MAFS 5030 — Quantitative Modeling of Derivative Securities

Topic 2 – Risk neutral measure and Fundamental Theorem of Asset Pricing

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2.1 Discrete securities models

- The initial prices of M risky securities, denoted by $S_1(0), \cdots, S_M(0)$, are positive scalars that are known at $t = 0$.

- Their values at $t = 1$ are random variables, which are defined with respect to a sample space $\Omega = \{\omega_1, \omega_2, \cdots, \omega_K\}$ of K possible outcomes (or states of the world).

- At $t = 0$, the investors know the list of all possible outcomes, but which outcome does occur is revealed only at the end of the investment period $t = 1$.

- A probability measure P satisfying $P(\omega) > 0$, for all $\omega \in \Omega$, is defined on Ω.

- We use $\boldsymbol{S}$ to denote the price process $\{\boldsymbol{S}(t) : t = 0, 1\}$, where $\boldsymbol{S}(t)$ is the row vector $\boldsymbol{S}(t) = (S_1(t)\ S_2(t) \cdots S_M(t))$.

Consider 3 risky assets with time-0 price vector

$$\boldsymbol{S}(0) = (S_1(0) \quad S_2(0) \quad S_3(0)) = (1 \quad 2 \quad 3).$$

At time 1, there are 2 possible states of the world:

$$\begin{aligned}
\omega_1 &= \text{Hang Seng index is at or above } 22,000 \\
\omega_2 &= \text{Hang Seng index falls below } 22,000.
\end{aligned}$$

If ω_1 occurs, then

$$\boldsymbol{S}(1;\omega_1) = (1.2 \quad 2.1 \quad 3.4);$$

otherwise, ω_2 occurs and

$$\boldsymbol{S}(1;\omega_2) = (0.8 \quad 1.9 \quad 2.9).$$

- The possible values of the asset price process at $t = 1$ are listed in the following $K \times M$ matrix

$$S(1;\Omega) = \begin{pmatrix} S_1(1;\omega_1) & S_2(1;\omega_1) & \cdots & S_M(1;\omega_1) \\ S_1(1;\omega_2) & S_2(1;\omega_2) & \cdots & S_M(1;\omega_2) \\ \cdots & \cdots & \cdots & \cdots \\ S_1(1;\omega_K) & S_2(1;\omega_K) & \cdots & S_M(1;\omega_K) \end{pmatrix}.$$

- Since the assets are limited liability securities, the entries in $S(1;\Omega)$ are non-negative scalars.

- Existence of a strictly positive riskless security or bank account, whose value is denoted by S_0. Without loss of generality, we take $S_0(0) = 1$ and the value at time 1 to be $S_0(1) = 1 + r$, where $r \geq 0$ is the deterministic interest rate over one period.

Discounted terminal payoff matrix

- We define the discounted price process by

$$S^*(t) = S(t)/S_0(t), \quad t = 0, 1,$$

  that is, we use the riskless security as the *numeraire* or *accounting unit*.

- The payoff matrix of the discounted price processes of the M risky assets and the riskless security can be expressed in the form

$$\widehat{S}^*(1;\Omega) = \begin{pmatrix} 1 & S_1^*(1;\omega_1) & \cdots & S_M^*(1;\omega_1) \\ 1 & S_1^*(1;\omega_2) & \cdots & S_M^*(1;\omega_2) \\ \cdots & \cdots & \cdots & \cdots \\ 1 & S_1^*(1;\omega_K) & \cdots & S_M^*(1;\omega_K) \end{pmatrix}.$$

Trading strategies

- An investor adopts a *trading strategy* by selecting a portfolio of the M assets at time 0. A trading strategy is characterized by asset holding in the portfolio.

- The number of units of asset m held in the portfolio from $t = 0$ to $t = 1$ is denoted by $h_m, m = 0, 1, \cdots, M$.

- The scalars h_m can be positive (long holding), negative (short selling) or zero (no holding).

- An investor is endowed with an initial endowment V_0 at time 0 to set up the trading portfolio. How do we choose the portfolio holding of the assets such that the expected portfolio value at time 1 is maximized?

Portfolio value process

- Let $V = \{V_t : t = 0, 1\}$ denote the value process that represents the total value of the portfolio over time. It is seen that

$$V_t = h_0 S_0(t) + \sum_{m=1}^{M} h_m S_m(t), \quad t = 0, 1.$$

- Let G be the random variable that denotes the total gain generated by investing in the portfolio. We then have

$$G = h_0 r + \sum_{m=1}^{M} h_m \Delta S_m, \quad \Delta S_m = S_m(1) - S_m(0).$$

Account balancing

- If there is no withdrawal or addition of funds into the portfolio within the investment horizon, then

$$V_1 = V_0 + G.$$

- Suppose we use the bank account as the numeraire, and define the discounted value process by $V_t^* = V_t/S_0(t)$ and discounted gain by $G^* = V_1^* - V_0^*$, we then have

$$\begin{aligned} V_t^* &= h_0 + \sum_{m=1}^{M} h_m S_m^*(t), \quad t = 0, 1; \\ G^* &= V_1^* - V_0^* = \sum_{m=1}^{M} h_m \Delta S_m^*. \end{aligned}$$

Asset span

- Consider two risky securities whose discounted payoff vectors are

$$\boldsymbol{S}_1^*(1) = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \quad \text{and} \quad \boldsymbol{S}_2^*(1) = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}.$$

- The payoff vectors are used to form the discounted terminal payoff matrix

$$S^*(1) = \begin{pmatrix} 1 & 3 \\ 2 & 1 \\ 3 & 2 \end{pmatrix}.$$

- Let the current prices be represented by the row vector $\boldsymbol{S}^*(0) = (1 \quad 2)$.

- We write $\boldsymbol{h}$ as the column vector whose entries are the portfolio holding of the securities in the portfolio. The trading strategy is characterized by specifying $\boldsymbol{h}$. The current portfolio value and the discounted portfolio payoff are given by $\boldsymbol{S}^*(0)\boldsymbol{h}$ and $S^*(1)\boldsymbol{h}$, respectively.

- The set of all portfolio payoffs via different holding of securities is called the *asset span* $\mathcal{S}$. The asset span is seen to be the column space of the payoff matrix $S^*(1)$, which is a subspace in $\mathbb{R}^K$ spanned by the columns of $S^*(1)$.

  Given the matrix $A = \begin{pmatrix} \boldsymbol{c_1} & \cdots & \boldsymbol{c_M} \end{pmatrix}$, where $\boldsymbol{c_i}$ is the i^{th} column in A, $i = 1, \ldots, M$. The column space $C(A)$ of A is given by

$$C(A) = \{\boldsymbol{x} : \lambda_1 \boldsymbol{c_1} + \cdots + \lambda_M \boldsymbol{c_M}\}, \text{ where } \lambda_i, \ldots, \lambda_M \text{ are scalars.}$$

$$\begin{aligned} \text{asset span} &= \text{column space of } S^*(1) \\ &= \text{span}(\boldsymbol{S}_1^*(1) \cdots \boldsymbol{S}_M^*(1)) \end{aligned}$$

Recall that

$$\begin{aligned} \text{column rank} &= \text{dimension of column space} \\ &= \text{number of independent columns.} \end{aligned}$$

It is well known that number of independent columns = number of independent rows, so column rank = row rank = rank $\leq \min(K, M)$.

- In the above numerical example, the asset span consists of all vectors of the form $h_1 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + h_2 \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$, where h_1 and h_2 are scalars.

Redundant security and complete securities model

- If the discounted terminal payoff vector of an added security lies inside $\mathcal{S}$, then its payoff can be expressed as a linear combination of $\boldsymbol{S}_1^*(1)$ and $\boldsymbol{S}_2^*(1)$. In this case, it is said to be a *redundant security*. The added security is said to be replicable by some combination of existing securities.

- A securities model is said to be *complete* if every payoff vector lies inside the asset span. That is, all new securities (contingent claims) can be replicated by existing securities. This occurs if and only if the dimension of the asset span equals the number of possible states, that is, the asset span becomes the whole $\mathbb{R}^K$.

Given the securities model with 4 risky securities and 3 possible states of world:

$$S^*(1;\Omega) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 5 & 7 \\ 3 & 5 & 8 & 11 \end{pmatrix}, \quad \boldsymbol{S}(0) = (1 \quad 2 \quad 4 \quad 7).$$

Note that asset span $=$ span$(\boldsymbol{S}_1^*(1), \boldsymbol{S}_2^*(1))$, which has dimension $= 2 < 3 =$ number of possible states. Hence, the securities model is not complete! For example, the following security (contingent claim)

$$\boldsymbol{S}_\beta^*(1;\Omega) = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$$

does not lie in the asset span of the securities model. This is confirmed by non-existence of solution to

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 5 & 7 \\ 3 & 5 & 8 & 11 \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \\ h_3 \\ h_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}.$$

Pricing problem

Given a contingent claim that is replicable by existing securities, its price with reference to a given securities model is given by the cost of setting up the replicating portfolio. *Replication enforces price*.

Consider a new security with discounted payoff at $t = 1$ as given by

$$\boldsymbol{S}_\alpha^*(1;\Omega) = \begin{pmatrix} 5 \\ 8 \\ 13 \end{pmatrix},$$

which is seen to be

$$\boldsymbol{S}_\alpha^*(1;\Omega) = \boldsymbol{S}_2^*(1;\Omega) + \boldsymbol{S}_3^*(1;\Omega) = \boldsymbol{S}_1^*(1;\Omega) + 2\boldsymbol{S}_2^*(1;\Omega).$$

This new security is redundant. Unfortunately, the price of this security can be either

$$\boldsymbol{S}_2(0) + \boldsymbol{S}_3(0) = 6 \quad \text{or} \quad \boldsymbol{S}_1(0) + 2\boldsymbol{S}_2(0) = 5.$$

There are two possible prices, corresponding to two different choices of the replicating portfolio.

Question

How to modify $\boldsymbol{S}^*(0)$ so as to avoid the above ambiguity that portfolios with the same terminal payoff have different initial prices (failure of law of one price)?

Note that $S_3^*(1;\Omega) = S_1^*(1;\Omega) + S_2^*(1;\Omega)$ and $S_4^*(1;\Omega) = S_1^*(1;\Omega) + S_3^*(1;\Omega)$, both the third and fourth security are redundant securities. To achieve the law of one price, we modify $S_3(0)$ and $S_4(0)$ such that

$$S_3(0) = S_1(0) + S_2(0) = 3 \quad \text{and} \quad S_4(0) = 2S_1(0) + S_2(0) = 4.$$

If we delete the last two securities (redundant securities), then the law of one price would not be violated. Non-existence of redundant securities means $S^*(1;\Omega)$ has full column rank. That is, column rank = number of columns. This gives a sufficient condition for "law of one price".

2.2 Law of one price and dominant trading strategies

1. The law of one price states that all portfolios with the same terminal payoff have the same initial price.

2. Consider two portfolios with different portfolio holdings $\boldsymbol{h}$ and $\boldsymbol{h}'$. Suppose these two portfolios have the same discounted payoff, that is, $S^*(1)\boldsymbol{h} = S^*(1)\boldsymbol{h}'$, then the law of one price infers that $\boldsymbol{S}(0)\boldsymbol{h} = \boldsymbol{S}(0)\boldsymbol{h}'$.

3. The trading strategy $\boldsymbol{h}$ is obtained by solving

$$S^*(1)\boldsymbol{h} = \boldsymbol{S}^*_\alpha(1).$$

Solution exists if $\boldsymbol{S}^*_\alpha(1)$ lies in the asset span. Uniqueness of solution is equivalent to null space of $S^*(1)$ having zero dimension. There is only one trading strategy that replicates the security with discounted terminal payoff $\boldsymbol{S}^*_\alpha(1)$. In this case, the law of one price always holds.

Dominant trading strategies

A trading strategy $\mathcal{H}$ is said to be *dominant* if there exists another trading strategy $\widehat{\mathcal{H}}$ such that

$$V_0 = \widehat{V}_0 \qquad \text{and} \quad V_1(\omega) > \widehat{V}_1(\omega) \quad \text{for all } \omega \in \Omega.$$

- Suppose $\mathcal{H}$ dominates $\widehat{\mathcal{H}}$, we define a new trading strategy $\widetilde{\mathcal{H}} = \mathcal{H} - \widehat{\mathcal{H}}$. Let $\widetilde{V}_0$ and $\widetilde{V}_1$ denote the portfolio value of $\widetilde{\mathcal{H}}$ at $t = 0$ and $t = 1$, respectively. We then have $\widetilde{V}_0 = 0$ and $\widetilde{V}_1(\omega) > 0$ for all $\omega \in \Omega$.

- This trading strategy is dominant since it dominates the strategy which starts with zero value and does no investment at all.

- Equivalent definition: A dominant trading strategy exists if and only if there exists a trading strategy satisfying $V_0 < 0$ and $V_1(\omega) \geq 0$ for all $\omega \in \Omega$.

