

- Every six months, the insurance company will pay LIBOR.
- Every six months, the insurance company will receive 8.45%.

Outcome

To be received	8.45% + 1.6%+ 6-month LIBOR
To be paid	9.00%+ 6-month LIBOR
spread income	1.05% or 105 basis points



Valuation of interest rate swaps

- At initiation of the interest rate swap, it typically has zero value.
- Valuation involves finding the fixed swap rate K such that the fixed and floating legs have equal value at inception.
- Consider a swap with payment dates $t_1, t_2, \cdots, t_n$ (tenor structure) set in the term of the swap. L_{i-1} is the LIBOR observed at t_{i-1} but payment is made at t_i. Write $\delta_i \approx t_i - t_{i-1}$ as the accrual period over $[t_{i-1}, t_i]$. Note that δ_i is in general not exactly the same as $t_i - t_{i-1}$ since some form of day count convention is used to compute δ_i (see below).
- The fixed payment at t_i is $KN\delta_i$ while the floating payment at t_i is $L_{i-1}N\delta_i, i = 1, 2, \cdots n$. Here, N is the notional.

Day count convention

For the 30/360 day count convention of the time period $(D_1, D_2]$ with D_1 excluded but D_2 included, the year fraction is

$$\frac{\max(30 - d_1, 0) + \min(d_2, 30) + 360 \times (y_2 - y_1) + 30 \times (m_2 - m_1 - 1)}{360}$$

where d_i, m_i and y_i represent the day, month and year of date $D_i, i = 1, 2$.

For example, the year fraction between *Feb 27, 2006* and *July 31, 2008*

$$= \frac{30 - 27 + 30 + 360 \times (2008 - 2006) + 30 \times (7 - 2 - 1)}{360}$$

$$= \frac{33}{360} + 2 + \frac{4}{12}.$$

Replication of cash flows

The fixed payment at T_i is $KN\delta_i$. The fixed payments are packages of discount bonds with par $KN\delta_i$ at maturity date $T_i, i = 1, 2, \cdots, n$.

To replicate the floating leg payments at current time $t, t < T_0$, we long T_0-maturity discount bond with par N and short T_n-maturity discount bond with par N. The N dollars collected at T_0 can generate the floating leg payments $L_{i-1}N\delta_i$ at all T_i by rolling over N dollars in a deposit bank account earning interest rate L_{i-1} over $[T_{i-1}, T_i]$, $i = 1, 2, \cdots, n$. Note that we always have N dollars at the beginning of each accrual period since the interests earned are used to honor the floating leg payments. The remaining N dollars at T_n is used to pay the par of the T_n-maturity bond.

By paying $NB(t, T_0)$ at time t to acquire the T_0-maturity discount bond, we can generate these floating leg payments at all time points and the par N at t_n.

- Let $B(t,T)$ be the time-t value of the discount bond with maturity t.

Present value of floating leg payments $= N[B(t,T_0) - B(t,T_n)]$

Sum of present value of fixed leg payments $= NK \sum_{i=1}^{n} \delta_i B(t,T_i)$.

The swap rate K is given by equating the present values of the two sets of payments:

$$K = \frac{B(t,T_0) - B(t,T_n)}{\sum_{i=1}^{n} \delta_i B(t,T_i)}.$$

The interest rate swap reduces to a FRA when $n = 1$. As a check, we obtain

$$K = \frac{B(t,T_0) - B(t,T_1)}{(T_1 - T_0)B(t,T_1)},$$

just the same formula as shown on p.50.

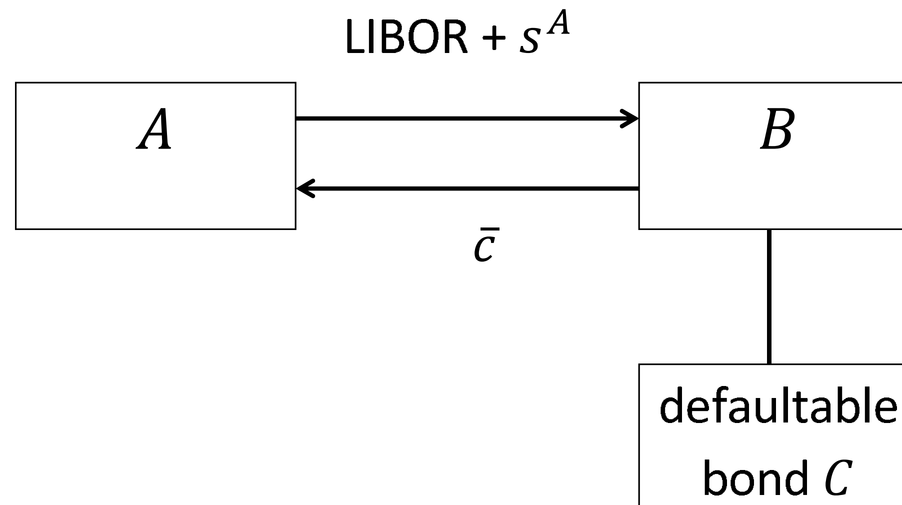
Asset swap

- Combination of a defaultable bond with an interest rate swap.

  B pays the notional amount upfront to acquire the asset swap package.

1. A fixed defaultable coupon bond issued by C with coupon $\bar{c}$ payable on coupon dates.

2. A fixed-for-floating swap.



Remarks

1. Asset swap transactions are driven by the desire to replace the fixed coupons by floating coupons. Asset swaps are more liquid than the underlying defaultable bond.

2. By market convention, the whole package is sold at par. The asset swap spread s^A is adjusted to ensure that the asset swap package has an initial value equal to the par value of the defaultable bond.

For convenience of our discussion, we set the par to be unity.

Default free bond price

$$C(t) = \text{time-}t \text{ price of default-free bond with fixed-coupon } \bar{c}$$
$$B(t,T) = \text{time-}t \text{ price of default-free zero-coupon bond with unit par}$$

Defaultable bond price

$$\overline{C}(t) = \text{time-}t \text{ price of defaultable bond with fixed-coupon } \bar{c}$$

The difference $C(t) - \overline{C}(t)$ represents the credit risk premium of the defaultable bond. Investors pay a lesser amount of $C(t) - \overline{C}(t)$ due to potential losses arising from bond default.

Forward swap rate

Recall that the swap rate is the fixed rate in an interest rate swap where the floating rate payer pays LIBOR and receives the swap rate (fixed). There is always a tenor structure that underlies an interest rate swap.

$s(t)$ = forward swap rate at time t of a standard fixed-for-floating over the tenor $[t_n, t_{n+1}, \ldots, t_N], \quad t \leq t_n$.

The first swap payment starts on t_{n+1} (based on the accrual period $[t_n, t_{n+1}]$) and the last payment date is t_N.

The theoretical forward swap rate can be determined in terms of discount bond prices based on replication. Interest rate swaps are highly liquid instruments and the forward swap rates are market observable. It may occur that the market swap rate may not agree exactly with the above theoretical formula based on traded bond prices. Arbitrage forces the market rates not to deviate too far from the theoretical prices.

Asset swap packages

An asset swap package consists of a defaultable coupon bond $\overline{C}$ with coupon $\overline{c}$ and an interest rate swap. The bond's coupon is swapped into LIBOR plus the asset swap rate s^A.

Payoff streams to the buyer of the asset swap package

time	defaultable bond	interest rate swap	net
$t = 0^\dagger$	$-\overline{C}(0)$	$-1 + \overline{C}(0)$	-1
$t = t_i$	$\overline{c}^*$	$-\overline{c} + L_{i-1} + s^A$	$L_{i-1} + s^A + (\overline{c}^* - \overline{c})$
$t = t_N$	$(1 + \overline{c})^*$	$-\overline{c} + L_{N-1} + s^A$	$1^* + L_{N-1} + s^A + (\overline{c}^* - \overline{c})$
default	recovery	unaffected**	recovery

$\star$ denotes payment contingent on survival.

† The value of the interest rate swap at $t = 0$ is not zero. The sum of the values of the interest rate swap and defaultable bond is set to be equal to par at $t = 0$ since the whole package is sold at par.

** The interest rate swap continues even after the bond defaults.

Remarks

- We do not need to model uncertainties in the coupons paid by the defaultable bond since the market has evaluated the expectation of the potential losses to the holder through the market price $\overline{C}(t)$ of the defaultable bond. Luckily, we are able to put all dusts (uncertainties of the time of arrival of default and recovery value) under the carpet (market price of the defaultable bond).

- However, if the interest rate swap discontinues after default of the underlying bond, we need to model the random time of arrival of default in the pricing procedure since there will be no exchange of interest payments after bond default.

The additional asset spread s^A above LIBOR serves as the compensation to the buyer for bearing the potential loss upon default.

$s(0)$ = fixed-for-floating swap rate (market quote)

$A(0)$ = value of an annuity paying at \$1 per annum over the same tenor as the interest rate swap (calculated based on observable default free bond prices)

Suppose the annuity stream of \$1 per annum makes payments quarterly, this would mean \$0.25 to be paid in each quarterly payment, making a total of $\$0.25 \times 4 = \$1$ per annum. Once the tenor is known, we can find the value of the annuity for a given tenor as sum of present values of the cash flows generated from the annuity.

Now, suppose the fixed coupon per annum is 3%, paid quarterly. Since the par is assumed to be unity, then each coupon payment is $\$0.03/4 = \$0.0075$. The value of the stream of the fixed coupons for the given tenor is $0.03A(0)$.

The value of the asset swap package is set at par at $t = 0$, so that

$$\overline{C}(0) + \underbrace{A(0)s(0) + A(0)s^A(0) - A(0)\overline{c}}_{\text{swap arrangement}} = 1. \tag{A}$$

The present value of the floating coupons is given by $A(0)s(0)$. This is because the value of the floating leg payments at LIBOR is the same as the value of the fixed leg payments at the market swap rate $s(0)$ under the market interest rate swap.

It has been assumed that the interest rate swap continues even after default so that $A(0)$ appears in all terms associated with the swap arrangement.

Solving for $s^A(0)$, we obtain the asset spread

$$s^A(0) = \frac{1}{A(0)}[1 - \overline{C}(0)] + \overline{c} - s(0).$$

Rearranging the terms in eq.(A), we obtain

$$\overline{C}(0) + A(0)s^A(0) = \underbrace{[1 - A(0)s(0)] + A(0)\overline{c}}_{\text{default-free bond}} \equiv C(0)$$

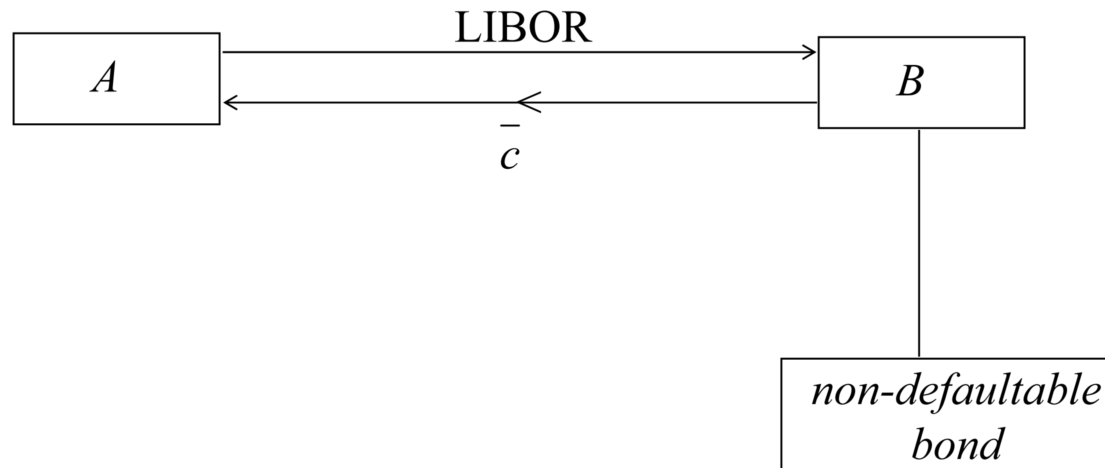
where the right-hand side gives the value of a default-free bond with coupon $\overline{c}$. A default free bond that pays LIBOR is called a par floater since its fair value at initiation equals par. Stripping all floating leg payments that pay LIBOR reduces the par floater to single payment of \$1 at maturity t_N. Therefore, $1 - A(0)s(0)$ is the present value of receiving \$1 at maturity t_N. We obtain

$$s^A(0) = \frac{1}{A(0)}[C(0) - \overline{C}(0)].$$

The extra asset swap s^A accounts for the potential loss upon default. This is quantified as $C(0) - \overline{C}(0)$ based on market bond prices at $t = 0$. For a fair deal between the two parties, the difference in the bond prices should be set equal to the present value of annuity stream at the fixed rate $s^A(0)$.

Alternative proof

A combination of the non-defaultable counterpart (bond with coupon rate $\overline{c}$) plus an interest rate swap (whose floating leg is LIBOR while the fixed leg is $\overline{c}$) becomes a par floater. Hence, the new asset package should also be sold at par.



The buyer is guaranteed to receive LIBOR floating rate interests plus par.

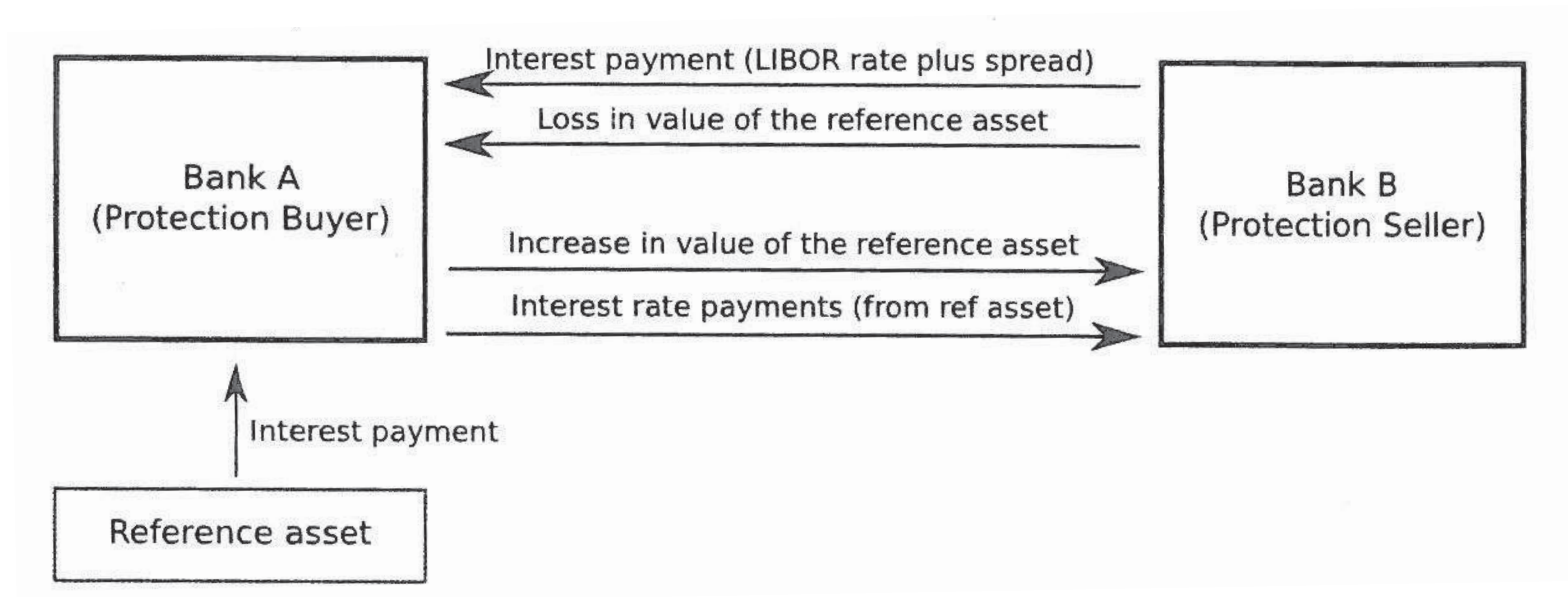
The two interest swaps with floating leg at LIBOR $+ s^A(0)$ and LIBOR, respectively, differ in values by $s^A(0)A(0)$.

Comparing the two asset swaps on P.62 and P.71, the earlier one has defaultable bond while the later one has default free bond. Both asset swaps have the same value as the par. The price difference is $C(0) - \overline{C}(0)$. This is compensated by the present value of the annuity at the rate $s^A(0)$ per annum. Therefore,

$$C(0) - \overline{C}(0) = s^A(0)A(0).$$

Total return swap

- Exchange the total economic performance of a specific asset for another cash flow.



Total return comprises the sum of interests, fees and any change-in-value payments with respect to the reference asset.

A commercial bank can hedge all credit risk on a bond/loan it has originated. The counterparty can gain access to the bond/loan on an off-balance sheet basis, without bearing the cost of originating, buying and administering the loan. The TRS terminates upon the default of the underlying asset.

Some essential features

1. The receiver is synthetically long the reference asset without having to fund the investment up front. He has almost the same payoff stream as if he had invested in risky bond directly and funded this investment at LIBOR $+ s^{TRS}$.
2. The TRS is marked to market at regular intervals, similar to a futures contract on the risky bond. The reference asset should be liquidly traded to ensure objective market prices for marking to market (determined using a dealer poll mechanism).
3. The TRS allows the receiver to leverage his position much higher than he would otherwise be able to (may require collateral). The TRS spread should not only be driven by the default risk of the underlying asset but also by the credit quality of the receiver.

The payments received by the total return receiver are:

1. The coupon $\overline{c}$ of the bond (if there were one since the last payment date T_{i-1}).
2. The price appreciation $(\overline{C}(T_i) - \overline{C}(T_{i-1}))^+$ of the underlying bond C since the last payment (if there were any).
3. The recovery value of the bond (if there were default).

The payments made by the total return receiver are:

1. A regular fee of LIBOR $+s^{TRS}$.
2. The price depreciation $(\overline{C}(T_{i-1}) - \overline{C}(T_i))^+$ of bond C since the last payment (if there were any).
3. The par value of the bond C (if there were a default in the meantime).

The coupon payments are netted and swap's termination date is earlier than bond's maturity.

Motivation of the receiver

1. Investors can create new assets with a specific maturity not currently available in the market.
2. Investors gain efficient off-balance sheet exposure to a desired asset class to which they otherwise would not have access.
3. Investors may achieve a higher leverage on capital – ideal for hedge funds. Otherwise, direct asset ownership is on on-balance sheet funded investment.
4. Investors can reduce administrative costs via an off-balance sheet purchase.
5. Investors can access entire asset classes by receiving the total return on an index.

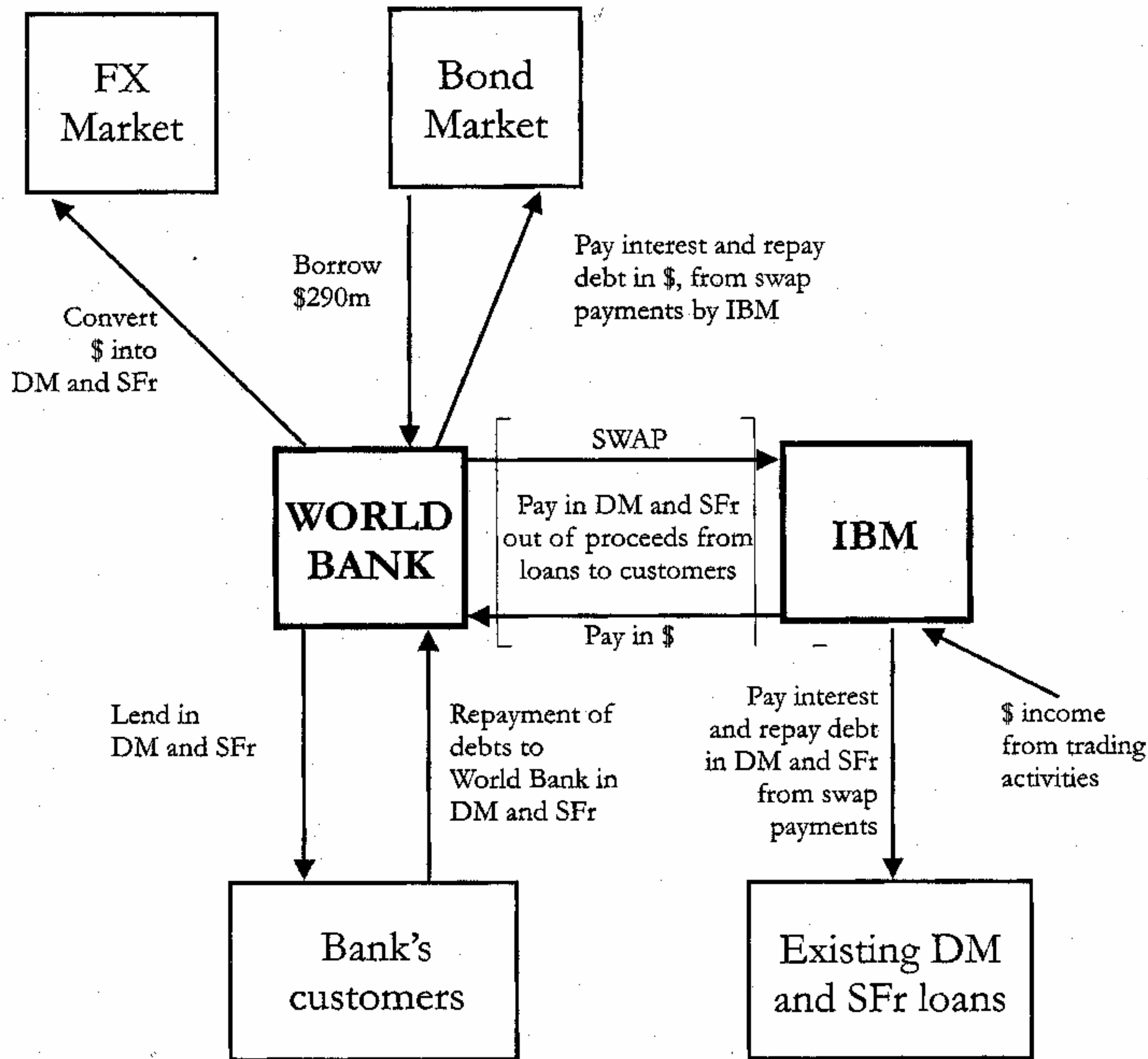
Motivation of the payer

- A long-term investor, who feels that a reference asset in the portfolio may *widen in spread* in the short term but will recover later, may enter into a total return swap that is shorter than the maturity of the asset. She can gain from the price depreciation. This structure is *flexible and does not require a sale of the asset* (thus accommodates a temporary *short-term negative view* on an asset).

IBM/World Bank – first currency swap structured in 1981

- IBM had existing debts in DM and Swiss francs. This had created a foreign exchange exposure since IBM had to convert USD into DM and Swiss Francs regularly to make the coupon payments. Due to a depreciation of the DM and Swiss franc against the dollar, IBM could realize a large foreign exchange gain, but only if it could eliminate its DM and Swiss franc liabilities and "lock in" the gain and remove any future exposure.

- The World Bank was raising most of its funds in DM (interest rate = 12%) and Swiss francs (interest rate = 8%). It did not borrow in dollars, for which the interest rate cost was about 17%. Though it wanted to lend out in DM and Swiss francs, the bank was concerned that saturation in the bond markets could make it difficult to borrow more in these two currencies at a favorable rate. Its objective, as always, was to raise cheap funds.


World Bank/IBM Currency Swap, 1981
FX Market
Bond Market
Convert $ into DM and SFr
Borrow $290m
Pay interest and repay debt in $, from swap payments by IBM
WORLD BANK
SWAP
Pay in DM and SFr out of proceeds from loans to customers
Pay in $
IBM
Lend in DM and SFr
Repayment of debts to World Bank in DM and SFr
Pay interest and repay debt in DM and SFr from swap payments
$ income from trading activities
Bank's customers
Existing DM and SFr loans

- IBM was willing to take on dollar liabilities and made dollar payments (periodic coupons and principal at maturity) to the World Bank since it could generate dollar income from its normal trading activities.
- The World Bank could borrow dollars, convert them into DM and SFr in FX market, and through the swap take on payment obligations in DM and SFr.

1. The foreign exchange gain on dollar appreciation is realized by IBM through the negotiation of a favorable swap rate in the currency swap contract.
2. The swap payments by the World Bank to IBM were scheduled so as to allow IBM to meet its debt obligations in DM and SFr.

Under the currency swap

- IBM pays regular US coupons and US principal at maturity.
- World Bank pays regular DM and SFr coupons together with DM and SFr principal at maturity.

Now IBM converted its DM and SFr liabilities into USD and pay these USD liabilities via US trading activities. The World Bank effectively raised DM and SFr at a cheap rate. Both parties achieved their objectives!

1.3 Options: Rational boundaries of option values

Financial options

- A *call* (or *put*) option is a contract which gives the holder the *right* to buy (or sell) a prescribed asset, known as the *underlying asset*, by a certain date (*expiration date*) for a predetermined price (commonly called the *strike price* or *exercise price*).

- The option is said to be *exercised* when the holder chooses to buy or sell the asset.