Law of one price and dominant trading strategy

If the law of one price fails, then it is possible to have two trading strategies $\boldsymbol{h}$ and $\boldsymbol{h}'$ such that $S^*(1)\boldsymbol{h} = S^*(1)\boldsymbol{h}'$ but $\boldsymbol{S}(0)\boldsymbol{h} > \boldsymbol{S}(0)\boldsymbol{h}'$.

Let $G^*(\omega)$ and $G^{*\prime}(\omega)$ denote the respective discounted gain corresponding to the trading strategies $\boldsymbol{h}$ and $\boldsymbol{h}'$. We then have $G^{*\prime}(\omega) > G^*(\omega)$ for all $\omega \in \Omega$, so there exists a dominant trading strategy. The corresponding dominant trading strategy is $\boldsymbol{h}' - \boldsymbol{h}$ so that $V_0 < 0$ but $V_1^*(\omega) = 0$ for all $\omega \in \Omega$.

Hence, the non-existence of dominant trading strategy implies the law of one price. However, the converse statement does not hold. The numerical example on p.45-46 shows that law of one price holds while dominant trading strategies exist.

Pricing functional

- Given that a contingent claim with discounted payoff $\boldsymbol{x}$ lies inside the asset span, the payoff can be generated by some linear combination of the securities in the securities model. We have $\boldsymbol{x} = S^*(1)\boldsymbol{h}$ for some $\boldsymbol{h} \in \mathbb{R}^M$. Existence of the solution $\boldsymbol{h}$ is guaranteed since $\boldsymbol{x}$ lies in the asset span, or equivalently, $\boldsymbol{x}$ lies in the column space of $S^*(1)$.

- The current value of the portfolio is $\boldsymbol{S}(0)\boldsymbol{h}$, where $\boldsymbol{S}(0)$ is the initial price vector. Be aware that $\boldsymbol{h}$ may not be unique.

- We may consider $\boldsymbol{S}(0)\boldsymbol{h}$ as a pricing functional $F(\boldsymbol{x})$ on the discounted payoff $\boldsymbol{x}$. If the law of one price holds, then the pricing functional is single-valued. Furthermore, it can be proven that it is a linear functional, that is,

$$F(\alpha_1\boldsymbol{x}_1 + \alpha_2\boldsymbol{x}_2) = \alpha_1 F(\boldsymbol{x}_1) + \alpha_2 F(\boldsymbol{x}_2)$$

  for any scalars α_1 and α_2 and payoffs $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$.

Arrow security and state price

- Let $\boldsymbol{e}_k$ denote the k^{th} coordinate vector in the vector space $\mathbb{R}^K$, where $\boldsymbol{e}_k$ assumes the value 1 in the k^{th} entry and zero in all other entries. The vector $\boldsymbol{e}_k$ can be considered as the discounted payoff vector of a security, and it is called the Arrow security of state k. The k^{th} Arrow security has unit discounted payoff when state k occurs and zero payoff otherwise.

- Suppose a pricing functional F exists such that F assigns unique value to each Arrow security. We write $s_k = F(\boldsymbol{e}_k)$, which is called the state price of state k. Note that state price must be non-negative. Consider the discounted terminal payoff vector

$$\boldsymbol{S}_\alpha^*(1) = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_K \end{pmatrix} = \sum_{k=1}^{K} \alpha_k \boldsymbol{e}_k,$$

then the initial price is given by

$$\begin{aligned} S_\alpha(0) &= F(\boldsymbol{S}^*_\alpha(1)) = F\left(\sum_{k=1}^{K} \alpha_k \boldsymbol{e}_k\right) \\ &= \sum_{k=1}^{K} \alpha_k F(\boldsymbol{e}_k) = \sum_{k=1}^{K} \alpha_k s_k. \end{aligned}$$

The above formula is consistent with the well known *Riesz Representation Theorem*. The Theorem states that any bounded linear functional $F(\boldsymbol{x})$ on $\mathbb{R}^n$ can be written as

$$F(\boldsymbol{x}) = \boldsymbol{v} \cdot \boldsymbol{x}.$$

That is, $F(\boldsymbol{x})$ is the dot product of some fixed $\boldsymbol{v} \in \mathbb{R}^n$ with $\boldsymbol{x}$. In the current context, the vector $\boldsymbol{v}$ is simply the vector whose components are the state prices, where

$$F(\boldsymbol{x}) = \begin{pmatrix} s_1 \\ \vdots \\ s_K \end{pmatrix} \cdot \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_K \end{pmatrix}.$$

Example – Calculation of the state prices

Given $F\left(\begin{pmatrix} 3 \\ 2 \end{pmatrix}\right) = 2.5$ and $F\left(\begin{pmatrix} 4 \\ 2 \end{pmatrix}\right) = 3$, find $F\left(\begin{pmatrix} 5 \\ 3 \end{pmatrix}\right)$.

By the linear property of pricing functional, we deduce that

$$F\left(\begin{pmatrix} 4 \\ 2 \end{pmatrix} - \begin{pmatrix} 3 \\ 2 \end{pmatrix}\right) = F\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) = 3 - 2.5 = 0.5 \text{ so that } s_1 = 0.5;$$

$$\begin{aligned} F\left(\frac{1}{2}\left[\begin{pmatrix} 3 \\ 2 \end{pmatrix} - 3\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right]\right) = F\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) &= \frac{1}{2}\left[F\left(\begin{pmatrix} 3 \\ 2 \end{pmatrix}\right) - 3F\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)\right] \\ &= \frac{1}{2}(2.5 - 3 \times 0.5) = 0.5 \end{aligned}$$

so that $s_2 = 0.5$.

By the linear property of pricing functional, the fair price of $\begin{pmatrix} 5 \\ 3 \end{pmatrix}$ is given by

$$F\left(\begin{pmatrix} 5 \\ 3 \end{pmatrix}\right) = 5F\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) + 3F\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) = 5s_1 + 3s_2 = 4.$$

The actual probabilities of occurrence of the two states are irrelevant in the pricing of the given contingent claim $\begin{pmatrix} 5 \\ 3 \end{pmatrix}$.

Lastly, we observe the following relation between the state price vector $(0.5 \quad 0.5)$, payoff matrix and initial price vector $(2.5 \quad 3)$.

$$\begin{pmatrix} 0.5 & 0.5 \end{pmatrix} \begin{pmatrix} 3 & 4 \\ 2 & 2 \end{pmatrix} = \begin{pmatrix} 2.5 & 3 \end{pmatrix}.$$

Later, we will establish the relation:

$$\boldsymbol{\pi} S^*(1) = \boldsymbol{S}(0),$$

where $\boldsymbol{\pi}$ is the row vector whose components are the state prices.

Replication: key concepts revisited

Given a securities model endowed with $S^*(1;\Omega)$ and $\boldsymbol{S}(0)$, can we find a trading strategy to form a portfolio that replicates a new security with discounted terminal payoff vector $\boldsymbol{S}^*_\alpha(1;\Omega)$ (also called a contingent claim) that is outside the universe of the M available risky securities in the securities model?

Replication means the terminal payoff of the replicating portfolio matches with that of the contingent claim under all scenarios of occurrence of the state of the world at $t = 1$.

Formation of the replicating portfolio is possible if we have *existence of solution* $\boldsymbol{h}$ to the following system

$$S^*(1;\Omega)\boldsymbol{h} = \boldsymbol{S}^*_\alpha(1;\Omega).$$

This is equivalent to the fact that "$\boldsymbol{S}^*_\alpha(1;\Omega)$ lies in the asset span (column space) of $S^*(1;\Omega)$". The solution $\boldsymbol{h}$ is the corresponding trading strategy. Note that $\boldsymbol{h}$ may not be unique.

Uniqueness of trading strategy

If $\boldsymbol{h}$ is unique, then there is only one trading strategy that generates the replicating portfolio. This occurs when the columns of $S^*(1;\Omega)$ are independent. Equivalently, column rank $= M$ and all securities are non-redundant. Mathematically, this is equivalent to observe that the homogeneous system

$$S^*(1;\Omega)\boldsymbol{h} = \boldsymbol{0}$$

admits only the trivial zero solution. In other words, the dimension of the null space of $S^*(1;\Omega)$ is zero.

When we have unique solution $\boldsymbol{h}$, the initial cost of setting up the replicating portfolio (price at time 0) as given by $\boldsymbol{S}(0)\boldsymbol{h}$ is unique. In this case, law of one price holds.

Completeness of securities model

If all contingent claims are replicable, then the securities model is said to be *complete*. Mathematically, the discount payoff vector of any contingent claim can be expressed as a linear combination of the columns in $S^*(1)$. This is equivalent to

$$\text{dim(asset span)} = K = \text{number of possible states},$$

that is, the column space spans $\mathbb{R}^K$. In this case, solution $\boldsymbol{h}$ always exists.

Orthogonal complement: null space and row space of matrix A

Given a matrix A, the null space of $\eta(A)$ is given by

$$\eta(A) = \{\boldsymbol{x} : \ A\boldsymbol{x} = \boldsymbol{0}\}.$$

Every vector in $\eta(A)$ is orthogonal to every row in A. If we define the orthogonal complement of $\eta(A)$ by

$$\eta(A)^{\perp} = \{\boldsymbol{y} : \ \boldsymbol{y} \cdot \boldsymbol{x} = 0 \text{ for all } \boldsymbol{x} \in \eta(A)\},$$

then $\eta(A)^{\perp}$ is seen to be the row space of A. Pick any vector $\boldsymbol{x}$ in the null space and any vector $\boldsymbol{y}$ in the row space, they must be orthogonal to each other.

By the orthogonal complement property, we deduce that

$$\text{row rank} + \text{dim(null space)} = M = \text{number of columns}.$$

Review of linear algebra property of existence and uniqueness of solution

Solution to the linear system of equations:

$$S^*(1)\boldsymbol{h} = \boldsymbol{S}_\alpha(1)$$

exists if and only if $\boldsymbol{S}_\alpha(1)$ lies in the column space of $S^*(1)$. If the column space of $S^*(1)$ is $\mathbb{R}^K$, then solution always exists.

The standard procedure is to check for *existence of solution* first, then explore whether *solution is unique*. If nonuniqueness of solution prevails, then the number of solutions is infinitely many. Actually, the number of solutions can only be either 0, 1 or ∞.

Note that dim(null space) $= 0$ does not mean existence of solution. It only means that if solution exists, then it must be unique.

Uniqueness of solution

Suppose nonzero solution, $\boldsymbol{h} \neq 0$, exists for the homogeneous system:

$$S^*(1)\boldsymbol{h} = \boldsymbol{0},$$

this means dim(null space) > 0.

If $\widehat{\boldsymbol{h}}$ is a solution to $S^*(1)\boldsymbol{x} = \boldsymbol{S}_\alpha(1)$, then $\widehat{\boldsymbol{h}} + \boldsymbol{h}$ is also a solution since

$$S^*(1)(\widehat{\boldsymbol{h}} + \boldsymbol{h}) = S^*(1)\widehat{\boldsymbol{h}} + S^*(1)\boldsymbol{h} = S^*(1)\widehat{\boldsymbol{h}} = \boldsymbol{S}_\alpha(1).$$

We have just established

dim(null space) $> 0 \Rightarrow$ nonuniqueness of solution.

On the other hand, nonuniqueness of solution means there exists distinct solution vectors $\boldsymbol{h}_1$ and $\boldsymbol{h}_2$ that satisfy

$$S^*(1)\boldsymbol{h}_1 = \boldsymbol{S}_\alpha(1) \quad \text{and} \quad S^*(1)\boldsymbol{h}_2 = \boldsymbol{S}_\alpha(1).$$

This implies

$$S^*(1)(\boldsymbol{h}_1 - \boldsymbol{h}_2) = \boldsymbol{0}, \text{ where } \boldsymbol{h}_1 \neq \boldsymbol{h}_2,$$

so nonzero solution exists for the homogeneous system. Therefore, we obtain dim(null space) > 0. We then establish

$$\text{nonuniqueness of solution} \Rightarrow \text{dim(null space)} > 0.$$

Combining the results together, we obtain

dim(null space) $= 0$ if and only if solution must be unique if solution exists.

Matrix properties of $S^*(1)$ and financial economics concepts

The securities model is endowed with

(i) discounted terminal payoff matrix $= \left(\boldsymbol{S}_1^*(1) \quad \cdots \quad \boldsymbol{S}_M^*(1)\right)$, and
(ii) initial price vector; $\boldsymbol{S}(0) = (S_1(0) \cdots S_M(0))$.

Recall that

$$\text{column rank } \leq \min(K, M)$$

where K = number of possible states, M = number of risky securities.

List of terms:	redundant securities, complete model, replicating portfolio, asset holding, asset span, law of one price, dominant trading strategy, pricing functional, Arrow securities, state prices

Given a risky security with the discounted terminal payoff vector $\boldsymbol{S}_\alpha^*(1)$, we are interested to explore the existence and uniqueness of solution to

$$S^*(1)\boldsymbol{h} = \boldsymbol{S}_\alpha^*(1).$$

Here, $\boldsymbol{h}$ is the asset holding of the portfolio that replicates $\boldsymbol{S}_\alpha^*(1)$.

(i) column rank $= K$

asset span $= \mathbb{R}^K$, so the securities model is complete. Any risky securities is replicable. In this case, solution $\boldsymbol{h}$ always exists.

(ii) column rank $= M$ [all columns of $S^*(1)$ are independent]

All securities are non-redundant. In this case, $\boldsymbol{h}$ may or may not exist. However, if $\boldsymbol{h}$ exists, then it must be unique. The price of any replicable security is unique.

(iii) column rank $< K$

Solution $\boldsymbol{h}$ exists if and only if $\boldsymbol{S}^*_\alpha(1)$ lies in the asset span. However, there is no guarantee for the uniqueness of solution.

(iv) column rank $< M$

Existence of redundant securities, so the replicating portfolio would not be unique. The law of one price may fail.

The numerical example on p.14-16 observes column rank $= 2$, $K = 3$ and $M = 4$. In one case, we observe failure of law of one price.

To explore "law of one price" and "existence of dominant trading strategies", one has to consider the nature of the solution to the linear system of equations: $\boldsymbol{S}(0) = \boldsymbol{\pi} S^*(1)$.

Solution exists if and only if $\boldsymbol{S}(0)$ lies in the row space of $S^*(1)$; or equivalently, $\boldsymbol{S}(0)$ can be expressed as a linear combination of the rows of $S^*(1)$.

Lemma on Law of one price

Law of one price holds if and only if solution to

$$\boldsymbol{\pi} S^*(1) = \boldsymbol{S}(0) \qquad (A)$$

exists.

1. Suppose solution to (A) exists, let $\boldsymbol{h}$ and $\boldsymbol{h'}$ be two trading strategies such that their respective discounted terminal payoff $\boldsymbol{V}$ and $\boldsymbol{V}'$ are the same. That is,

$$S^*(1)\boldsymbol{h} = \boldsymbol{V} = \boldsymbol{V}' = S^*(1)\boldsymbol{h'}.$$

Since $\boldsymbol{\pi}$ exists, we multiply both sides by $\boldsymbol{\pi}$ and obtain

$$\boldsymbol{\pi} S^*(1)(\boldsymbol{h} - \boldsymbol{h'}) = 0.$$

Noting that $\boldsymbol{\pi} S^*(1) = \boldsymbol{S}(0)$, we obtain

$$\boldsymbol{S}(0)(\boldsymbol{h} - \boldsymbol{h'}) = 0 \quad \text{so that } V_0 = V_0'.$$

Hence, law of one price is established.

2. Suppose solution to (A) does not exist for the given $\boldsymbol{S}(0)$, this implies that $\boldsymbol{S}(0)$ that does not lie in the row space of $S^*(1)$. The row space of $S^*(1)$ does not span the whole $\mathbb{R}^M$. Therefore, dim(row space of $S^*(1)) < M$, where M is the number of securities = number of columns in $S^*(1)$.

   Recall that

$$\text{dim(null space of } S^*(1)) + \text{rank}(S^*(1)) = M$$

   so that dim(null space of $S^*(1)) > 0$.

   Hence, there exists non-zero solution $\boldsymbol{h}$ to

$$S^*(1)\boldsymbol{h} = \boldsymbol{0}.$$

   Note that $\boldsymbol{h}$ is orthogonal to all rows of $S^*(1)$, and recall that row space = orthogonal complement of null space.

Also, there always exists a non-zero solution $\boldsymbol{h}$ that is not orthogonal to $\boldsymbol{S}(0)$. If otherwise, $\boldsymbol{S}(0)$ is orthogonal to every vector in the null space, so it lies in the orthogonal complement of the null space (that is, row space). This leads to a contradiction since $\boldsymbol{S}(0)$ does not lie in the row space.

Consider the above choice of $\boldsymbol{h}$ that lies in the null space, where $\boldsymbol{S}(0)\boldsymbol{h} \neq 0$; we let $\boldsymbol{h} = \boldsymbol{h}_1 - \boldsymbol{h}_2$, where $\boldsymbol{h}_1 \neq \boldsymbol{h}_2$. Since $S^*(1)(\boldsymbol{h}_1 - \boldsymbol{h}_2) = \boldsymbol{0}$, then there exist two distinct trading strategies such that

$$S^*(1)\boldsymbol{h}_1 = S^*(1)\boldsymbol{h}_2.$$

The two strategies have the same discounted terminal payoff under all states of the world. However, their initial prices are unequal since

$$\boldsymbol{S}(0)\boldsymbol{h}_1 \neq \boldsymbol{S}(0)\boldsymbol{h}_2,$$

by virtue of the property: $\boldsymbol{S}(0)\boldsymbol{h} \neq 0$. Hence, the law of one price does not hold.

Linear pricing measure

We consider securities models with the inclusion of the riskfree security. A non-negative row vector $\boldsymbol{q} = (q(\omega_1) \cdots q(\omega_K))$ is said to be a linear pricing measure of the securities model if for every trading strategy the portfolio values at $t = 0$ and $t = 1$ are related by

$$V_0 = \sum_{k=1}^{K} q(\omega_k) V_1^*(\omega_k).$$

Note that $\boldsymbol{q}$ is not required to be unique.

Note that the definition requires that the same initial price V_0 is always resulted. Also, there is no dependence of V_0 on the asset holding of the portfolio so that two portfolios with different holdings of securities but the same terminal payoff for all states of the world would have the same price. Implicitly, this implies that the law of one price holds. The rigorous justification of the above statement will be presented later.

If the actual probability of occurrence $P(\omega_k)$ can be determined without ambiguity and inconsistency among different investors (like casino games), then the fair price is given by the expectation of discounted terminal payoff under the actual probability. Story about replication and knowledge of securities model are *NOT* required.

Suppose we take the holding amount of every risky security to be zero, thereby $h_1 = h_2 = \cdots = h_M = 0$, leaving only the riskfree security. This gives

$$V_0 = h_0 = \sum_{k=1}^{K} q(\omega_k) h_0$$

so that

$$\sum_{k=1}^{K} q(\omega_k) = 1.$$

Since we have taken $q(\omega_k) \geq 0, k = 1, \cdots, K$, and their sum is one, we may interpret $q(\omega_k)$ as a probability measure on the sample space Ω.

Relation between linear pricing measure and Arrow security

Suppose that the securities model is complete. By taking the portfolio to have the same terminal payoff as that of the k^{th} Arrow security, we obtain

$$s_k = q(\omega_k), \quad k = 1, 2, \cdots, K.$$

That is, the state price of the k^{th} state is simply $q(\omega_k)$. This is not surprising when we observe $\alpha_k = V_1^*(\omega_k)$ and compare the two different expressions for the initial value of the security and that of its replicating portfolio:

$$V_0 = \sum_{k=1}^{K} q(\omega_k) V_1^*(\omega_k) \quad \text{and} \quad S_\alpha(0) = \sum_{k=1}^{K} \alpha_k s_k.$$

Later, we show that uniqueness of $q(\omega_k)$ is equivalent to market completeness. Under market completeness and state prices are nonnegative, then state prices are the same as the linear pricing probabilities.

By taking the portfolio weights to be zero except for the m^{th} security, we have

$$S_m(0) = \sum_{k=1}^{K} q(\omega_k) S_m^*(1; \omega_k), \quad m = 0, 1, \cdots, M.$$

Putting these $m + 1$ components into the matrix form, we obtain

$$\widehat{\boldsymbol{S}}(0) = \boldsymbol{q} \widehat{S}^*(1; \Omega), \quad \boldsymbol{q} \geq \boldsymbol{0}.$$

The system of algebraic equations resembles that in law of one price discussion except that we require (i) nonnegative solution $\boldsymbol{q} \geq \mathbf{0}$, (ii) inclusion of the riskfree asset.

We may rewrite the $M+1$ equations as

$$\sum_{k=1}^{K} q(w_k)\Delta S_m^*(1;\omega_k) = 0, \quad m = 0, 1, \ldots, M.$$

Note that $q(\omega_k)$ is not related to the actual probability of occurrence of the state k. These probability values are determined using information of the securities model: $\boldsymbol{S}(0)$ and $S^*(1)$. The underlying principle is replication. Prices of existing securities dictate the fair price of a contingent claim.

Numerical example

Take $\widehat{S}^*(1) = \begin{pmatrix} 1 & 2 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}$ and $\widehat{\boldsymbol{S}}(0) = (1 \quad 1\frac{1}{3})$, then

$$\boldsymbol{q} = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}$$

is a linear pricing measure. The linear pricing measure is not unique! Actually, we have $q(\omega_1) = \frac{1}{3}$ and $q_2(\omega_2) + q(\omega_3) = \frac{2}{3}$.

- The securities model is not complete. Though $\boldsymbol{e}_1$ is replicable and its initial price is $\frac{1}{3}$, but $\boldsymbol{e}_2$ and $\boldsymbol{e}_3$ are not replicable so the state price of ω_2 and ω_3 do not exist. Incompleteness of securities model is equivalent to nonuniqueness of linear pricing measure.

Suppose we add the new risky security with discounted terminal payoff $\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ and initial price $\frac{2}{3}$ into the securities model, then the securities model becomes complete. It is unreasonable to set the initial price of $\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ to be below that of $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. This leads to negative state price of the second Arrow security. Indeed, we obtain the following state prices

$$s_1 = \frac{1}{3}, \quad s_2 = -\frac{1}{3} \quad s_3 = 1.$$

In this case, law of one price holds but dominant trading strategy exists. For example, we may take

$$V_1^*(\omega) = \begin{pmatrix} 3 \\ 6 \\ 1 \end{pmatrix} > \mathbf{0}, \quad V_0 = 3s_1 + 6s_2 + s_3 = 0.$$

Law of one price and existence of solution

Take $\widehat{S}^*(1;\Omega) = \begin{pmatrix} 1 & 2 & 6 & 9 \\ 1 & 3 & 3 & 7 \\ 1 & 6 & 12 & 19 \end{pmatrix}$, the sum of the first 3 columns gives the fourth column. The first column corresponds to the discounted terminal payoff of the riskfree security under the 3 possible states of the world. The third risky security is a redundant security.

Let $\widehat{\boldsymbol{S}}(0) = (1 \quad 2 \quad 3 \quad k)$. We observe that solution to

$$(1 \quad 2 \quad 3 \quad k) = (\pi_1 \quad \pi_2 \quad \pi_3) \begin{pmatrix} 1 & 2 & 6 & 9 \\ 1 & 3 & 3 & 7 \\ 1 & 6 & 12 & 19 \end{pmatrix} \qquad (A)$$

exists if and only if $k = 6$. That is, $S_3(0) = S_0(0) + S_1(0) + S_2(0)$.

When $k \neq 6$, the law of one price does not hold. The last equation: $9\pi_1 + 7\pi_2 + 19\pi_3 = k \neq 6$ is inconsistent with the first 3 equations. One may check that $(1 \quad 2 \quad 3 \quad 6)$ can be expressed as a linear combination of the rows of $\widehat{S}^*(1;\Omega)$.

We consider the linear system

$$\widehat{\boldsymbol{S}}(0) = \boldsymbol{\pi}\widehat{S}^*(1;\Omega),$$

solution exists if and only if $\widehat{\boldsymbol{S}}^*(0)$ lies in the row space of $\widehat{S}^*(1;\Omega)$. Uniqueness follows if the rows of $\widehat{S}^*(1;\Omega)$ are independent. Since

$$\boldsymbol{S}_3^*(1;\Omega) = \boldsymbol{S}_0^*(1;\Omega) + \boldsymbol{S}_1^*(1;\Omega) + \boldsymbol{S}_2^*(1;\Omega),$$

the third risky security is replicable by holding one unit of each of the riskfree security and the first two risky securities. The initial price must observe the same relation in order that the law of one price holds.

Here, we have $\dim(\widehat{S}(1;\Omega)) = 3 =$ number of independent rows. Recall that the law of one price holds if and only if we have existence of solution to the linear system. In this example, when $k = 6$, we obtain

$$\boldsymbol{\pi} = \begin{pmatrix} \frac{1}{2} & \frac{2}{3} & -\frac{1}{6} \end{pmatrix}.$$

The solution $\boldsymbol{\pi}$ is unique since the rows are independent. This is *not* a linear pricing measure.

Law of one price holds while dominant trading strategies exist

Consider a securities model with 2 risky securities and the riskfree security, and there are 3 possible states. The current discounted price vector $\widehat{\boldsymbol{S}}(0)$ is $(1 \quad 4 \quad 2)$ and the discounted payoff matrix at $t = 1$ is $\widehat{S}^*(1) = \begin{pmatrix} 1 & 4 & 3 \\ 1 & 3 & 2 \\ 1 & 2 & 4 \end{pmatrix}$.