- If the option can only be exercised on the expiration date, then the option is called a *European* option.

- If the exercise is allowed at any time prior to the expiration date, then it is called an *American* option

Terminal payoff

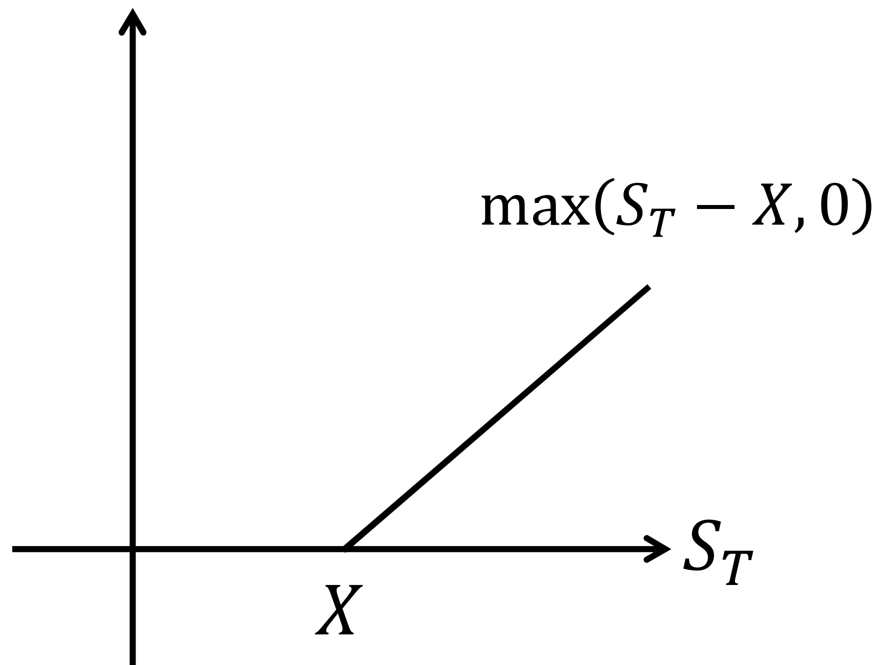
- Let S_T denote the asset price at maturity date T and X be the strike price.
- The terminal payoff from the long position (holder's position) of a European call is then

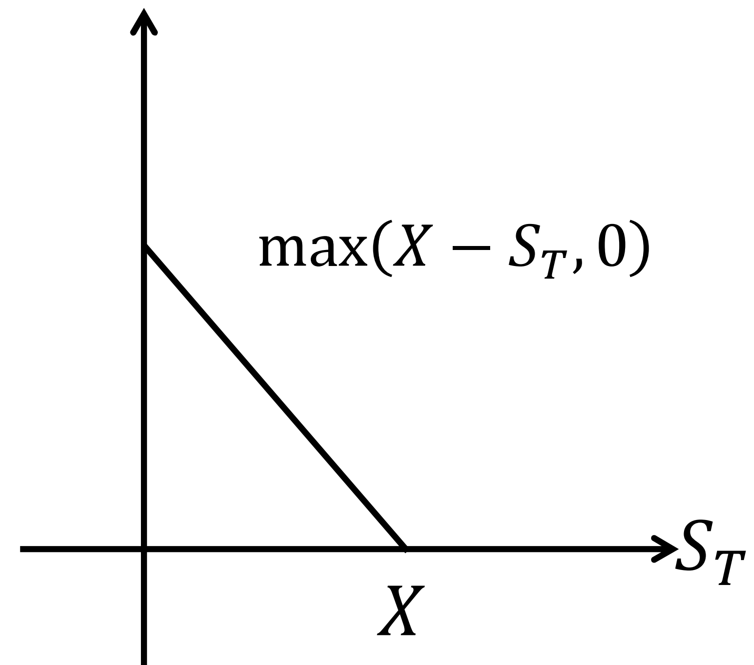
$$\max(S_T - X, 0).$$

- The terminal payoff from the long position in a European put can be shown to be

$$\max(X - S_T, 0),$$

since the put will be exercised at expiry only if $S_T < X$, whereby the asset worth S_T is sold at a higher price of X.





Note that call's terminal payoff minus put's terminal payoff equals the terminal payoff of a forward contract: $S_T - X$.

Questions and observations

What should be the fair option premium (usually called option price or option value) so that the deal is fair to both writer and holder?

What should be the optimal strategy to exercise prior to the expiration date for an American option?

At least, the option price is easily seen to depend on the strike price, time to expiry and current asset price. The less obvious factors for the pricing models are the prevailing *interest rate* and the degree of randomness of the asset price, commonly called the *volatility*.

Hedging a short position in a call

- If the writer of a call does not simultaneously own any amount of the underlying asset, then he is said to be in a *naked position*.
- Suppose the call writer owns some amount of the underlying asset, the loss in the short position of the call when asset price rises can be compensated by the gain in the long position of the underlying asset.
- This strategy is called *hedging*, where the risk in a portfolio is monitored by taking opposite directions in two securities which are highly negatively correlated.
- In a *perfect hedge* situation, the *hedger* combines a risky option and the corresponding underlying asset in an appropriate proportion to form a riskless portfolio.

Swaptions – Product nature

- The buyer of a swaption has the right to enter into an interest rate swap by some specified date. The swaption also specifies the maturity date of the swap. The buyer of the swaption pays the premium upfront.
- The buyer can be the fixed-rate receiver or the fixed-rate payer. The writer becomes the counterparty to the swap if the buyer exercises. The strike rate indicates the fixed rate that will be swapped versus the floating rate.
- Suppose the buyer of the swaption is the fixed rate payer in the underlying swap, she chooses to exercise the swaption when the prevailing swap rate on swaption's maturity date is higher than the strike rate. This is because the swaption buyer would pay a lower fixed rate in the interest rate swap under the swaption contract when compared with the higher prevailing fixed rate in a newly negotiated interest rate swap. This swaption is in-the-money when the market swap rate is higher than the strike rate.

- To the buyer (fixed rate payer), the exercise of the swaption is shorting the fixed rate bond and longing the floating rate bond.
- The value of the floating rate bond equals the par at initiation of the swap, so it may be viewed as having fixed value (visualized as the strike price of the swaption). This swaption is thus seen as a put swaption since the holder sells a fixed rate bond with floating value to receive a floating rate bond with fixed value upon exercise. Recall that the holder of a put option has the right to forfeit an asset with floating value to receive fixed amount of dollars.
- Note that the value of the fixed rate bond with coupon rate same as the prevailing swap rate would be equal to the par value. Obviously, the value of the fixed rate bond with coupon rate lower than the prevailing swap rate would be below par. When the swaption (fixed rate payer) holder exercises the swaption, the floating value asset (fixed rate bond) is below the strike price (par value of the floating rate bond) of the put swaption.

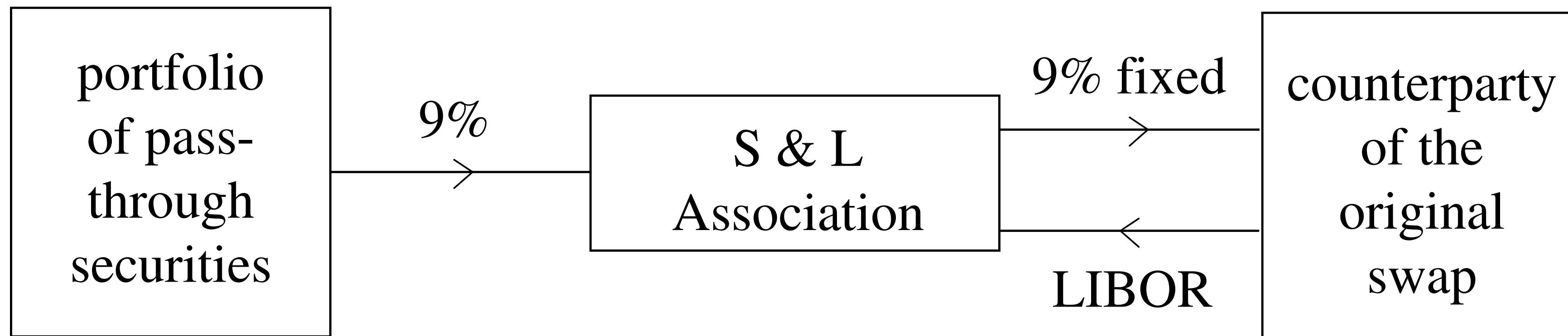
Uses of swaptions

Used to hedge a portfolio strategy that involves the use of an interest rate swap while the cash flow of the underlying asset or liability is uncertain. Uncertainties come from (i) early termination of cash flows due to callability in a callable bond or prepayment of mortgage loans, (ii) exposure to default risk.

Example

Consider a Savings & Loans Association entering into a 4-year swap in which it agrees to pay 9% fixed and receive LIBOR.

- The fixed rate payments come from a portfolio of mortgage pass-through securities with a coupon rate of 9%. One year later, mortgage rates decline, resulting in large prepayments.
- The purchase of a call swaption with a strike rate of 9% would be useful to offset the original swap position.



Due to decline in the interest rate, large prepayments are resulted in the mortgage pass-through securities. The source of 9% fixed payment dissipates.

The call swaption is in-the-money since the interest rate declines, so does the swap rate.

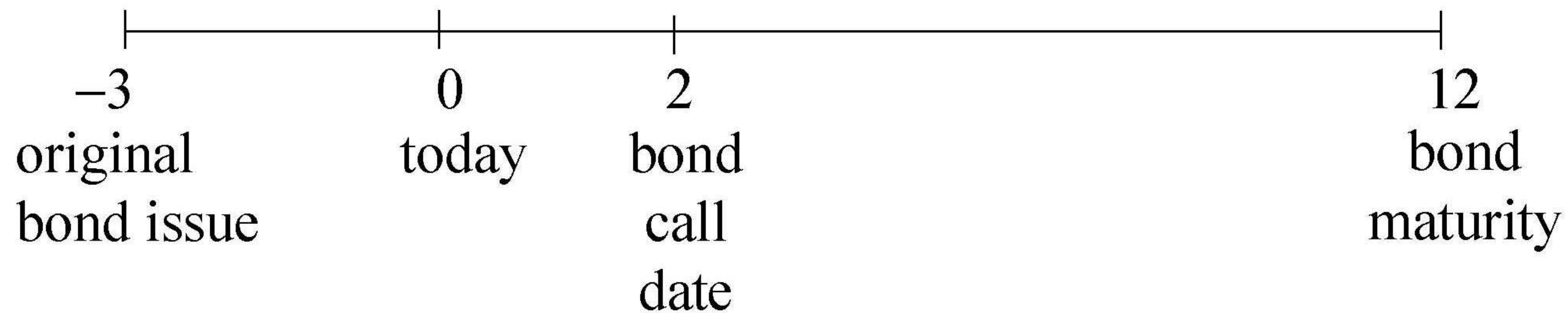
Hedging by call swaption

By exercising the call swaption, the Savings & Loans Association receives a fixed rate of 9%. The exposure to prepayment risk due to decline in interest rates is hedged via the purchase of a swaption.

- Treating the fixed rate bond as the underlying asset in the swaption and the floating rate bond as the fixed par value, the "pay-float" swaption is visualized as a call swaption since the holder pays the fixed value (floating rate bond exhibits fixed value) to receive the fixed rate bond with floating value.

Management of callable debts

Three years ago, XYZ issued 15-year fixed rate callable debt with a coupon rate of 12%.