Here, the law of one price holds since the only solution to $\widehat{S}^*(1)\boldsymbol{h} = \boldsymbol{0}$ is $\boldsymbol{h} = \boldsymbol{0}$. This is because the columns of $\widehat{S}^*(1)$ are independent so that the dimension of the nullspace of $\widehat{S}^*(1)$ is zero.

The linear pricing probabilities $q(\omega_1), q(\omega_2)$ and $q(\omega_3)$, if exist, should satisfy the following linear system of equations:

$$\begin{aligned}1 &= q(\omega_1) + q(\omega_2) + q(\omega_3)\\ 4 &= 4q(\omega_1) + 3q(\omega_2) + 2q(\omega_3)\\ 2 &= 3q(\omega_1) + 2q(\omega_2) + 4q(\omega_3).\end{aligned}$$

Solving the above equations, we obtain $q(\omega_1) = q(\omega_2) = 2/3$ and $q(\omega_3) = -1/3$.

- Since not all the pricing probabilities are non-negative, the linear pricing measure does not exist for this securities model.

Theorem

There exists a linear pricing measure if and only if there are no dominant trading strategies.

The above linear pricing measure theorem can be seen to be a direct consequence of the Farkas Lemma.

Farkas Lemma

There does not exist $\boldsymbol{h} \in \mathbb{R}^M$ such that

$$\widehat{S}^*(1;\Omega)\boldsymbol{h} > \boldsymbol{0} \quad \text{and} \quad \widehat{\boldsymbol{S}}(0)\boldsymbol{h} = 0$$

if and only if there exists $\boldsymbol{q} \in \mathbb{R}^K$ such that

$$\widehat{\boldsymbol{S}}(0) = \boldsymbol{q}\widehat{S}^*(1;\Omega) \quad \text{and} \quad \boldsymbol{q} \geq \boldsymbol{0}.$$

Given that the security lies in the asset span, or equivalently replicable, we can deduce that law of one price holds by observing either

(i) null space of $S^*(1)$ has zero dimension, or

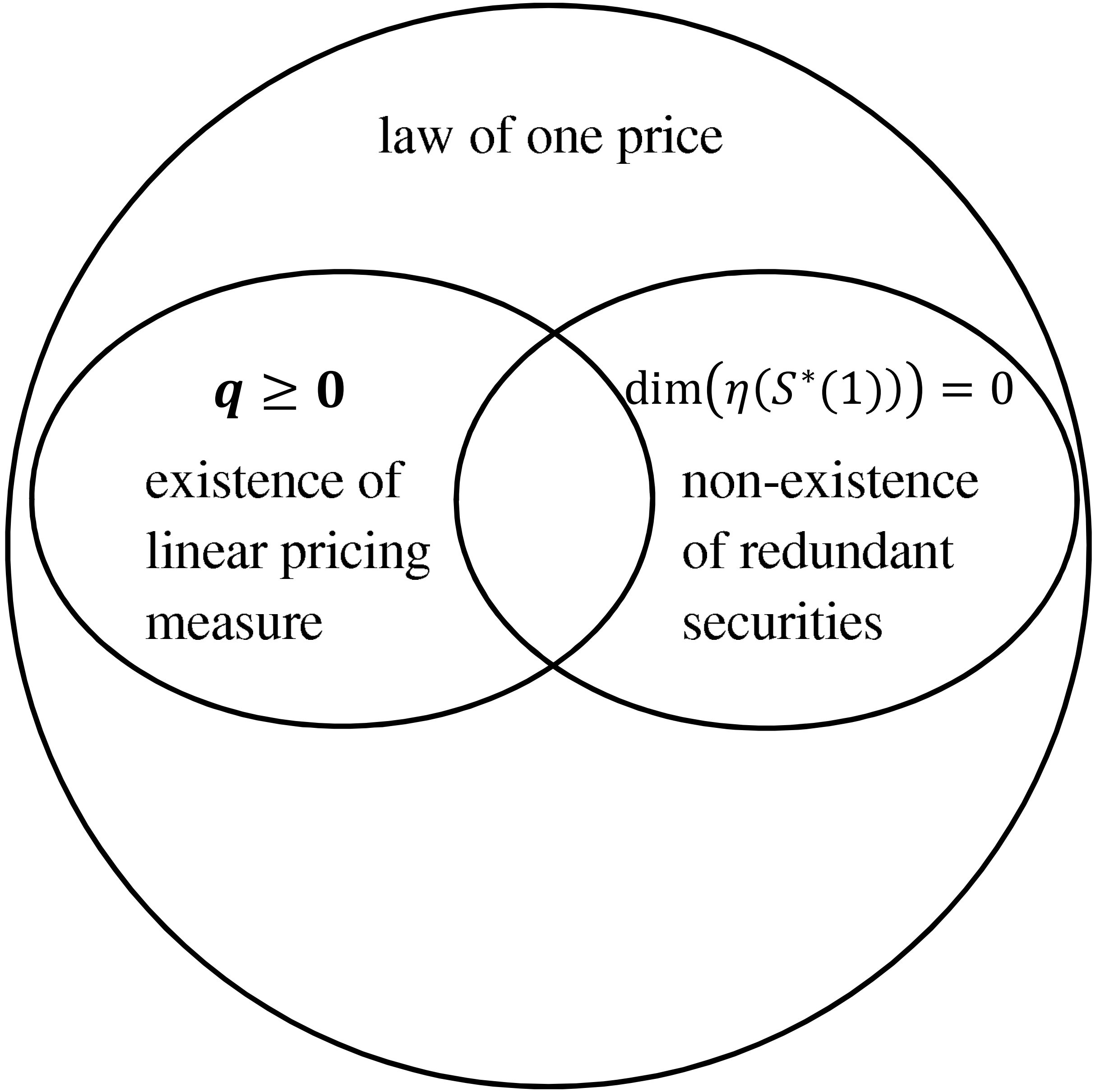
(ii) existence of a linear pricing measure.

Both (i) and (ii) represent the various forms of sufficient condition for the law of one price.

Remarks

1. Condition (i) is equivalent to non-existence of redundant securities.

2. Condition (i) and condition (ii) are not equivalent, nor one implies the other.



Various forms of sufficient condition for the law of one price, given that the security is replicable.

2.3 Fundamental Theorem of Asset Pricing: arbitrage free and risk neutral measure

Absence of arbitrage opportunities

- An *arbitrage opportunity* is some trading strategy that has the following properties: (i) $V_0 = 0$, (ii) $V_1^*(\omega) \geq 0$ with strict inequality at least for one state.

- The existence of a dominant strategy requires a portfolio with initial zero wealth to end up with a *strictly* positive wealth in all states.

- The existence of a dominant trading strategy implies the existence of an arbitrage opportunity, but the converse is not necessarily true.

Risk neutral probability measure

A probability measure Q on Ω is a risk neutral probability measure if it satisfies

(i) $Q(\omega) > 0$ for all $\omega \in \Omega$, and

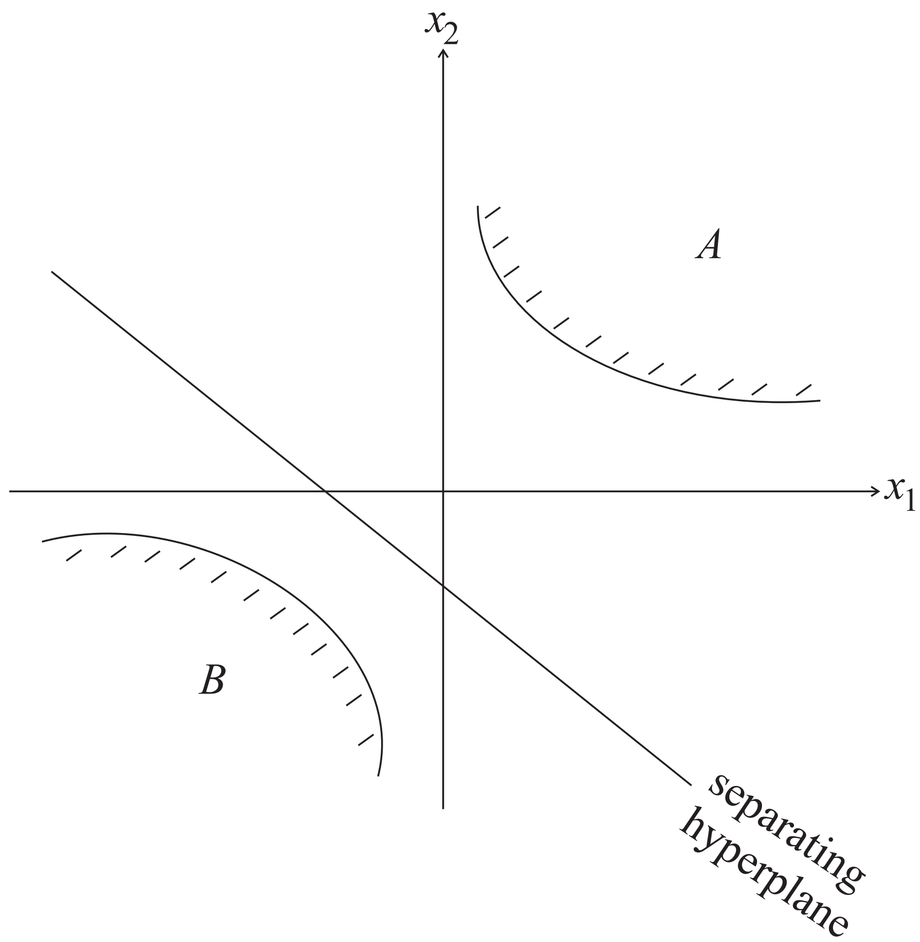
(ii) $E_Q[\Delta S_m^*] = 0, m = 0, 1, \cdots, M$, where E_Q denotes the expectation under Q. The expectation of the discounted gain of any security in the securities model under Q is zero.

Note that $E_Q[\Delta S_m^*] = 0$ is equivalent to $S_m(0) = \sum_{k=1}^{K} Q(\omega_k) S_m^*(1; \omega_k)$.

When $m = 0$, this leads to $\sum_{k=1}^{K} Q(\omega_k) = 1$.

Under absence of arbitrage opportunities, every investor should use $Q(\omega)$ (though not necessarily unique) to find the fair value of an attainable contingent claim, paying no regard to $P(\omega)$.

Separating hyperplane



The hyperplane (represented by a line in $\mathbb{R}^2$) separates the two convex sets A and B in $\mathbb{R}^2$. A set C is convex if any convex combination $\lambda \boldsymbol{x} + (1 - \lambda)\boldsymbol{y}, 0 \le \lambda \le 1$, of a pair of vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ in C also lies in C.

The hyperplane $[\boldsymbol{f}, \alpha]$ separates the sets A and B in $\mathbb{R}^n$ if there exists α such that $\boldsymbol{f} \cdot \boldsymbol{x} \geq \alpha$ for all $\boldsymbol{x} \in A$ and $\boldsymbol{f} \cdot \boldsymbol{y} < \alpha$ for all $\boldsymbol{y} \in B$.

For example, the hyperplane $\left[\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, 0 \right]$ separates the two disjoint convex sets $A = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} : x_1 \geq 0, x_2 \geq 0, x_3 \geq 0 \right\}$

and $B = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} : x_1 < 0, x_2 < 0, x_3 < 0 \right\}$ in $\mathbb{R}^3$.

Fundamental Theorem of Asset Pricing

No arbitrage opportunities exist if and only if there exists a risk neutral probability measure Q.

Proof of Theorem "$\Leftarrow$ part".
Assume that a risk neutral probability measure Q exists, that is, $\widehat{\boldsymbol{S}}(0) = \boldsymbol{\pi}\widehat{S}^*(1;\Omega)$, where $\boldsymbol{\pi} = (Q(\omega_1)\cdots Q(\omega_K))$ and $\boldsymbol{\pi} > \mathbf{0}$. We would like to show that it is never possible to construct a trading strategy that represents an arbitrage opportunity.

Suppose we claim that an arbitrage opportunity exists where there exists a trading strategy $\boldsymbol{h} = (h_0\ h_1\ \cdots\ h_M)^T \in \mathbb{R}^{M+1}$ such that $\widehat{\boldsymbol{S}}(0)\boldsymbol{h} = 0$ and $\widehat{S}^*(1;\Omega)\boldsymbol{h} \geq 0$ in all $\omega \in \Omega$ and with strict inequality in at least one state. Now consider $\widehat{\boldsymbol{S}}(0)\boldsymbol{h} = \boldsymbol{\pi}\widehat{S}^*(1;\Omega)\boldsymbol{h}$, it is seen that $\widehat{\boldsymbol{S}}(0)\boldsymbol{h} > 0$ since all entries in $\boldsymbol{\pi}$ are strictly positive and entries in $\widehat{S}^*(1;\Omega)\boldsymbol{h}$ are either zero or strictly positive (at least for one state). This leads to a contradiction since it is impossible to have $\widehat{\boldsymbol{S}}(0)\boldsymbol{h} = 0$ and $\widehat{S}^*(1;\Omega)\boldsymbol{h} \geq \mathbf{0}$ in all $\omega \in \Omega$, with strict inequality in at least one state.