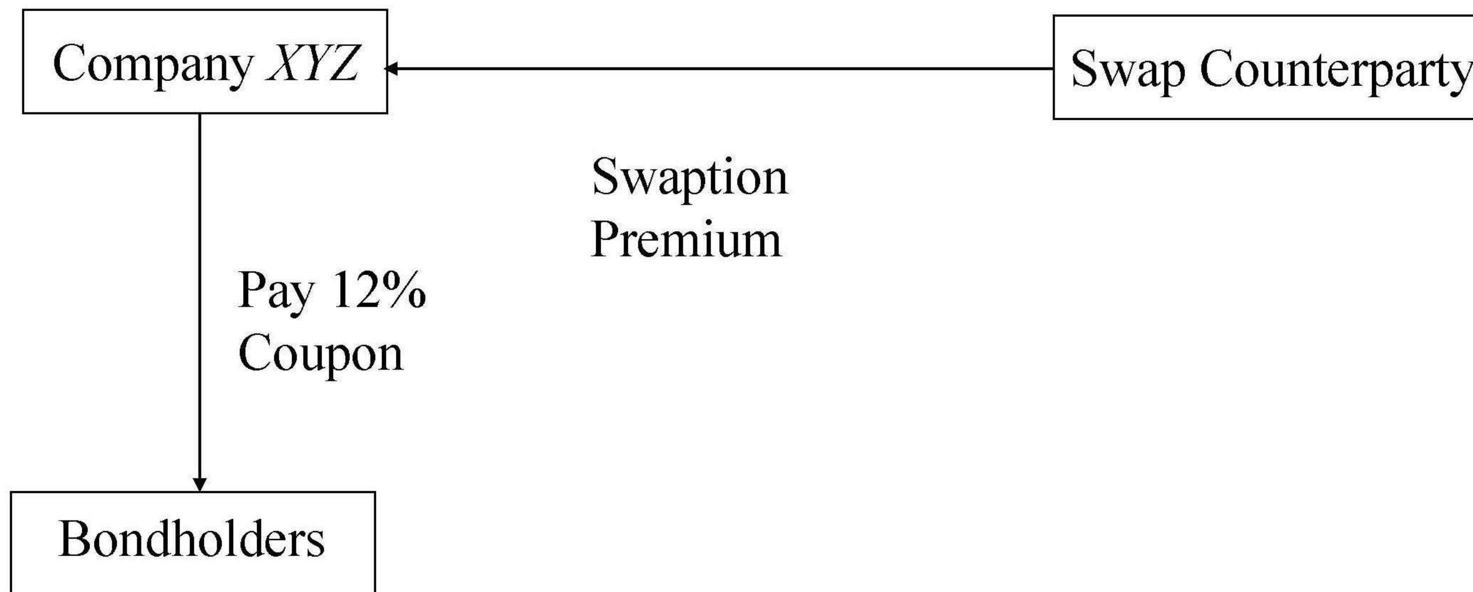
Strategy

The bond issuer XYZ sells a two-year fixed rate receiver option on a 10-year swap that gives the holder the right but not the obligation to receive the fixed rate of 12%. By selling the European call swaption with two-year maturity today, the company has committed itself to paying fixed 12% coupon for the remaining life of the original bond.

Call monetization

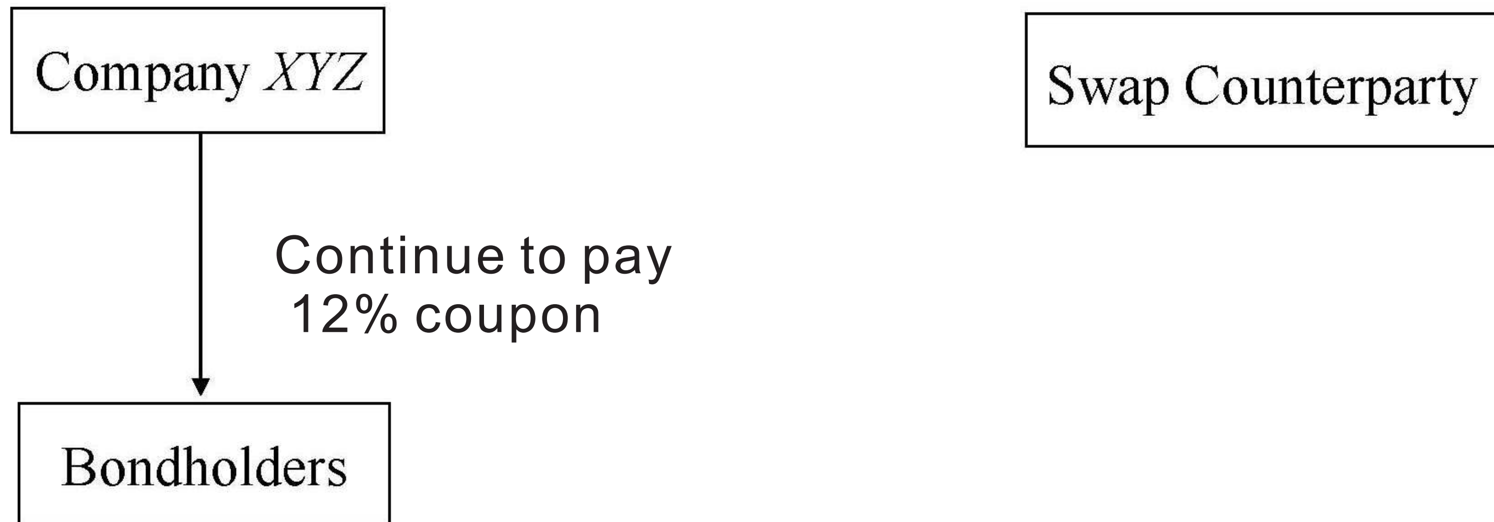
The callable right has intrinsic value. However, the callable right can be exercised only two years later. How to cash in the intrinsic value at the current time?

- The call swaption was sold by XYZ in exchange for an upfront swaption premium received at date 0. The monetization of the callable right is realized via the swaption premium received.



Call-Monetization cash flow: Swaption expiration date

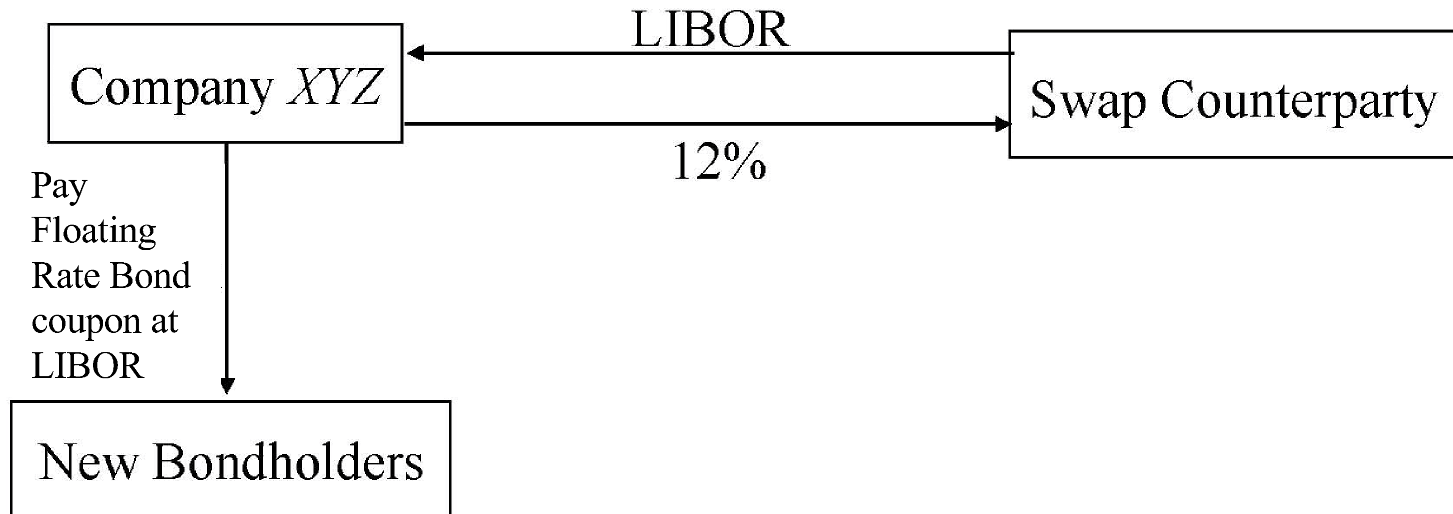
Interest rate (swap rate) $\geq$ 12%



- The call swaption holder (fixed rate receiver) does not exercise the European swaption on expiration date since receiving the prevailing swap rate is better than strike rate of 12%.

- XYZ earns the full proceed of the swaption premium. The swaption premium serves as the upfront (today) monetization of the callable right (exercisable 2 years later) held by XYZ in the callable bond.

Interest rate (swap rate) $< 12\%$



- Counterparty of the European swaption exercises the swaption
- XYZ calls the bond since the prevailing swap rate is below the promised 12% coupon rate in the bond. Once the old bond is retired, XYZ issues a new floating rate bond that pays floating rate LIBOR (funding rate would also depend on the creditworthiness of XYZ at that time). The cash required to retire the old fixed rate bond is financed by the issuance of the new floating rate bond.

Rational boundaries for option values

- Mathematical properties of the option values as functions of the strike price X, asset price S and time to expiry τ are derived. We do not specify the probability distribution of the movement of the asset price so that the *fair* option value cannot be derived. We study the impact of dividends on these rational boundaries.

- The optimal early exercise policy of American options on a non-dividend paying asset can be inferred from the analysis of these bounds on option values.

- The relations between put and call prices (called the *put-call parity relations*) are also deduced. These relations are distribution free.

Non-negativity of option prices

All option prices are non-negative, that is,

$$C \geq 0, \quad P \geq 0, \quad c \geq 0, \quad p \geq 0,$$

as derived from the non-negativity of the payoff structure of option contracts.

If the price of an option were negative, this would mean an option buyer receives cash up-front. In addition, he is guaranteed to have a non-negative terminal payoff. In this way, he can always lock in a riskless profit.

Intrinsic values of American options

- $\max(S - X, 0)$ and $\max(X - S, 0)$ are commonly called the *intrinsic value* of a call and a put, respectively.

- Since American options can be exercised at any time before expiration, their values must be worth at least their intrinsic values, that is,

$$
\begin{aligned}
C(S, \tau; X) &\geq \max(S - X, 0) \\
P(S, \tau; X) &\geq \max(X - S, 0).
\end{aligned}
$$

- Suppose C is less than $S - X$ when $S \geq X$, then an arbitrageur can lock in a riskless profit by borrowing $C + X$ dollars to purchase the call and exercise it immediately to receive the asset worth S. The riskless profit would be $S - X - C > 0$. Market frictions may lead to violation of the lower bound on call value (like near-maturity warrants).

American options are worth at least their European counterparts

An American option confers all the rights of its European counterpart plus the privilege of early exercise. The additional privilege cannot have negative value.

$$\begin{aligned} C(S,\tau;X) &\geq c(S,\tau;X) \\ P(S,\tau;X) &\geq p(S,\tau;X). \end{aligned}$$

- The European put value can be below the intrinsic value $X - S$ at sufficiently low asset value. In this case, the European put is almost sure to expire in-the-money. However, the par is received only at maturity for the European put.
- The value of a European call on a dividend paying asset can be below the intrinsic value $S - X$ at sufficiently high asset value. This is because holding of the European call does not entitle the holder to receive the dividends while the asset price drops in value when sizable dividends are paid during the life of the call.

Values of options with different dates of expiration

Consider two American options with different times to expiration τ_2 and τ_1 ($\tau_2 > \tau_1$), the one with the longer time to expiration must be worth at least that of the shorter-lived counterpart since the longer-lived option has the additional right to exercise between the two expiration dates.

$$
\begin{aligned}
C(S, \tau_2; X) &> C(S, \tau_1; X), \quad \tau_2 > \tau_1, \\
P(S, \tau_2; X) &> P(S, \tau_1; X), \quad \tau_2 > \tau_1.
\end{aligned}
$$

The above argument cannot be applied to European options due to lack of the early exercise privilege.

Values of options with different strike prices

$$\begin{aligned} c(S,\tau;X_2) &< c(S,\tau;X_1), \quad X_1 < X_2, \\ C(S,\tau;X_2) &< C(S,\tau;X_1), \quad X_1 < X_2. \end{aligned}$$

and

$$\begin{aligned} p(S,\tau;X_2) &> p(S,\tau;X_1), \quad X_1 < X_2, \\ P(S,\tau;X_2) &> P(S,\tau;X_1), \quad X_1 < X_2. \end{aligned}$$

Values of options at varying asset value levels

$$\begin{aligned} c(S_2,\tau;X) &> c(S_1,\tau;X), \quad S_2 > S_1, \\ C(S_2,\tau;X) &> C(S_1,\tau;X), \quad S_2 > S_1; \end{aligned}$$

and

$$\begin{aligned} p(S_2,\tau;X) &< p(S_1,\tau;X), \quad S_2 > S_1, \\ P(S_2,\tau;X) &< P(S_1,\tau;X), \quad S_2 > S_1. \end{aligned}$$

Upper bounds on call and put values

- A call option is said to be a *perpetual call* if its date of expiration is infinitely far away. The asset itself can be considered as an American perpetual call with zero strike price plus additional privileges such as voting rights and receipt of dividends, so we deduce that $S \geq C(S, \infty; 0)$.

$$S \geq C(S, \infty; 0) \geq C(S, \tau; X) \geq c(S, \tau; X).$$

- The price of an American put equals the strike value when the asset value is zero; otherwise, it is bounded above by the strike price.

$$X \geq P(S, \tau; X) \geq p(S, \tau; X).$$

Lower bounds on the values of call options on a non-dividend paying asset

Portfolio A consists of a European call on a non-dividend paying asset plus a discount bond with a par value of X whose date of maturity coincides with the expiration date of the call. Portfolio B contains one unit of the underlying asset.