"$\Rightarrow$ part".

First, we define the subset U in $\mathbb{R}^{K+1}$ which consists of vectors of the form $\begin{pmatrix} -\widehat{\boldsymbol{S}}^*(0)\boldsymbol{h} \\ \widehat{\boldsymbol{S}}^*(1;\omega_1)\boldsymbol{h} \\ \vdots \\ \widehat{\boldsymbol{S}}^*(1;\omega_K)\boldsymbol{h} \end{pmatrix}$, where $\widehat{\boldsymbol{S}}^*(1;\omega_k)$ is the k^{th} row in $\widehat{S}^*(1;\Omega)$ and $\boldsymbol{h} \in \mathbb{R}^{M+1}$ represents a trading strategy. This subset is seen to be a subspace. The convexity property of U is obvious.

Consider another subset $\mathbb{R}_+^{K+1}$ defined by

$$\mathbb{R}_+^{K+1} = \{\boldsymbol{x} = (x_0\ x_1 \cdots x_K)^T \in \mathbb{R}^{K+1} : x_i \geq 0 \quad \text{for all} \quad 0 \leq i \leq K\},$$

which is a convex set in $\mathbb{R}^{K+1}$.

We claim that the non-existence of arbitrage opportunities implies that U and $\mathbb{R}_+^{K+1}$ can only have the zero vector in common.

Assume the contrary, suppose there exists a non-zero vector $\boldsymbol{x} \in U \cap \mathbb{R}_{+}^{K+1}$. Since there is a trading strategy vector $\boldsymbol{h}$ associated with every vector in U, it suffices to show that the trading strategy $\boldsymbol{h}$ associated with $\boldsymbol{x}$ always represents an arbitrage opportunity.

We consider the following two cases: $-\widehat{\boldsymbol{S}}^*(0)\boldsymbol{h} = 0$ or $-\widehat{\boldsymbol{S}}^*(0)\boldsymbol{h} > 0$.

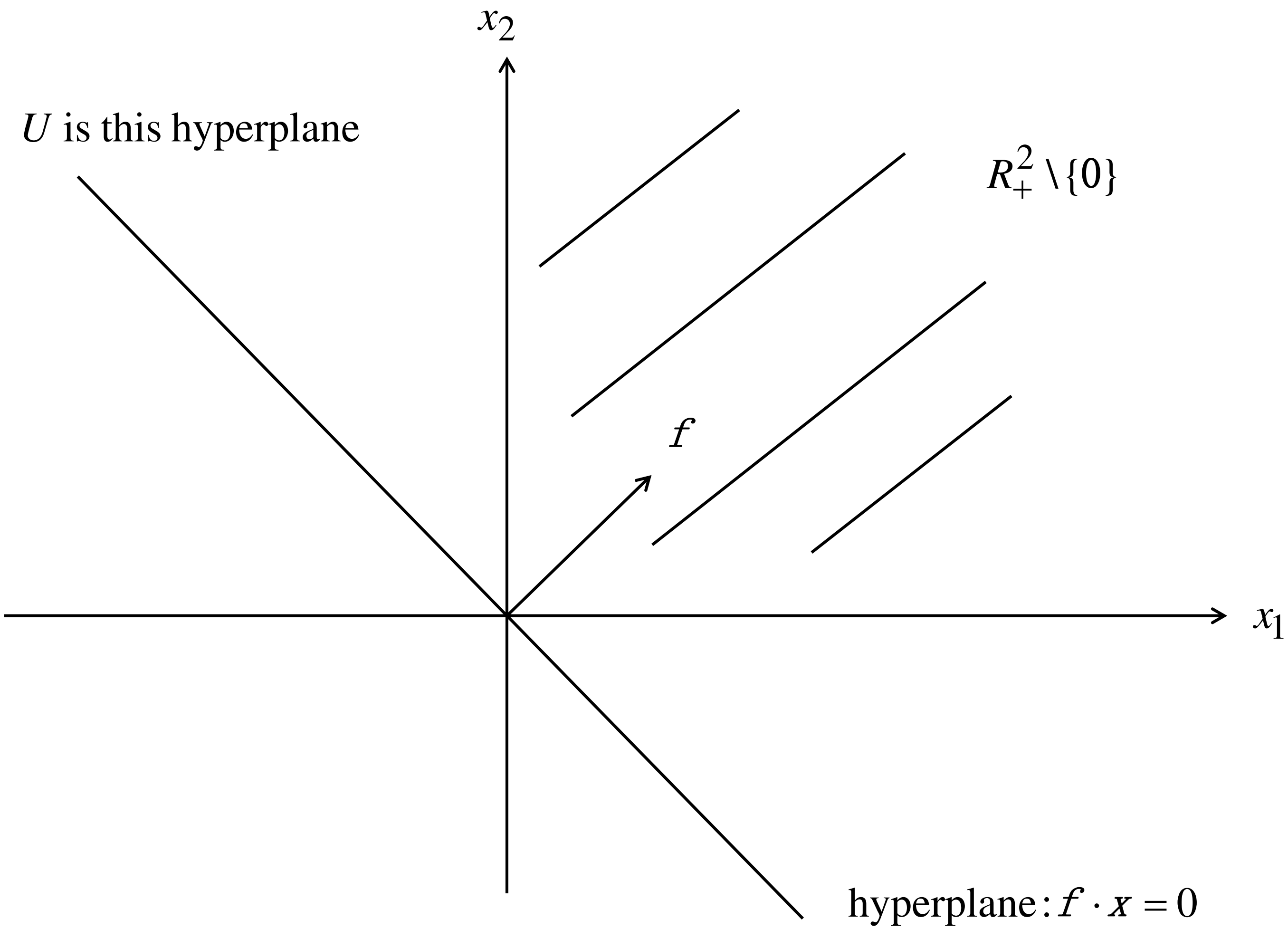
(i) When $\widehat{\boldsymbol{S}}^*(0)\boldsymbol{h} = 0$, since $\boldsymbol{x} \neq \mathbf{0}$ and $\boldsymbol{x} \in R_{+}^{K+1}$, then the entries $\widehat{\boldsymbol{S}}(1;\omega_k)\boldsymbol{h}, k = 1, 2, \cdots K$, must be all greater than or equal to zero, with at least one strict inequality. In this case, $\boldsymbol{h}$ is seen to represent an arbitrage opportunity.

(ii) When $\widehat{\boldsymbol{S}}^*(0)\boldsymbol{h} < 0$, all the entries $\widehat{\boldsymbol{S}}(1;\omega_k)\boldsymbol{h}, k = 1, 2, \cdots, K$ must be all non-negative. Correspondingly, $\boldsymbol{h}$ represents a dominant trading strategy and in turns $\boldsymbol{h}$ is an arbitrage opportunity.

Since $U \cap R_+^{K+1} = \{\mathbf{0}\}$, by the Separating Hyperplane Theorem, there exists a hyperplane that separates the pair of disjoint convex sets: $\mathbb{R}_+^{K+1} \backslash \{\mathbf{0}\}$ and U. This hyperplane must go through the origin, so its equation is of the form $[f, 0]$. Let $\boldsymbol{f} \in \mathbb{R}^{K+1}$ be the normal to this hyperplane, then we have $\boldsymbol{f} \cdot \boldsymbol{x} > \boldsymbol{f} \cdot \boldsymbol{y}$, for all $\boldsymbol{x} \in \mathbb{R}_+^{K+1} \backslash \{\mathbf{0}\}$ and $\boldsymbol{y} \in U$.

Remark: We may have $\boldsymbol{f} \cdot \boldsymbol{x} < \boldsymbol{f} \cdot \boldsymbol{y}$, depending on the orientation of the normal vector $\boldsymbol{f}$. However, the final conclusion remains unchanged.

Two-dimensional case



(i) $\boldsymbol{f} \cdot \boldsymbol{y} = 0$ for all $y \in U$;

(ii) $\boldsymbol{f} \cdot \boldsymbol{x} > 0$ for all $x \in R^2_+ \setminus \{\mathbf{0}\}$.

Since U is a linear subspace so that a negative multiple of $\boldsymbol{y} \in U$ also belongs to U. Note that $\boldsymbol{f} \cdot \boldsymbol{x} > \boldsymbol{f} \cdot \boldsymbol{y}$ and $\boldsymbol{f} \cdot \boldsymbol{x} > \boldsymbol{f} \cdot (-\boldsymbol{y})$ both holds only if $\boldsymbol{f} \cdot \boldsymbol{y} = 0$ for all $\boldsymbol{y} \in U$.

Remark

Interestingly, all vectors in U lie in the hyperplane $\boldsymbol{f} \cdot \boldsymbol{y} = 0$ through the origin. This hyperplane separates U (hyperplane itself) and $R_+^{K+1} \backslash \{0\}$.

We have $\boldsymbol{f} \cdot \boldsymbol{x} > 0$ for all $\boldsymbol{x}$ in $\mathbb{R}_+^{K+1} \backslash \{\mathbf{0}\}$. This requires all entries in $\boldsymbol{f}$ to be strictly positive. Note that if at least one of the components (say, the i^{th} component) of $\boldsymbol{f}$ is zero or negative, then we choose $\boldsymbol{x}$ to be the i^{th} coordinate vector. This gives $\boldsymbol{f} \cdot \boldsymbol{x} \leq 0$, a violation of $\boldsymbol{f} \cdot \boldsymbol{x} > 0$.

From $\boldsymbol{f} \cdot \boldsymbol{y} = 0$, we have

$$-f_0 \widehat{\boldsymbol{S}}^*(0)\boldsymbol{h} + \sum_{k=1}^{K} f_k \widehat{\boldsymbol{S}}^*(1;\omega_k)\boldsymbol{h} = 0$$

for all $\boldsymbol{h} \in \mathbb{R}^{M+1}$, where $f_j, j = 0, 1, \cdots, K$ are the entries of $\boldsymbol{f}$. We then deduce that

$$\widehat{\boldsymbol{S}}^*(0) = \sum_{k=1}^{K} Q(\omega_k)\widehat{\boldsymbol{S}}^*(1;\omega_k), \text{ where } Q(\omega_k) = f_k/f_0.$$

Consider the first component in the vectors on both sides of the above equation. They both correspond to the current price and discounted payoff of the riskless security, and all are equal to one. We then obtain

$$1 = \sum_{k=1}^{K} Q(\omega_k).$$

We obtain the risk neutral probabilities $Q(\omega_k), k = 1, \cdots, K$, whose sum is equal to one and they are all strictly positive since $f_j > 0, j = 0, 1, \cdots, K$.