Asset value at expiry	$S_T < X$	$S_T \geq X$
Portfolio A	X	$(S_T - X) + X = S_T$
Portfolio B	S_T	S_T
Result of comparison	$V_A > V_B$	$V_A = V_B$

The present value of Portfolio A (dominant portfolio) must be equal to or greater than that of Portfolio B (dominated portfolio). If otherwise, an arbitrage opportunity can be secured by buying Portfolio A and selling Portfolio B. This is called the principle of dominance (generalization of the law of one price).

Write $B(\tau)$ as the price of the unit-par discount bond with time to expiry τ. Then

$$c(S, \tau; X) + XB(\tau) \geq S.$$

Together with the non-negativity property of option value.

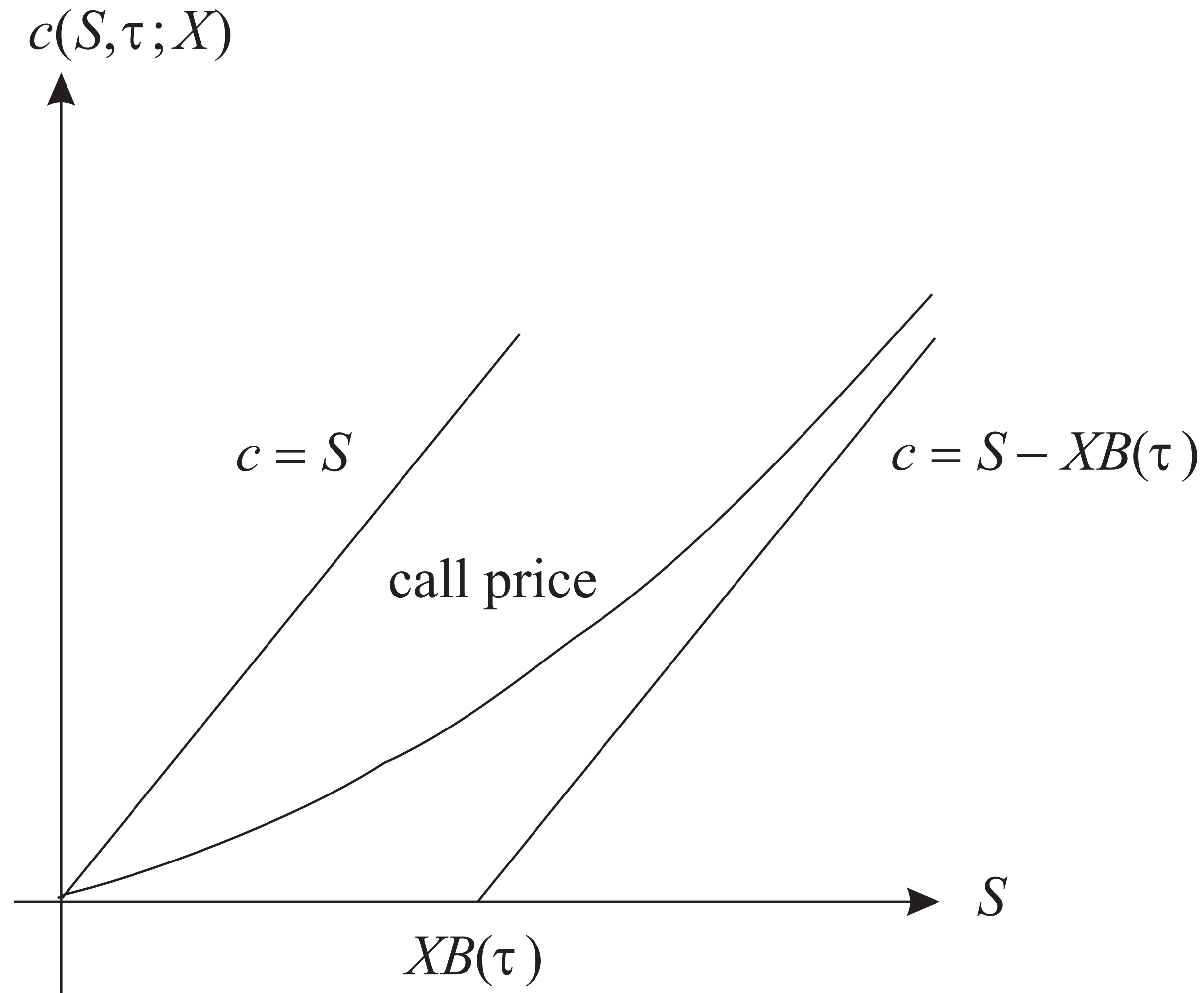
$$c(S, \tau; X) \geq \max(S - XB(\tau), 0).$$

The upper and lower bounds of the value of a European call on a non-dividend paying asset are given by (see Figure)

$$S \geq c(S, \tau; X) \geq \max(S - XB(\tau), 0).$$

Remark

Earlier, we have established $C(S, \tau; X) \geq \max(S - X, 0)$ for an American call option. Note that $C(S, \tau; X) \geq c(S, \tau; X)$ and $\max(S - XB(\tau), 0) \geq \max(S - X, 0)$, so the lower bound for a European call $\max(S - XB(\tau), 0)$ is even sharper.

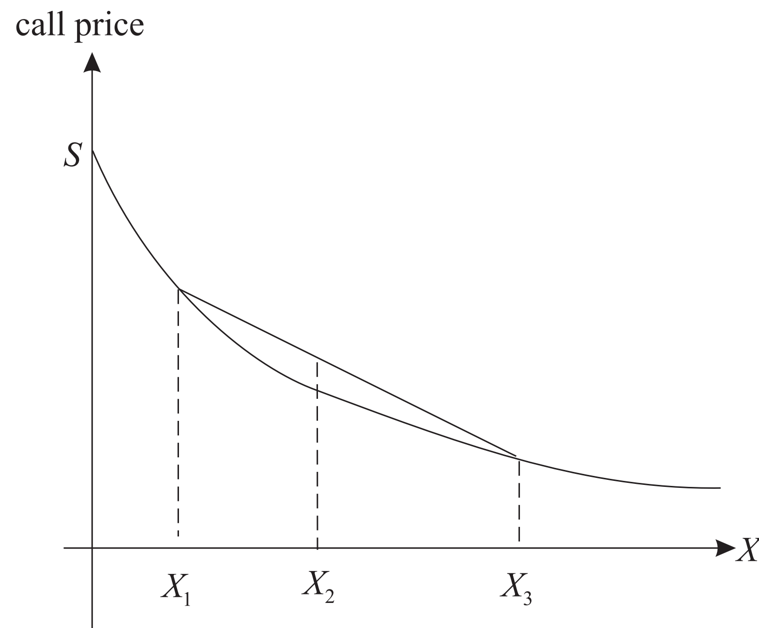


max$(S - XB(\tau), 0)$ is the sharper lower bound of the call price function compared to max$(S - X, 0)$.

Convexity properties of the option price functions

The call prices are convex functions of the strike price. Write $X_2 = \lambda X_3 + (1-\lambda)X_1$, where $0 \leq \lambda \leq 1, X_1 \leq X_2 \leq X_3$, the convexity property gives

$$c(S, \tau; X_2) \leq \lambda c(S, \tau; X_3) + (1-\lambda)c(S, \tau; X_1)$$



$c(S, \tau; X)$ is a decreasing function of X; furthermore, $\left|\frac{\partial c}{\partial X}\right| < B(\tau)$.

Proof

The drop in the European call value for one dollar increase in the strike price should be less than one dollar. The loss in the terminal payoff of the call due to the increase in the strike price is realized only when the call expires in-the-money. More precisely, we have $\left|\frac{\partial c}{\partial X}\right| < B(\tau)$. The factor $B(\tau)$ appears since the potential loss of paying extra one dollar in the strike price occurs at maturity so its present value is $B(\tau)$.

Consider the payoffs of the following two portfolios at expiry. Portfolio C contains λ units of call with strike price X_3 and $(1-\lambda)$ units of call with strike price X_1, and Portfolio D contains one call with strike price X_2.

Since $V_C \geq V_D$ for all possible values of S_T, Portfolio C is dominant over Portfolio D. Therefore, the present value of Portfolio C must be equal to or greater than that of Portfolio D.

Payoff at expiry of Portfolios C and D.

Asset value at expiry	$S_T \leq X_1$	$X_1 \leq S_T \leq X_2$	$X_2 \leq S_T \leq X_3$	$X_3 \leq S_T$
Portfolio C	0	$(1-\lambda)(S_T - X_1)$	$(1-\lambda)(S_T - X_1)$	$\lambda(S_T - X_3) +$ $(1-\lambda)(S_T - X_1)$
Portfolio D	0	0	$S_T - X_2$	$S_T - X_2$
Result of comparison	$V_C = V_D$	$V_C \geq V_D$	$V_C \geq V_D^*$	$V_C = V_D$

* Recall $X_2 = \lambda X_3 + (1-\lambda)X_1$, and observe

$$
\begin{aligned}
& (1-\lambda)(S_T - X_1) \geq S_T - X_2 \\
\Leftrightarrow \quad & X_2 - (1-\lambda)X_1 \geq \lambda S_T \\
\Leftrightarrow \quad & X_3 \geq S_T.
\end{aligned}
$$

- By changing the call options in the above two portfolios to the corresponding put options, it can be shown that European put prices are also convex functions of the strike price.

- By using the linear homogeneity property of the call and put option functions with respect to the asset price and strike price, where for the case of an European call option:

$$c(\lambda S, \tau, \lambda X) = \lambda c(S, \tau, X), \qquad \lambda > 0,$$

  one can show that the European call prices are convex functions of the asset price (see Qn 7, Hw 1). That is,

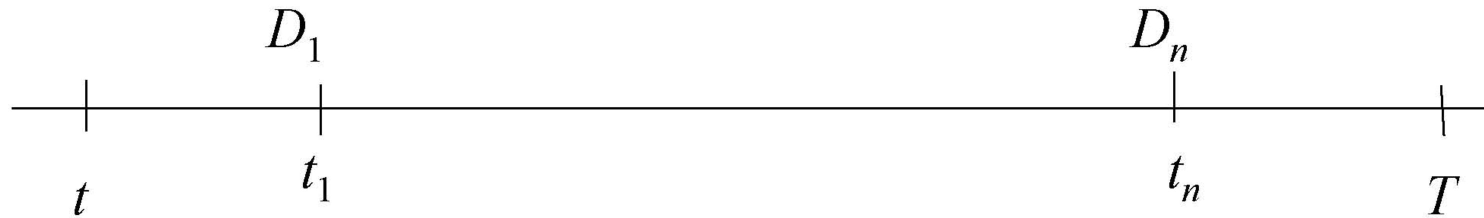
$$c(\lambda S_1 + (1-\lambda)S_2, \tau; X) \leq \lambda c(S_1, \tau, X) + (1-\lambda)c(S_2, \tau; X), \ \lambda > 0.$$

It is necessary in the first step to establish convexity property of option price functions on strike via the dominance principle since we can hold portfolios of call options with different strikes and same expiration date. We then use the linear homogeneity property to establish the convexity property of call price functions on asset price.

Impact of dividends on the asset price

- When an asset pays a certain amount of dividend, no-arbitrage argument dictates that the asset price falls by an amount same as the dividend (assuming there exist no other factors affecting the income proceeds, like taxation and transaction costs).

- Suppose the asset price falls by an amount less than the dividend, an arbitrageur can lock in a riskless profit by borrowing money to buy the asset right before the dividend date, selling the asset right after the dividend payment and returning the loan.



It is assumed that the deterministic dividend amount D_i is paid at time t_i, $i = 1, 2, \cdots, n$. The current time is t and write $\tau_i = t_i - t, i = 1, 2, \cdots, n$. The sum of the present value of the dividends within the life of the option is

$$D = D_1 e^{-r\tau_1} + \cdots + D_n e^{-r\tau_n}.$$

The dividends stream may be visualized as a portfolio of bonds with par value D_i maturing at $t_i, i = 1, 2, \cdots, n$.

Weakness in the assumption

One may query whether the asset can honor the deterministic dividend payments when the asset value becomes very low.

Put-call parity relations

For a pair of European put and call on the same underlying asset and with the same expiration date and strike price, we have

$$p = c - S + D + XB(\tau).$$

Recall that D is the sum of present values of all future deterministic dividends. When the underlying asset is non-dividend paying, we set $D = 0$. Recall that the value of the forward $f = S - D - XB(\tau)$, so $c - p = f$.