Remark

Corresponding to each risky asset, we have

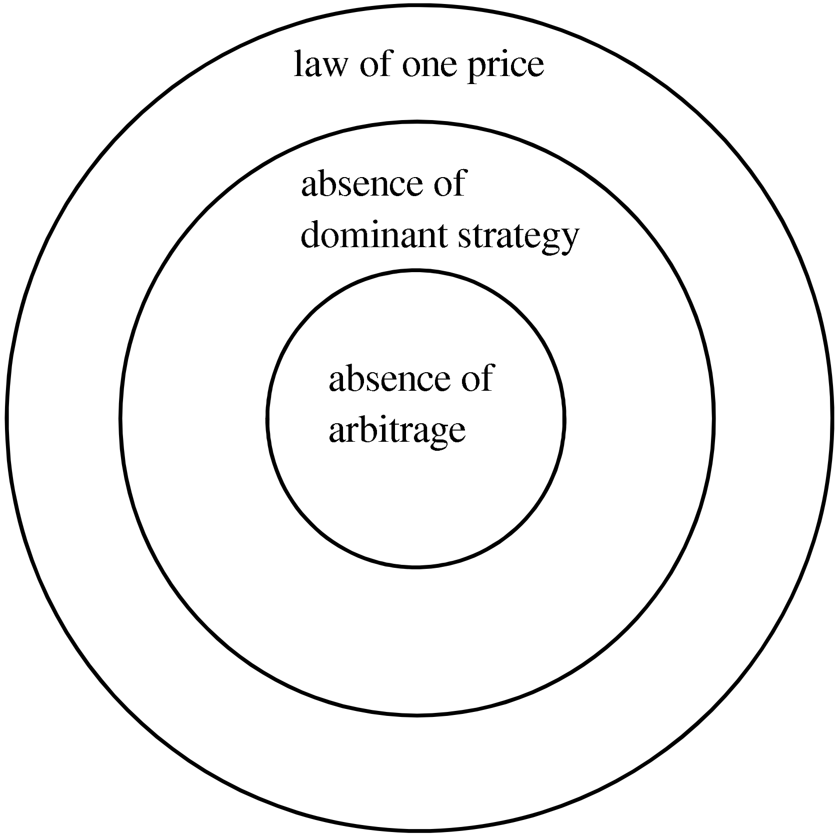
$$S_m(0) = \sum_{k=1}^{K} Q(\omega_k) S_m^*(1; \omega_k), \quad m = 1, 2, \cdots, M.$$

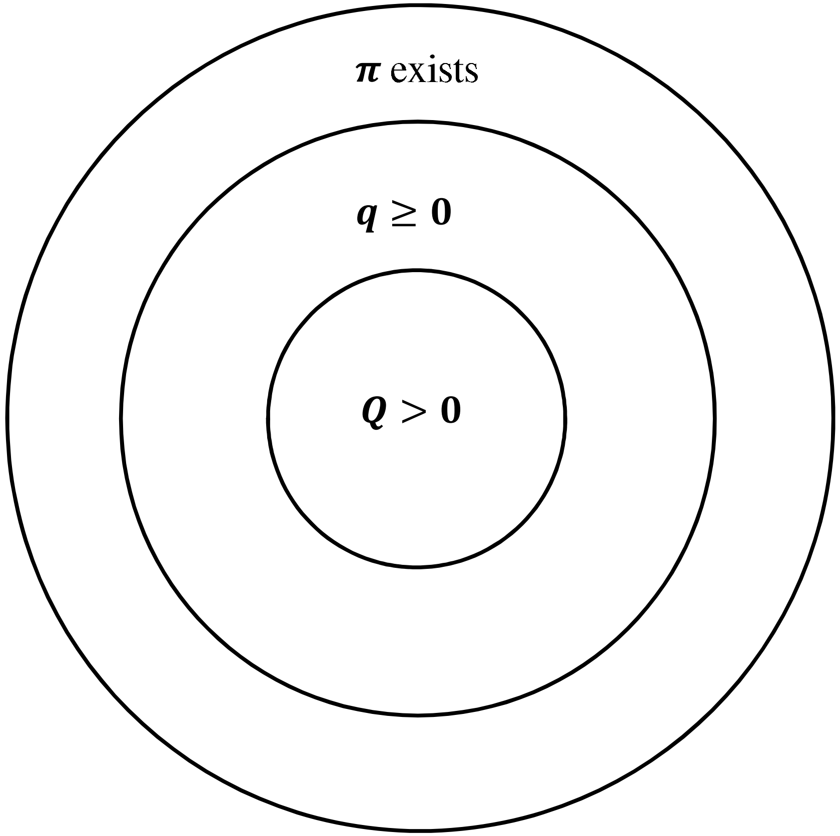
Hence, the current price of any one of risky securities in the securities model is given by the expectation of the discounted terminal payoff under the risk neutral measure Q.

Financial Economics　　　*Linear Algebra*: $\boldsymbol{\pi} S^*(1) = \boldsymbol{S}(0)$

$$\boldsymbol{q}\widehat{S}^*(1) = \widehat{\boldsymbol{S}}(0)$$

$$\boldsymbol{Q}\widehat{S}^*(1) = \widehat{\boldsymbol{S}}(0)$$





Remark The one-period discrete securities model must contain the riskfree asset when we consider the linear pricing measure $\boldsymbol{q}$ and martingale pricing measure $\boldsymbol{Q}$.

Existence of arbitrage opportunities but non-existence of dominant trading strategies

Consider the securities model

$$(1 \quad 2 \quad 3 \quad 6) = (\pi_1 \quad \pi_2 \quad \pi_3)\begin{pmatrix} 1 & 2 & 3 & 6 \\ 1 & 3 & 4 & 8 \\ 1 & 6 & 7 & 14 \end{pmatrix},$$

where the number of non-redundant securities is only 2. Note that

$$\boldsymbol{S}_2^*(1;\Omega) = \boldsymbol{S}_0^*(1;\Omega) + \boldsymbol{S}_1^*(1;\Omega) \quad \text{and}$$

$$\boldsymbol{S}_3^*(1;\Omega) = \boldsymbol{S}_0^*(1;\Omega) + \boldsymbol{S}_1^*(1;\Omega) + \boldsymbol{S}_2^*(1;\Omega),$$

and the initial prices have been set such that

$$S_2(0) = S_0(0) + S_1(0) \quad \text{and} \quad S_3(0) = S_0(0) + S_1(0) + S_2(0),$$

so we expect to have the law of one price. However, since $2 =$ number of non-redundant securities $<$ number of states $= 3$, we do not have uniqueness of solution. Indeed, we obtain

$$(\pi_1 \quad \pi_2 \quad \pi_3) = (1 \quad 0 \quad 0) + t(3 \quad -4 \quad 1), \quad t \text{ any value.}$$

For example, when we take $t = 1$, then

$$(\pi_1 \quad \pi_2 \quad \pi_3) = (4 \quad -4 \quad 1).$$

In terms of linear algebra, we have existence of solution if the equations are consistent. Consider the present example, we have

$$\begin{aligned}
\pi_1 + \pi_2 + \pi_3 &= 1 \\
2\pi_1 + 3\pi_2 + 6\pi_3 &= 2 \\
3\pi_1 + 4\pi_2 + 7\pi_3 &= 3 \\
6\pi_1 + 8\pi_2 + 14\pi_3 &= 6
\end{aligned}$$

Note that the last two redundant equations are consistent. Alternatively, we can interpret that the row vector $\boldsymbol{S}^*(0) = (1 \quad 2 \quad 3 \quad 6)$ lies in the row space of $\widehat{S}^*(1;\Omega)$, which is spanned by $\{(1 \quad 2 \quad 3 \quad 6), (1 \quad 3 \quad 4 \quad 8)\}$.

The above calculations confirm with the result:

$$\text{law of one price} \Leftrightarrow \text{existence of solution.}$$

In this securities model, we cannot find a risk neutral measure where $(Q_1 \quad Q_2 \quad Q_3) > \mathbf{0}$. This is easily seen since $\pi_2 = -4t$ and $\pi_3 = t$, and they always have opposite sign. However, a linear pricing measure exists. By setting $t = 0$, we obtain the linear pricing measure

$$(q_1 \quad q_2 \quad q_3) = (1 \quad 0 \quad 0) \geq \mathbf{0}.$$

Since Q does not exist, the securities model admits arbitrage opportunities. One such example is $\boldsymbol{h} = (-11 \quad 1 \quad 1 \quad 1)^T$, where

$$\widehat{\boldsymbol{S}}(0)\boldsymbol{h} = 0 \quad \text{and} \quad \widehat{S}^*(1;\Omega)\boldsymbol{h} = \begin{pmatrix} 1 & 2 & 3 & 6 \\ 1 & 3 & 4 & 8 \\ 1 & 6 & 7 & 14 \end{pmatrix} \begin{pmatrix} -11 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \\ 16 \end{pmatrix}.$$

We start with zero portfolio value at $t = 0$ while the discounted portfolio value at $t = 1$ is guaranteed to be non-negative, with strict positivity for at least one state. This signifies an arbitrage opportunity.

However, the securities model does not admit any dominant trading strategy since a linear pricing measure $\boldsymbol{\pi} = (1 \quad 0 \quad 0)$ exists. That is, one cannot find a trading strategy $\boldsymbol{h} = (h_0 \quad h_1 \quad h_2 \quad h_3)^T$ such that

$$h_0 + 2h_1 + 3h_2 + 6h_3 = 0$$

while

$$\begin{aligned} h_0 + 2h_1 + 3h_2 + 6h_3 &> 0, \\ h_0 + 3h_1 + 4h_2 + 8h_3 &> 0, \\ h_0 + 6h_1 + 6h_2 + 14h_3 &> 0. \end{aligned}$$

The first inequality can never be satisfied when we impose $h_0 + 2h_1 + 3h_2 + 6h_3 = 0$.

Indeed, when $\boldsymbol{S}(0) = \boldsymbol{S}^*(1; \omega_k)$ for some ω_k, then a linear pricing measure exists, where $\boldsymbol{q} = \boldsymbol{e}_k^T$. In this numerical example, we have

$$\boldsymbol{S}(0) = (1 \quad 2 \quad 3 \quad 6) \quad \text{and} \quad \boldsymbol{q} = (1 \quad 0 \quad 0) = \boldsymbol{e}_1^T$$

Definition of martingale*

In the context of one-period model, given the information on the initial prices and terminal payoff values of the security prices at $t = 0$, we have

$$S_m(0) = E_Q[S_m^*(1;\Omega)] = \sum_{k=1}^{K} S_m^*(1;\omega_k)Q(\omega_k), \quad m = 1, 2, \cdots, M.$$

The discounted security price process $S_m^*(t)$ is said to be a martingale† under Q.

Martingale is associated with the wealth process of a gambler in a fair game. In a fair game, the expected value of the gambler's wealth after any number of plays is always equal to her initial wealth.

*A stochastic process is adapted to a filtration with respect to a measure. Say S_m^* is adapted to $\mathbb{F} = \{\mathcal{F}_t; t = 0, 1, \cdots, T\}$, then $S_m^*(t)$ is $\mathcal{F}_t$-measurable.

†Martingale property with respect to Q and $\mathbb{F}$:

$$S_m^*(t) = E_Q[S_m^*(s + t)|\mathcal{F}_t] \text{ for all } t \geq 0, s \geq 0.$$

Equivalent martingale measure

The risk neutral probability measure Q is commonly called the equivalent martingale measure. "Equivalent" refers to the equivalence between the physical measure P and martingale measure Q [observing $P(\omega) > 0 \Leftrightarrow Q(\omega) > 0$ for all $\omega \in \Omega$]*. The linear pricing measure falls short of this equivalence property since $q(\omega)$ can be zero for some ω.

*P and Q may not agree on the assignment of probability values to individual events, but they always agree as to which events are possible or impossible.

Martingale property of discounted portfolio value (assuming the existence of Q or equivalently, the absence of arbitrage in the securities model)

- Let $V_1^*(\Omega)$ denote the discounted payoff of a portfolio. Since $V_1^*(\Omega) = \widehat{S}^*(1;\Omega)\boldsymbol{h}$ for the portfolio holding $\boldsymbol{h} = (h_0 \ \cdots \ h_M)^T$, we have

$$\begin{aligned} V_0 &= (S_0(0) \cdots S_M(0))\boldsymbol{h} \\ &= (E_Q[S_0^*(1;\Omega)] \cdots E_Q[S_M^*(1;\Omega)])\boldsymbol{h} \\ &= \sum_{m=0}^{M} \left[\sum_{k=1}^{K} S_m^*(1;\omega_k) Q(\omega_k) \right] h_m \\ &= \sum_{k=1}^{K} Q(\omega_k) \left[\sum_{m=0}^{M} S_m^*(1;\omega_k) h_m \right] = E_Q[V_1^*(\Omega)]. \end{aligned}$$

- The equivalent martingale measure Q is not necessarily unique. Since "absence of arbitrage opportunities " implies "law of one price", the expectation value $E_Q[V_1^*(\Omega)]$ must be single-valued under all equivalent martingale measures.

Finding the set of risk neutral measures

Consider the earlier securities model with the riskfree security and only one risky security, where $\widehat{S}^*(1;\Omega) = \begin{pmatrix} 1 & 4 \\ 1 & 3 \\ 1 & 2 \end{pmatrix}$ and $\widehat{\boldsymbol{S}}(0) = (1 \quad 3)$. The risk neutral probability measure

$$\boldsymbol{Q} = (Q(\omega_1) \quad Q(\omega_2) \quad Q(\omega_3)),$$

if exists, will be determined by the following system of equations

$$(Q(\omega_1) \quad Q(\omega_2) \quad Q(\omega_3)) \begin{pmatrix} 1 & 4 \\ 1 & 3 \\ 1 & 2 \end{pmatrix} = (1 \quad 3).$$

Since there are more unknowns than the number of equations, the solution is not unique. The solution is found to be $\boldsymbol{Q} = (\lambda \quad 1-2\lambda \quad \lambda)$, where λ is a free parameter. Since all risk neutral probabilities are all strictly positive, we must have $0 < \lambda < 1/2$.

Lemma: completeness and uniqueness

Under market completeness, if the set of risk neutral measures is non-empty, then it must be a singleton.

Under market completeness, column rank of $\widehat{S}^*(1;\Omega)$ equals the number of states. Since column rank = row rank, then all rows of $\widehat{S}^*(1;\Omega)$ are independent. If solution $\boldsymbol{Q}$ exists for

$$\boldsymbol{Q}\widehat{S}^*(1;\Omega) = \widehat{\boldsymbol{S}}(0),$$

then it must be unique. Note that $\boldsymbol{Q} > \boldsymbol{0}$.