The first portfolio involves long holding of a European call, a cash amount of $D + XB(\tau)$ and short selling of one unit of the asset. The second portfolio contains only one European put. The cash amount D in the first portfolio is used to compensate the dividends paid to the lender of the asset due to the short position of the asset. At expiry, both portfolios are worth $\max(X - S_T, 0)$. Since both options are European, they cannot be exercised prior to expiry. By virtue of law of one price, both portfolios have the same value throughout the life of the options.

Impact of dividends on the lower bound on a European call value and the early exercise policy of an American call option

Recall the put-call parity relation: $c = S - XB(\tau) - D + p$. Since $p \geq 0$ and $c \geq 0$, we deduce that

$$c(S, \tau; X, D) \geq \max(S - XB(\tau) - D, 0).$$

- When S is sufficiently large, the European call almost behaves like a forward whose value is $S - D - XB(\tau)$. Using the put-call parity relation: $c = p + S - D - XB(\tau)$, we obtain

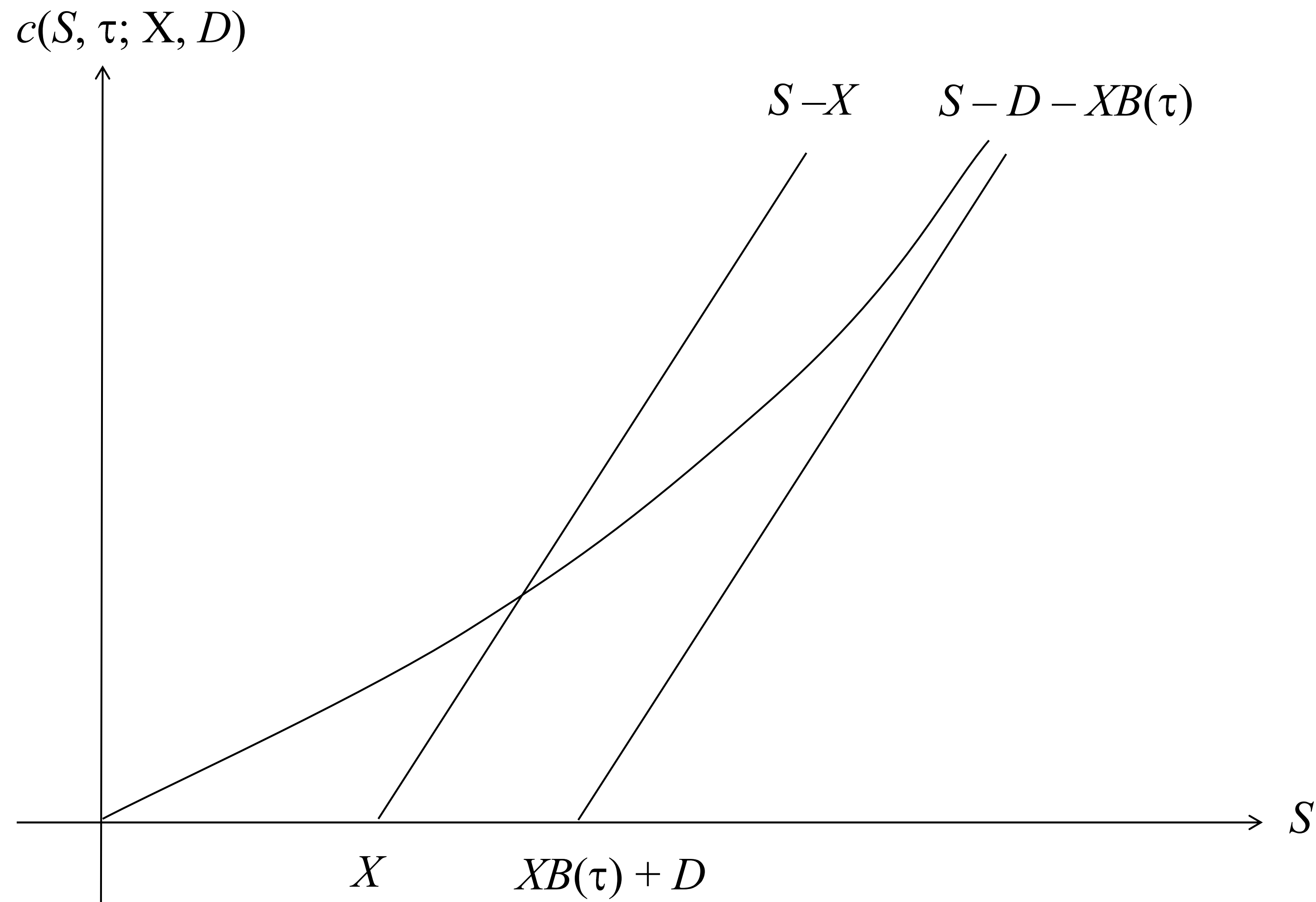
$$p(S, \tau; X, D) \to 0 \text{ as } S \to \infty.$$

- Since the call price is lowered due to the dividends of the underlying asset, it may be possible that the call price becomes less than the intrinsic value $S - X$ when the lumped dividend D is deep enough.

- A necessary condition on D such that $c(S, \tau; X, D)$ may fall below the intrinsic value $S - X$ is given by

$$S - X > S - XB(\tau) - D \text{ or } D > X[1 - B(\tau)].$$

  As a necessary condition, suppose D does not satisfy the above condition (that is, the dividends are not sufficiently deep), then it is never optimal to exercise the dividend-paying American call prematurely. This is because $C(S, \tau; X, D) \geq c(S, \tau; X, D) > S - XB(\tau) - D > S - X$ when $D \leq X[1 - B(\tau)]$.



When $D > X[1 - B(\tau)]$, the lower bound $S - D - XB(\tau)$ becomes less than the intrinsic value $S - X$. As $S \to \infty$, the European call price curve falls below the intrinsic value line.

Bounds on put price functions

The bounds for American and European puts can be shown to be

$$P(S,\tau;X,D) \geq p(S,\tau;X,D) \geq \max(XB(\tau)+D-S,0).$$

The second inequality is easily seen by virtue of the put-call parity relation and observing $c \geq 0$ and $p \geq 0$.

- When $XB(\tau)+D < X \Leftrightarrow D < X[1-B(\tau)]$, the lower bound $XB(\tau)+D-S$ may become less than the intrinsic value $X-S$ when the put is sufficiently deep in-the-money (corresponding to low value for S). It becomes sub-optimal for the holder of an American put option to continue holding the put option when the put value falls below the intrinsic value, so the American put should be exercised prematurely.
- The presence of dividends makes the early exercise of an American put option less likely since the holder loses the future dividends when the asset is sold upon exercising the American put. When $D \geq X[1-B(\tau)]$, it is never optimal to exercise an American put.

Lower and upper bounds on the difference of the prices of American call and put options

- The parity relation cannot be applied to American options due to the embedded early exercise feature.

First, we assume the underlying asset to be non-dividend paying. From the put-call parity relation, since $P > p$ and $C = c$, and putting $D = 0$, we have

$$C - P < S - XB(\tau),$$

giving the upper bound on $C - P$.

- Consider the following two portfolios: one contains a European call plus cash of amount X, and the other contains an American put together with one unit of underlying asset.

If there were no early exercise of the American put prior to maturity, the terminal value of the first portfolio is always higher than that of the second portfolio. If the American put is exercised prior to maturity, the second portfolio's value becomes X, which is always less than $c + X$. The first portfolio dominates over the second portfolio, so we have

$$c + X > P + S.$$

Further, since $c = C$ when the asset does not pay dividends, the lower bound on $C - P$ is given by

$$S - X < C - P.$$

Remark This result is sharper than the earlier result: $C > S - X$.

Combining the two bounds, the difference of the American call and put option values on a non-dividend paying asset is bounded by

$$S - X < C - P < S - XB(\tau).$$

How to modify the bounds when the underlying asset pays dividends?

- The right side inequality: $C - P < S - XB(\tau)$ also holds for options on a dividend paying asset since dividends decrease call value and increase put value.

- The left side inequality has to be modified as: $S - D - X < C - P$.

- Combining the results, the difference of the American call and put option values on a dividend paying asset is bounded by

$$S - D - X < C - P < S - XB(\tau).$$

1.4 American options: Optimal early exercise policies

Non-dividend paying asset

- At any moment when an American call is exercised, its value immediately becomes max$(S - X, 0)$. The exercise value is less than max$(S - XB(\tau), 0)$, the lower bound of the call value if the American call remains alive. The act of exercising prior to expiry causes a decline in value of the American call.

- Since the early exercise privilege is forfeited, the American and European call values should be the same when the underlying asset does not pay any dividend within the life of the American call option.

- For an American put, it may become optimal to exercise prematurely when S falls to sufficiently low value. Since the gain in time value of strike is zero when the interest rate is zero, an American put is never exercised prematurely under zero interest rate.

Dividend paying asset

- When the underlying asset pays dividends, the early exercise of an American call prior to expiry may become optimal when (i) S is very high and (ii) the dividends are sizable. Under these circumstances, it then becomes more attractive for the investor to acquire the asset rather than holding the option.
  - When S is high, the chance of regret of early exercise is low; equivalently, the insurance value of holding the call is lower.
  - When the dividends are sizable, it is more attractive to hold the asset directly instead of holding the call.
  - Paying X at maturity date T means present value is $XB(\tau)$. Paying X at the current time t means the loss in the time value of strike valued at t is $X[1 - B(\tau)]$.
  - A necessary condition for optimal early exercise of an American call is $D > X[1 - B(\tau)]$. Note that when $D = 0$, this necessary condition cannot be satisfied. To be sufficient for optimal early exercise, the asset price has to be sufficiently high.

- For an American put, when D is sufficiently high, it may become non-optimal to exercise prematurely even at very low value of S (even when the put is very deep-in-the-money). The gain in the time value of the strike cannot offset the losses in insurance value of holding the put and the dividends received through holding the asset.
  - A necessary condition for optimal early exercise of an American put is $D < X[1-B(\tau)]$. Note that this necessary condition is satisfied automatically when $D = 0$. To be sufficient for optimal early exercise, the asset price has to be sufficiently low.

The necessary condition for early exercise for an American call or put is dictated by the relative sizes of dividend amount D and time value of strike as quantified by $X[1 - B(\tau)]$.

	American call	American put
time value of strike, X	loss	gain
dividend, D_{total}	gain	loss
insurance value associated with holding of the option	loss	loss
Necessary condition for early exercise	$D_{\text{total}} > X[1-B(\tau)]$	$D_{\text{total}} < X[1-B(\tau)]$
Sufficiently deep-in-the-money	Sufficiently high asset price	Sufficiently low asset price

(1) When $D = 0$, $D_{\text{total}} > X[1 - B(\tau)]$ cannot be satisfied, so the holder never chooses to early exercise an American call. Also, under $D = 0$, gain on dividend is not a contributing factor for early exercise for the American call.

(2) When $r = 0$, $B(\tau) = 1$, so $X[1 - B(\tau)] = 0$. Since $D_{\text{total}} \geq 0$, so $D_{\text{total}} < X[1 - B(\tau)]$ cannot be satisfied. Therefore, the holder never chooses to early exercise an American put. Also, there is no gain on the time value of strike for the American put.

American call on an asset paying discrete dividends

- Since the holder of an American call on an asset paying discrete dividends will not receive any dividend in between the dividend times, so within these periods, it is never optimal to exercise the American call.
- It may be optimal to exercise the American call *immediately before* the asset goes ex-dividend. What are the necessary and sufficient conditions on the size of the dividend and the asset price level right before the dividend time t_d for optimal early exercise?

One-dividend model – Amount of D is paid out at t_d

- If the American call is exercised at t_d^-, the call value becomes $S_d^- - X$. The asset price drops to $S_d^+ = S_d^- - D$ right after the dividend payout.

- Holding of the American call exhibits zero net cash flow even the dividend is paid at t_d, so the American call value has no jump across t_d. If the call has a drop in value, we short the call at t_d^- and buy back the call at t_d^+ to close out the shorting position. This represents an arbitrage opportunity.

- It behaves like an ordinary European option for $t > t_d^+$. This is because when there is no further dividend, it becomes always non-optimal to exercise the American call.

- The lower bound of the one-dividend American call value at t_d^+ is the same lower bound for a European call, which is given by $S_d^+ - Xe^{-r(T-t_d^+)}$, where $T - t_d^+$ is the time to expiry.

- By virtue of the continuity of the call value across the dividend date, the lower bound B for the call value at time t_d^- should remain to be $S_d^+ - Xe^{-r(T-t_d)}$. In terms of S_D^-, the lower bound B is equal to $(S_d^- - D) - Xe^{-r(T-t_d)}$.

By comparing the American call value (continuation meaning staying alive across t_d) with the early exercise proceed: $S_d^- - X$, we deduce

(i) If $S_d^- - X \leq B$, the exercise proceed is less than or equal to the lower bound; that is,

$$S_d^- - X \leq (S_d^- - D) - Xe^{-r(T-t_d)}$$

$$\text{or} \qquad D \leq X[1 - e^{-r(T-t_d)}]$$

it is never optimal to exercise at t_d^- since exercising leads to a drop in call value.

(ii) When the discrete dividend D is sufficiently sizable such that

$$D > X[1 - e^{-r(T-t_d)}],$$

it may become optimal to exercise at t_d^- when the asset price S_d^- is above some threshold value S_d^*.

Suppose the American call stays alive across t_d, the call value right after the dividend date is given by

$$c(S_d^+, T - t_d^+) \text{ or equivalently } c(S_d^- - D, T - t_d^+),$$

where $c(S, T-t)$ denotes the European call value with asset price S and time to expiry $T - t$.

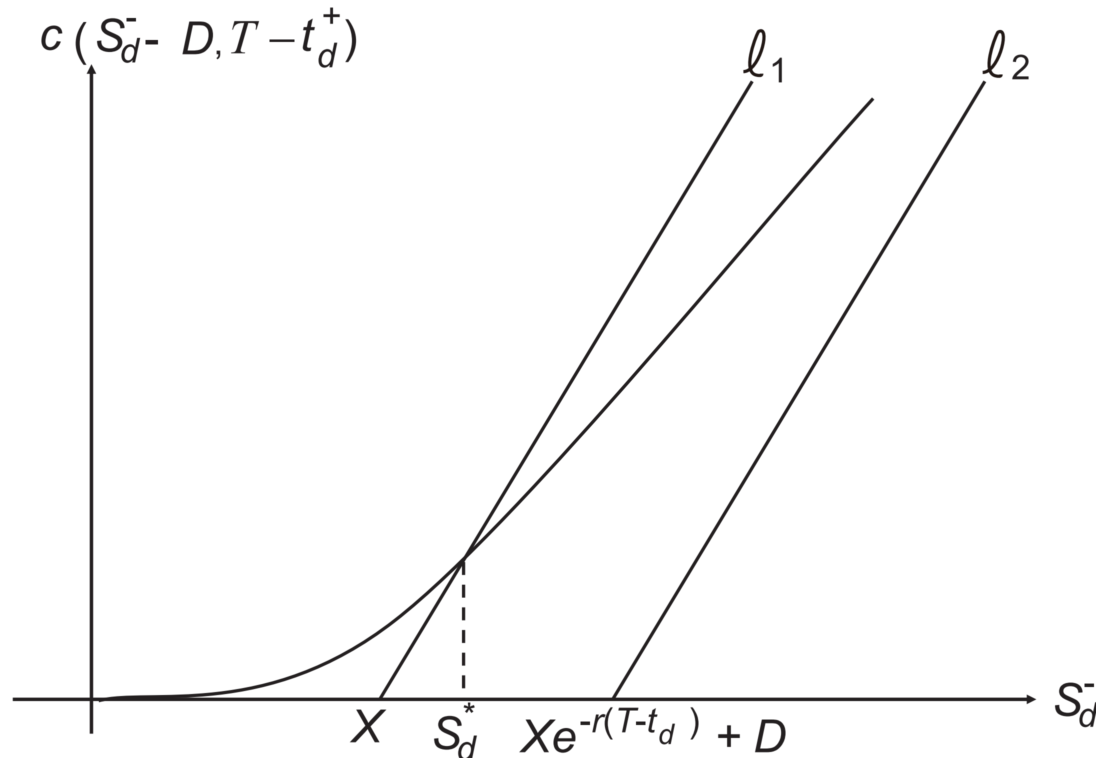
Indifferent to early exercise and continuation

Suppose D is sizable such that $D > X[1 - e^{-r(T-t_d)}]$ is satisfied. The holder can make two choices at t_d^-: (i) exercise the American call to receive payoff $S_d^- - X$, (ii) continue to hold the call option. The critical asset price S_d^* at which the holder is indifferent to early exercise or continuation is given by the solution to

$$c(S_d^- - D, T - t_d^+) = S_d^- - X.$$

When $S_d^- < S_d^*$, we observe $c(S_d^- - D, T - t_d^+) > S_d^- - X$, so it is optimal to continue to hold the American call. Otherwise, when $S_d^- \geq S_d^*$, it is optimal to early exercise the American call at t_d^-.

Determination of S_d^* (potential early exercise at t_d^- when D is sufficiently deep)



The European call price function $V = c(S_d^- - D, T - t_d^+)$ falls below the exercise payoff line $\ell_1 : E = S_d^- - X$. Note that ℓ_1 lies to the left of the lower bound value line $\ell_2 : B = S_d^- - D - Xe^{-r(T-t_d)}$ when $D > X[1 - e^{-r(T-t_d)}]$. Here, S_d^* is the value of S_d^- at which the European call price curve cuts the exercise payoff line ℓ_1.

Let $C_d(S_d^-, T - t_d^-)$ denote the American call price at asset price level S_d^- and time t_d^-, where time to expiry is $T - t_d^-$. When $D > X[1 - e^{-r(T-t_d)}]$, then the American call price function right before t_d consists of two segments:

$$C_d(S_d^-, T - t_d^-) = \begin{cases} c(S_d^- - D, T - t_d^+) & \text{when } S_d^- < S_d^* \\ S_d^- - X & \text{when } S_d^- \geq S_d^* \end{cases}.$$

The American call price at $t < t_d$ can be obtained by the risk neutral valuation formula with the known value at t_d.

Property of the critical asset price, S_d^*

S_d^* depends on D, which decreases when D increases. This is because the price curve of $c(S_d^- - D, T - t_d^+)$ is lowered and it cuts the intrinsic value line ℓ_1 at a lower value of S_d^*. Financially, the holder chooses to receive the asset even at a lower asset value when the dividend is deeper.

Summary of the early exercise policies of American calls

- With no dividend, the decision of early exercise of an American option (call or put) depends on the competition between the time value of X and the loss of insurance value associated with the holding of the option.
- Early exercise of non-dividend paying American call is non-optimal since this leads to the loss of insurance value of the call plus the loss of time value of X.
- For an American call on a discrete dividend paying asset, it may become optimal to exercise at time right before the ex-dividend time, provided that the dividend amount is sizable and the call is sufficiently deep in-the-money ($S_d^- > S_d^*$). The critical asset price S_d^* is a decreasing function of the size of dividend. Early exercise of the American call at a lower asset price level leads to a greater loss of insurance value but the loss is offset by the more sizable dividend received since the asset is held at t_d via exercising the American call at t_d^-.

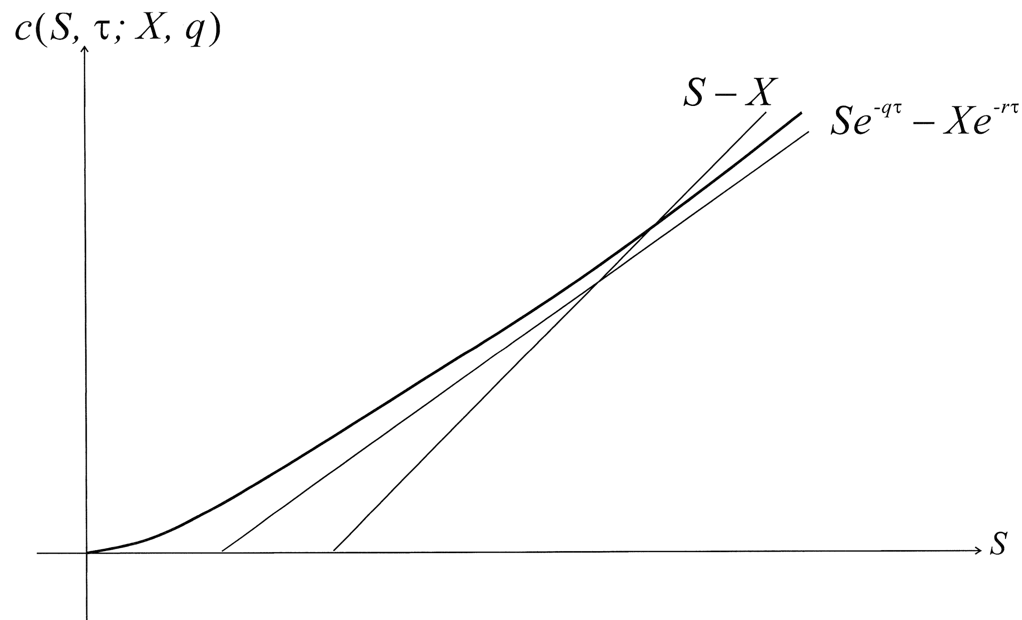
Continuous dividend model

Under constant dividend yield q, the dividend amount received during $(t, t+dt)$ from holding one unit of asset is $qS_t\,dt$. Also, $e^{-q(T-t)}$ unit of the asset at time t will become one unit at time T through the accumulation of the dividends into purchase of the asset.

Why do we consider dividend yield model?

- It is considered as a continuous approximation to the discrete dividends model. Otherwise, pricing under the discrete n-dividend model requires the joint distribution of asset prices on all dividend dates: $S_{t_{d_1}}, S_{t_{d_2}}, \cdots, S_{t_{d_n}}$.

- The foreign money market account, which serves as the underlying asset in exchange options, earns the foreign interest rate r_f that can be visualized as the dividend yield.



When the underlying asset pays dividend yield q, the lower bound of a European call $c(S, \tau; X, q)$ becomes $\max\left(Se^{-q\tau} - Xe^{-r\tau}, 0\right)$. As S becomes sufficiently high, $Se^{-q\tau} - Xe^{-r\tau}$ becomes less than $S - X$. Actually, as $S \to \infty$, the value of the European call tends to that of the forward, where

$$c(S, \tau; X, q) \to Se^{-q\tau} - Xe^{-r\tau}.$$

Smooth pasting condition at $S^*(\tau)$ under continuous dividend yield model

Value matching at $S^*(\tau) : C(S^*(\tau), \tau) = S^*(\tau) - X$. (i)

Smooth pasting at $S^*(\tau) : \dfrac{\partial C}{\partial S}(S^*(\tau), \tau) = 1$. (ii)

$S^*(\tau)$ can be visualized as the lowest asset price at which the American call does not depend on the time to expiry. That is,

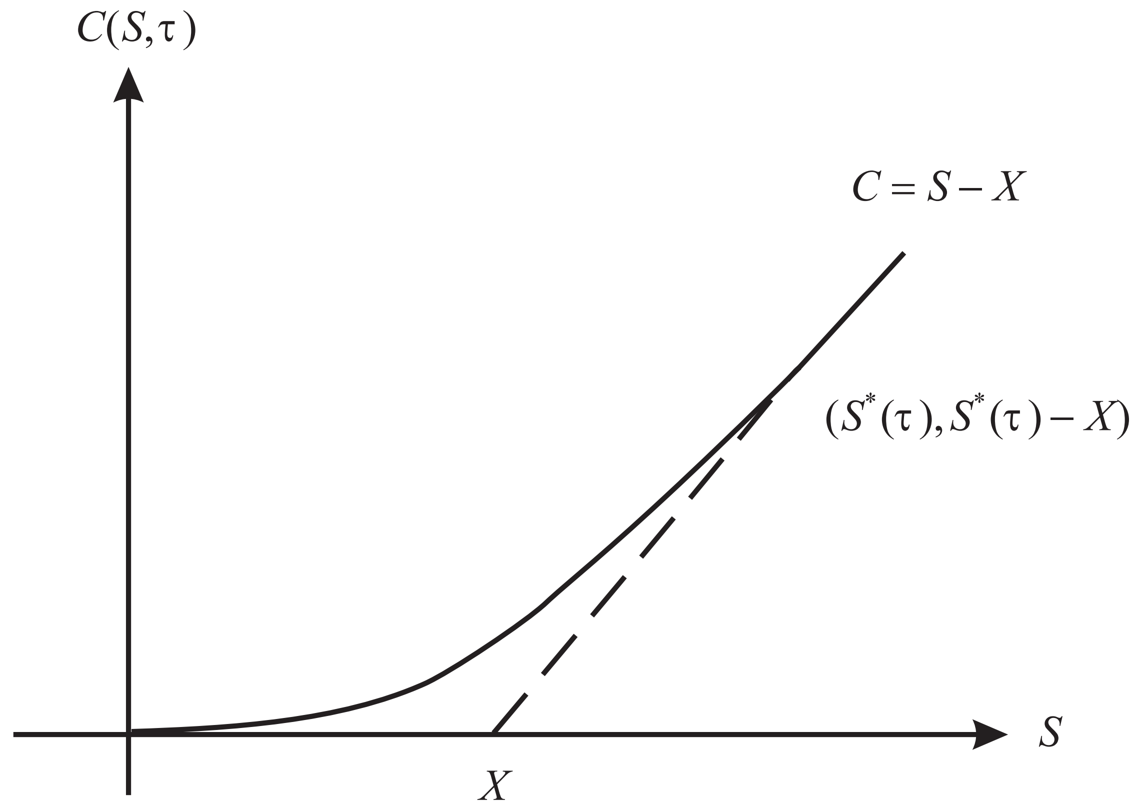
$$\frac{\partial C}{\partial \tau} = 0 \quad \text{at} \quad S = S^*(\tau). \tag{iii}$$