Conversely, suppose the set of risk neutral measures is a singleton, one can show that the securities model is complete (see later discussion).

Numerical example

Suppose we add the second risky security with discounted payoff $\boldsymbol{S}_2^*(1) = \begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix}$ and current value $S_2(0) = 3$. With this new addition, the securities model becomes complete.

With the new equation $3Q(\omega_1) + 2Q(\omega_2) + 4Q(\omega_3) = 3$ added to the system, this new securities model is seen to have the unique risk neutral measure $(1/3 \quad 1/3 \quad 1/3)$.

Indeed, when the securities model is complete, all Arrow securities are replicable. Their prices (state prices that are positive) are simply equal to the risk neutral measures. In this example, we have

$$s_1 = Q(\omega_1) = \frac{1}{3}, \quad s_2 = Q(\omega_2) = \frac{1}{3}, \quad s_3 = Q(\omega_3) = \frac{1}{3}.$$

Subspace of discounted gains

Let W be a subspace in $\mathbb{R}^K$ which consists of discounted gains corresponding to some trading strategy $\boldsymbol{h}$. Note that W is spanned by the set of vectors representing discounted gains of the risky securities.

In the above securities model, the discounted gains of the first and second risky securities are $\begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix} - \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix} - \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$, respectively.

The discounted gain subspace is given by

$$W = \left\{ h_1 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + h_2 \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}, \text{ where } h_1 \text{ and } h_2 \text{ are scalars} \right\}.$$

Orthogonality of the discounted gain vector and $\boldsymbol{Q}$

Let G^* denote the discounted gain of a portfolio. For any risk neutral probability measure Q, we have

$$\begin{aligned} E_Q G^* &= \sum_{k=1}^{K} Q(\omega_k) \left[\sum_{m=1}^{M} h_m \Delta S_m^*(\omega_k) \right] \\ &= \sum_{m=1}^{M} h_m E_Q[\Delta S_m^*] = 0. \end{aligned}$$

Under the absence of arbitrage opportunities, the expected discounted gain from any risky portfolio is simply zero. Apparently, there is no risk premium derived from the risky investment. Therefore, the financial economics term "risk neutrality" is adopted under this framework of asset pricing.

For any $\boldsymbol{G}^* = (G(\omega_1) \cdots G(\omega_K))^T \in W$, we have

$$\boldsymbol{Q}\boldsymbol{G}^* = 0, \text{ where } \boldsymbol{Q} = (Q(\omega_1) \cdots Q(\omega_K)).$$

Characterization of the set of neutral measures

Since the sum of risk neutral probabilities must be one and all probability values must be positive, any risk neutral probability vector $\boldsymbol{Q}$ must lie in the following subset

$$P^+ = \{\boldsymbol{y} \in \mathbb{R}^K : y_1 + y_2 + \cdots + y_K = 1 \quad \text{and} \quad y_k > 0, k = 1, \cdots K\}.$$

Also, the risk neutral probability vector $\boldsymbol{Q}$ must lie in the orthogonal complement $W^\perp$, where W is the discounted gain subspace. Let R denote the set of all risk neutral measures, then

$$R = P^+ \cap W^\perp.$$

In the above numerical example, $W^\perp$ is the line through the origin in $\mathbb{R}^3$ which is perpendicular to $(1 \quad 0 \quad -1)$ and $(0 \quad -1 \quad 1)$. The line should assume the form $\lambda(1 \quad 1 \quad 1)$ for some scalar λ. We obtain the risk neutral probability vector $\boldsymbol{Q} = (1/3 \quad 1/3 \quad 1/3)$.

2.4 Valuation of contingent claims and complete markets

- A contingent claim can be considered as a random variable Y that represents the terminal payoff whose value depends on the occurrence of a particular state ω_k, where $\omega_k \in \Omega$.

- Suppose the holder of the contingent claim is promised to receive the preset contingent payoff, how much should the writer of such contingent claim charge at $t = 0$ so that the price is *fair* to both parties.

- Consider the securities model with the riskfree security whose values at $t = 0$ and $t = 1$ are $S_0(0) = 1$ and $S_0(1) = 1.1$, respectively, and the risky security with $S_1(0) = 3$ and $\boldsymbol{S}_1(1) = \begin{pmatrix} 4.4 \\ 3.3 \\ 2.2 \end{pmatrix}$.

The set of $t = 1$ payoffs that can be generated by certain trading strategy is given by $h_0 \begin{pmatrix} 1.1 \\ 1.1 \\ 1.1 \end{pmatrix} + h_1 \begin{pmatrix} 4.4 \\ 3.3 \\ 2.2 \end{pmatrix}$ for some scalars h_0 and h_1.

For example, the contingent claim $\begin{pmatrix} 5.5 \\ 4.4 \\ 3.3 \end{pmatrix}$ can be generated by the trading strategy: $h_0 = 1$ and $h_1 = 1$, while the other contingent claim $\begin{pmatrix} 5.5 \\ 4.0 \\ 3.3 \end{pmatrix}$ cannot be generated by any trading strategy associated with the given securities model.

Attainability of contingent claim

A contingent claim Y is said to be *attainable* if there exists some trading strategy $\boldsymbol{h}$, called the *replicating portfolio*, such that $V_1 = Y$ for all possible states occurring at $t = 1$.

The price at $t = 0$ of the replicating portfolio is given by

$$V_0 = h_0 S_0(0) + h_1 S_1(0) = 1 \times 1 + 1 \times 3 = 4.$$

Suppose there are no arbitrage opportunities (equivalent to the existence of a risk neutral probability measure), then the law of one price holds and so V_0 is unique.

Pricing of attainable contingent claims

Let $V_1^*(1;\Omega)$ denote the discounted value of the replicating portfolio that matches with the payoff of the attainable contingent claim at every state of the world. Suppose the associated trading strategy to generate the replicating portfolio is $\boldsymbol{h}$, then

$$V_1^* = \widehat{S}^*(1;\Omega)\boldsymbol{h}.$$

The initial cost of setting up the replicating portfolio is

$$V_0 = \widehat{\boldsymbol{S}}(0)\boldsymbol{h}.$$

Assuming $\boldsymbol{\pi}$ exists, where $\widehat{\boldsymbol{S}}(0) = \boldsymbol{\pi}\widehat{S}^*(1;\Omega)$ so that

$$\begin{aligned} V_0 &= \boldsymbol{\pi}\widehat{S}^*(1;\Omega)\boldsymbol{h} = \boldsymbol{\pi}V_1^*(1;\Omega) \\ &= \sum_{k=1}^{K} \pi_k V_1^*(1;\omega_k), \text{ independent of } \boldsymbol{h}. \end{aligned}$$

Even when $\boldsymbol{\pi}$ is not a risk neutral measure or linear pricing measure, the above pricing relation remains valid. Note that $\boldsymbol{\pi}$ may not be unique, however we always have the same value for V_0 by virtue of law of one price.

- Consider a given attainable contingent claim Y which is generated by certain trading strategy. The associated discounted gain G^* of the trading strategy is given by $G^* = \sum_{m=1}^{M} h_m \Delta S_m^*$. Now, suppose a risk neutral probability measure Q associated with the securities model exists, we have

  $$V_0 = E_Q V_0 = E_Q[V_1^* - G^*].$$

  Since $E_Q[G^*] = E_Q[\sum_{m=1}^{M} h_m \Delta S_m^*] = \sum_{m=1}^{M} h_m E_Q[\Delta S_m^*] = 0$, and $V_1^* = Y^*$ by virtue of replication. We obtain

  $$V_0 = E_Q[Y^*].$$

Risk neutral valuation principle:

The price at $t = 0$ of an attainable claim Y is given by the expectation under any risk neutral measure Q of the discounted payoff of the contingent claim.

Recall that the existence of the risk neutral probability measure implies the law of one price. Provided that Y is attainable, $E_Q[Y^*]$ assumes the same value for every risk neutral probability measure Q by virtue of the law of one price. What happen to $E_Q[Y^*]$ when Y is non-attainable? We will show that $E_Q[Y^*]$ does not take the same value for all $Q \in M$.

Theorem (Attainability of a contingent claim and uniqueness of $E_Q[Y^*]$)

Suppose the securities model admits no arbitrage opportunities. The contingent claim Y is attainable if and only if $E_Q[Y^*]$ takes the same value for every $Q \in M$, where M is the set of risk neutral measures.

Proof

$\Longrightarrow$ part

existence of $Q \Longleftrightarrow$ absence of arbitrage $\Longrightarrow$ law of one price. For an attainable Y, $E_Q[Y^*]$ is constant with respect to all $Q \in M$, otherwise this leads to violation of the law of one price.

$\Longleftarrow$ part

It suffices to show that if the contingent claim Y is not attainable then $E_Q[Y^*]$ does not take the same value for all $Q \in M$.

Let $\boldsymbol{y}^* \in \mathbb{R}^K$ be the discounted payoff vector corresponding to Y^*. Since Y is not attainable, then there is no solution to

$$\widehat{S}^*(1)\boldsymbol{h} = \boldsymbol{y}^*$$

(non-existence of trading strategy $\boldsymbol{h}$). It then follows that there exists a non-zero row vector $\boldsymbol{\pi} \in \mathbb{R}^K$ such that

$$\boldsymbol{\pi}\widehat{S}^*(1) = \mathbf{0} \quad \text{and} \quad \boldsymbol{\pi}\boldsymbol{y}^* \neq 0.$$

Justification

Recall that the orthogonal complement of the column space is the left null space. The dimension of the left null space equals $K-$ column rank, and it is non-zero since the column space does not span the whole $\mathbb{R}^K$. The above result indicates that when $\boldsymbol{y}^*$ is not in the column space of $S^*(1)$, then there exists a non-zero vector $\boldsymbol{\pi}$ in the left null space of $S^*(1)$ such that $\boldsymbol{y}^*$ and $\boldsymbol{\pi}$ are not orthogonal. If otherwise, $\boldsymbol{y}^*$ is orthogonal to every vector in the left null space, then $\boldsymbol{y}^*$ lies in the column space. This leads to a contradiction.

Write $\boldsymbol{\pi} = (\pi_1 \cdots \pi_K)$. Let $\widehat{Q} \in M$ be an arbitrary risk neutral measure, and let $\lambda > 0$ be small enough such that

$$Q(\omega_k) = \widehat{Q}(\omega_k) + \lambda \pi_k > 0, \quad k = 1, 2, \cdots, K.$$

To show that $Q(\omega_k)$ is a risk neutral measure, it suffices to show that $\boldsymbol{Q}$ satisfies

$$\boldsymbol{Q}\widehat{S}^*(1) = \widehat{\boldsymbol{S}}(0).$$

Consider

$$\boldsymbol{Q}\widehat{S}^*(1) = \widehat{\boldsymbol{Q}}\widehat{S}^*(1) + \lambda\boldsymbol{\pi}\widehat{S}^*(1)$$

and observe that

$$\widehat{\boldsymbol{Q}}\widehat{S}^*(1) = \widehat{\boldsymbol{S}}(0) \quad \text{and} \quad \boldsymbol{\pi}\widehat{S}^*(1) = \mathbf{0},$$

so the required condition is checked. Note that $\sum_{k=1}^K Q(w_k) = 1$ is satisfied since $\sum_{k=1}^K \pi_k = 0$ [as enforced by zero value of the product of $\boldsymbol{\pi}$ with the first column of $\widehat{S}^*(1)$].

Lastly, we consider

$$\begin{aligned} E_Q[Y^*] &= \sum_{k=1}^{K} Q(\omega_k)Y^*(\omega_k) \\ &= \sum_{k=1}^{K} \widehat{Q}(\omega_k)Y^*(\omega_k) + \lambda \sum_{k=1}^{K} \pi_k Y^*(\omega_k). \end{aligned}$$

The last term is non-zero since $\boldsymbol{\pi y}^* \neq 0$ and $\lambda > 0$. Therefore, we have

$$E_Q[Y^*] \neq E_{\widehat{Q}}[Y^*].$$

Thus, when Y is not attainable, $E_Q[Y^*]$ does not take the same value for all risk neutral measures.

Corollary Given that the set of risk neutral measures R is non-empty. The securities model is complete if and only if R consists of exactly one risk neutral measure.

completeness of securities model $\Leftrightarrow$ uniqueness of Q

An earlier proof of "$\Longrightarrow$ part" has been shown on p.59. Alternatively, we may prove by contradiction: non-uniqueness of $Q \Rightarrow$ non-completeness.

Suppose there exist two distinct Q and $\widehat{Q}$, that is, $Q(\omega_k) \neq \widehat{Q}(\omega_k)$ for some state ω_k. Let $Y^* = \begin{cases} 1 & \text{if } \omega = \omega_k \\ 0 & \text{otherwise} \end{cases}$, which is the k^{th} Arrow security. Obviously,

$$E_Q[Y^*] = Q(\omega_k) \neq \widehat{Q}(\omega_k) = E_{\widehat{Q}}[Y^*],$$

so $E_Q[Y^*]$ is not unique. By the theorem, the k^{th} Arrow security is not attainable so the securities model is not complete.