Find the total derivative of the American call price function with respect to τ at $(S^*(\tau), \tau)$ satisfying the value matching condition:

$$\frac{d}{d\tau}\left[C(S^*(\tau), \tau)\right] = \frac{\partial C}{\partial \tau}(S^*(\tau), \tau) + \frac{\partial C}{\partial S}(S^*(\tau), \tau)\frac{dS^*(\tau)}{d\tau} = \frac{dS^*(\tau)}{d\tau}$$

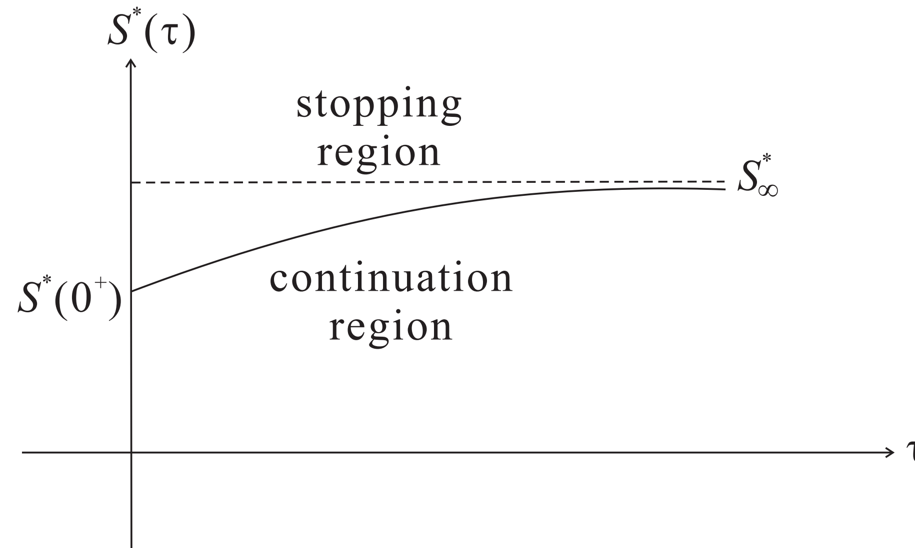
By eq(iii), we obtain $\dfrac{\partial C}{\partial S}(S^*(\tau), \tau) = 1$. The smooth pasting condition (ii) can also be derived from optimality of early exercise (to be visualized as the first order derivative condition).

American call on a continuous dividend yield paying asset



The option price curve of a longer-lived American call will be above that of its shorter-lived counterpart for all values of S. The upper price curve cuts the intrinsic value tangentially at a higher critical asset value $S^*(\tau)$. Hence, $S^*(\tau)$ for an American call is an increasing function of τ.

Properties of the optimal early exercise boundary $S^*(\tau)$ of an American call under continuous dividend yield model



X = strike price, r = riskfree interest rate, q = constant dividend yield, σ = volatility of asset price

$$S^*(0^+) = X \max\left(1, \frac{r}{q}\right), S^*_\infty = \frac{\mu_+}{\mu_+ - 1}X,$$

$$0 < \mu_+ = \frac{-\left(r - q - \frac{\sigma^2}{2}\right) + \sqrt{\left(r - q - \frac{\sigma^2}{2}\right)^2 + 2\sigma^2 r}}{\sigma^2}$$

Stopping region $= \{(S, \tau) : S \geq S^*(\tau)\}$, inside which the American call should be optimally exercised. When $S < S^*(\tau)$, it is optimal for the holder to continue holding the American call option.

1. At a higher value of τ, critical asset price $S^*(\tau)$ is larger since the American call option has to be deeper in-the-money to induce optimal early exercise. $S^*(\tau)$ is monotonically increasing with respect to τ with

   $S^*(0^+) = X \max\left(1, \frac{r}{q}\right)$ and $S^*_\infty = \frac{\mu_+}{\mu_+ - 1} X$. The determination of S^*_∞ requires a pricing model of the perpetual American call option.

2. $S^*(\tau)$ is a continuous function of τ when the asset price process is continuous.

3. $S^*(\tau) \geq X$ for $\tau \geq 0$

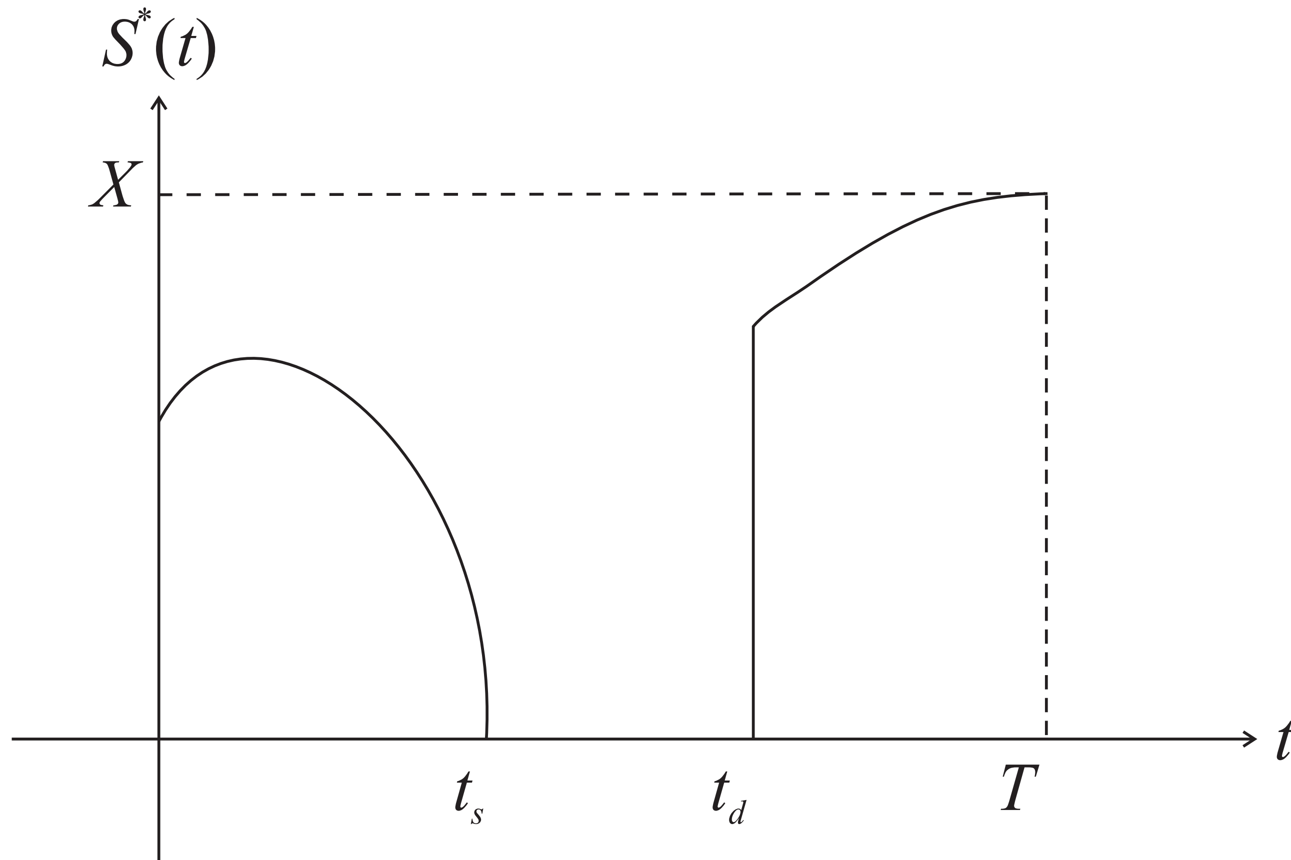
   Suppose $S^*(\tau) < X$, then the early exercise proceed $S^*(\tau) - X$ becomes negative. This must be ruled out.

One-dividend paying model for an American put

- A single dividend D is paid at t_d.
- Never exercise immediately prior to the dividend payment for time $t < t_d$, since gain in the time value of X is $X[e^{r(t_d - t)} - 1]$ cannot offset the loss in dividend. We would like to find t_s such that the holder of the American put is indifferent to the time value of strike and the dividend at t_d. Paying X later at t_d, the strike X at t_s grows to $X[e^{r(t_d - t_s)} - 1]$. Equating the time value of strike X with D at t_d, we obtain

$$X[e^{r(t_d - t_s)} - 1] = D \text{ giving } t_s = t_d - \frac{\ln\left(1 + \frac{D}{X}\right)}{r}.$$

- When $t < t_s$, early exercise is optimal when S falls below some critical asset price $S^*(t)$.



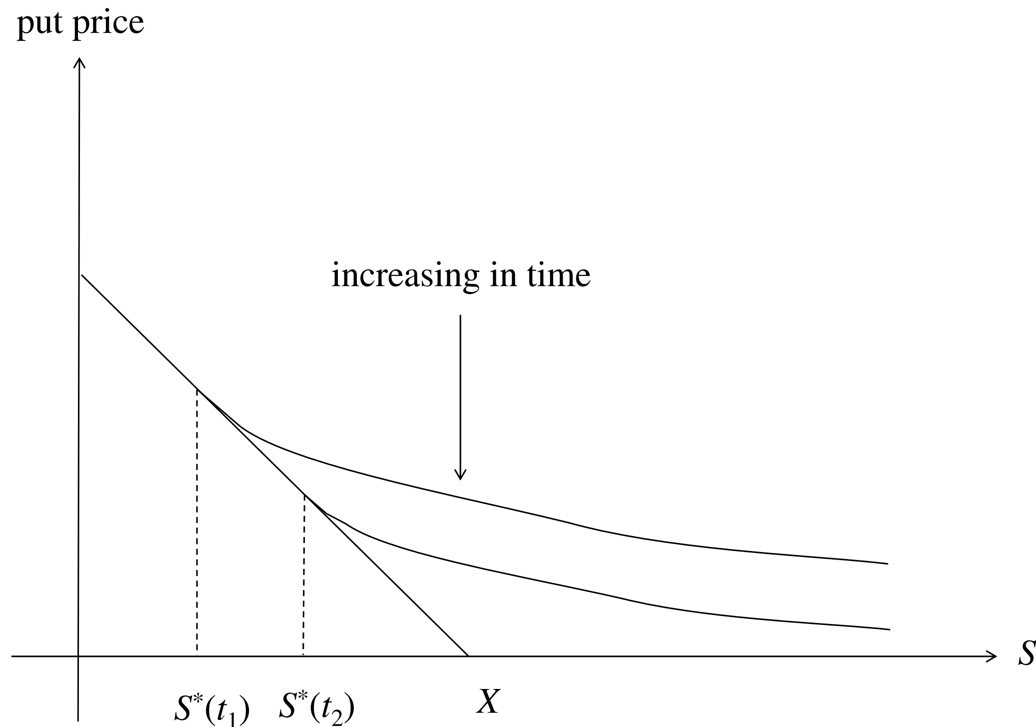
The behavior of the optimal exercise boundary $S^*(t)$ as a function of t for a one-dividend American put option. Note that $S^*(T) = X$ for an American put option whose underlying asset becomes non-dividend paying after t_d.

In summary, the optimal exercise boundary $S^*(t)$ of the one-dividend American put model exhibits the following behavior.

(i) When $t < t_s$, $S^*(t)$ first increases then decreases smoothly with increasing t until it drops to the zero value at t_s. When the calendar time is well before t_s, $S^*(t)$ is increasing as it follows a general trend of increasing monotonically in time. However, when the calendar time comes closer to t_s, $S^*(t)$ decreases in time in order to adopt the trend that $S^*(t)$ falls to zero when t reaches t_s.

(ii) $S^*(t)$ stays at the zero value in the time interval $[t_s, t_d]$.

(iii) When $t \in (t_d, T]$, the American put behaves like the European put counterpart since there will be no more dividend. As a result, $S^*(t)$ is a monotonically increasing function of t with $S^*(T) = X$.



Here, $t_d < t_1 < t_2 < T$. The put price curve at time t_1 intersects the intrinsic value line tangentially at $S^*(t_1)$. We observe: $S^*(t_1) < S^*(t_2) < X$. At longer time to expiry, the American put has to be deeper in-the-money (lower asset price) in order to induce optimal early exercise.