$\Longleftarrow$ part: If the risk neutral measure is unique, then for any contingent claim $Y, E_Q[Y^*]$ takes the same value for any Q (actually single Q). Hence, any contingent claim is attainable so the market is complete.

Remarks

- When the securities model is complete and admits no arbitrage opportunities, then all Arrow securities lie in the asset span and risk neutral measure is unique. The state price of state ω_k exists for any state and it is equal to the unique risk neutral probability $Q(\omega_k)$. The risk neutral valuation procedure can be applied for pricing any contingent claim (which is always attainable due to completeness).

- On the other hand, suppose there are two risk neutral probability values for the same state ω_k, the state price of that state cannot be defined properly without contradicting the law of one price. Actually, by the theorem, the Arrow security of that state would not be attainable, so the securities model cannot be complete.

Example

Suppose

$$Y^* = \begin{pmatrix} 5 \\ 4 \\ 3 \end{pmatrix}, \ \widehat{\boldsymbol{S}}(0) = (1 \quad 3) \quad \text{and} \quad \widehat{S}^*(1;\Omega) = \begin{pmatrix} 1 & 4 \\ 1 & 3 \\ 1 & 2 \end{pmatrix},$$

Y^* is seen to be attainable. We have seen that the risk neutral probability is given by

$$\boldsymbol{Q} = (\lambda \quad 1-2\lambda \quad \lambda), \text{ where } 0 < \lambda < 1/2.$$

The price at $t = 0$ of the contingent claim is given by

$$V_0 = 5\lambda + 4(1-2\lambda) + 3\lambda = 4,$$

which is independent of λ. This verifies the earlier claim that $E_Q[Y^*]$ assumes the same value for any risk neutral measure Q. Suppose Y^* is changed to $(5 \quad 4 \quad 4)^T$, then $V_0 = E_Q[Y^*] = 4 + \lambda$, which is not unique. This is expected since the new Y^* is non-attainable.

Complete markets - summary of results

Recall that a securities model is complete if every contingent claim Y lies in the asset span, that is, Y can be generated by some trading strategy.

Consider the augmented terminal payoff matrix

$$\widehat{S}(1;\Omega) = \begin{pmatrix} S_0(1;\omega_1) & S_1(1;\omega_1) & \cdots & S_M(1;\omega_1) \\ \vdots & \vdots & & \vdots \\ S_0(1;\omega_K) & S_1(1;\omega_K) & \cdots & S_M(1;\omega_K) \end{pmatrix},$$

Y always lies in the asset span if and only if the column space of $\widehat{S}(1;\Omega)$ is equal to $\mathbb{R}^K$.

- Since the dimension of the column space of $\widehat{S}(1;\Omega)$ cannot be greater than $M+1$, a necessary condition for market completeness is that $M+1 \geq K$. In this case, number of columns $= M+1 \geq K =$ number of rows.

- When $\widehat{S}(1;\Omega)$ has independent columns and the asset span is the whole $\mathbb{R}^K$, then rank $= M + 1 = K$. Now, the trading strategy that generates Y must be unique since there are no redundant securities. In this case, any contingent claim is replicable and its price is unique.

- When the asset span is the whole $\mathbb{R}^K$ but some securities are redundant, the trading strategy that generates Y would not be unique.

- Suppose a risk neutral measure Q exist, then the price at $t = 0$ of an attainable contingent claim is unique under risk neutral valuation, independent of the chosen trading strategy. This is a consequence of the law of one price, which holds since a risk neutral measure exists.

Bounds on arbitrage prices for non-attainable contingent claim

Suppose a risk neutral measure Q exists, risk neutral valuation fails when we price a *non-attainable contingent claim*. However, we may specify an interval $[V_-(Y), V_+(Y)]$ where a reasonable price at $t = 0$ of the contingent claim should lie. The lower and upper bounds are given by

$$\begin{aligned} V_+(Y) &= \inf\{E_Q[\widetilde{Y}/S_0(1)] : \widetilde{Y} \succeq Y \text{ and } \widetilde{Y} \text{ is attainable}\} \\ V_-(Y) &= \sup\{E_Q[\widetilde{Y}/S_0(1)] : \widetilde{Y} \preceq Y \text{ and } \widetilde{Y} \text{ is attainable}\}. \end{aligned}$$

Here, inf is the greatest lower bound and sup is the smallest upper bound. For example, $\inf\{x : 0 < x < 2\} = 0$ and $\sup\{x : 0 < x < 2\} = 2$.

$\widetilde{Y} \succeq Y$ means $\widetilde{Y}$ weakly dominates Y in the sense that $\widetilde{Y}(\omega) \geq Y(\omega)$ for all $\omega \in \Omega$ and strict inequality holds at least for one state. Here, $V_+(Y)$ is the minimum value among all prices of attainable contingent claims that weakly dominate the non-attainable claim Y, while $V_-(Y)$ is the maximum value among all prices of attainable contingent claims that are weakly dominated by Y.

Proof of the upper bound

Suppose $V(Y) > V_+(Y)$, then an arbitrageur can lock in riskless profit by selling the contingent claim to receive $V(Y)$ and use $V_+(Y)$ to construct the replicating portfolio that generates the attainable $\widetilde{Y}$. This $\widetilde{Y}$ is the attainable claim that gives inf $E_Q[\widetilde{Y}/S_0(1)]$ and observe $\widetilde{Y} \succeq Y$. The upfront positive gain is $V(Y) - V_+(Y)$ and the terminal gain is $\widetilde{Y} - Y$.

Alternatively, based on the linear programming duality theory, we have the following results: If the set of risk neutral measures $R \neq \phi$, then for any contingent claim Y, we have

$$V_+(Y) = \sup\{E_Q[Y^*] : Q \in R\},$$

$$V_-(Y) = \inf\{E_Q[Y^*] : Q \in R\}.$$

If Y is attainable, by virtue of the theorem that attainability of Y is equivalent to uniqueness of $E_Q[Y^*]$ for all $Q \in R$, then $V_+(Y) = V_-(Y)$.

Example

Consider the securities model: $\widehat{\boldsymbol{S}}(0) = (1 \quad 3)$ and $\widehat{S}^*(1;\Omega) = \begin{pmatrix} 1 & 4 \\ 1 & 3 \\ 1 & 2 \end{pmatrix}$, and the non-attainable discounted contingent claim $Y^* = \begin{pmatrix} 5 \\ 4 \\ 4 \end{pmatrix}$. The set of risk neutral measures is given by

$$\boldsymbol{Q} = (\lambda \quad 1 - 2\lambda \quad \lambda), \quad \text{where } 0 < \lambda < 1/2.$$

Note that $E_Q[Y^*] = 4 + \lambda$ so that

$$V_+ = \sup\{E_Q[Y^*] : Q \in R\} = 9/2 \quad \text{and} \quad V_- = \inf\{E_Q[Y^*] : Q \in R\} = 4.$$

The discounted payoff vector of the attainable contingent claim corresponding to V_+ is

$$\widetilde{Y}_+^* = \begin{pmatrix} 5 \\ 4.5 \\ 4 \end{pmatrix} = 3\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + 0.5\begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix}, \quad \text{where } E_Q[\widetilde{Y}_+^*] = 4.5.$$

Note that $\widetilde{Y}_+^* \geq Y^*$.

On the other hand, the discounted payoff vector of the attainable contingent claim corresponding to V_- is

$$\widetilde{Y}_-^* = \begin{pmatrix} 5 \\ 4 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix}, \quad \text{where } E_Q[\widetilde{Y}_-^*] = 4.$$

Note that $\widetilde{Y}_-^* \leq Y^*$.

Any reasonable initial price of the non-attainable discounted contingent claim $Y^* = (5 \quad 4 \quad 4)^T$ should lie between the interval $[4, 4.5]$.

Summary *Arbitrage opportunity* 無風險套利機會

An arbitrage strategy requires no initial investment, no probability of negative value at expiration, and yet some possibility of a positive terminal portfolio value.

- It is commonly assumed that there are no arbitrage opportunities in well functioning and competitive financial markets.

1. absence of arbitrage opportunities
   $\Rightarrow$ absence of dominant trading strategies
   $\Rightarrow$ law of one price

2. absence of arbitrage opportunities $\Leftrightarrow$ existence of a risk neutral measure

   absence of dominant trading strategies $\Leftrightarrow$ existence of a linear pricing measure.

3. Provided that the securities model is complete and law of one price holds, then all coordinate vectors lie in the asset span so that state prices always exist and single-valued. However, state prices may be negative. The state prices are non-negative when a linear pricing measure exists and they become strictly positive when a risk neutral measure exists. Indeed, a sensible state price must be positive since the discounted terminal payoff of an Arrow security is nonnegative and strictly positive for one state.

4. Under the absence of arbitrage opportunities, the risk neutral valuation principle can be applied to find the fair price of an attainable contingent claim. When the contingent claim is not attainable, we can specify the range of prices that reasonable prices should lie in order to avoid arbitrage opportunities.

2.5 Binomial option pricing models

By buying the underlying asset and shorting riskless money market account in appropriate proportions, one can replicate the position of a call.

Under the binomial random walk model, the asset price $S^{\Delta t}$ after one period Δt will be either uS or dS with probability q and $1 - q$, respectively. Note that $S^{\Delta t}$ is a Bernoulli random variable that assumes only two discrete values.

We assume $u > 1 > d$ so that uS and dS represent the up-move and down-move of the asset price, respectively. The jump parameters u and d will be related to the asset price dynamics.

Let R denote the growth factor of riskless money market account over one period so that \$1 invested in a riskless money market account will grow to \$$R$ after one period. In order to avoid riskless arbitrage opportunities, we must have $u > R > d$.

For example, suppose $u > d > R$, then we borrow as much as possible for the riskfree asset and use the loan to buy the risky asset. Even the downward move of the risky asset generates a return better than the riskfree rate. This represents an arbitrage.

Suppose we form a replicating portfolio which consists of α units of the asset and cash amount M in the form of riskless money market account. After one period $\triangle t$, the value of the portfolio becomes

$$\begin{cases} \alpha uS + RM & \text{with probability } q \\ \alpha dS + RM & \text{with probability } 1 - q. \end{cases}$$

Valuation of a call option using the approach of replication

The portfolio is used to replicate the long position of a call option on a non-dividend paying asset.

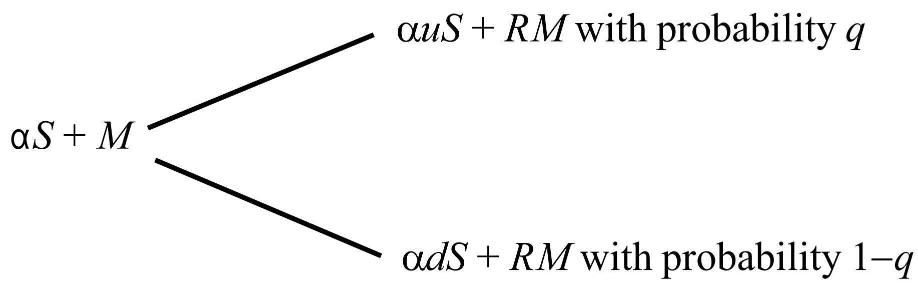
As there are two possible states of the world: asset price goes up or down, the call is thus a contingent claim.

Suppose the current time is only one period $\triangle t$ prior to expiration. Let c denote the current call price, and c_u and c_d denote the call price after one period (which is the expiration time in the present context) corresponding to the up-move and down-move of the asset price, respectively.

Let X denote the strike price of the call. The payoff of the call at expiry is given by

$$\begin{cases} c_u = \max(uS - X, 0) & \text{with probability } q \\ c_d = \max(dS - X, 0) & \text{with probability } 1 - q. \end{cases}$$

One can establish algebraically that $uc_d - dc_u \leq 0 \Leftrightarrow u/d < c_u/c_d$.



Evolution of the asset price S and money market account M after one time period under the binomial model.

Concept of replication revisited

The above portfolio containing the risky asset and money market account is said to replicate the long position of the call if and only if the values of the replicating portfolio and the call option match for each possible outcome, that is,

$$\alpha uS + RM = c_u \quad \text{and} \quad \alpha dS + RM = c_d.$$

Solving the equations, we obtain

$$\alpha = \frac{c_u - c_d}{(u-d)S} > 0, \qquad M = \frac{uc_d - dc_u}{(u-d)R} < 0.$$

Apparently, the binomial model is fortunate to have two instruments in the replicating portfolio and two states of the world (up and down moves) so that the number of equations equals the number of unknowns. The securities model is complete. At some point, people thought that the trinomial (3-jump) model is not feasible since there are only 2 instruments but 3 states to be matched: 3 equations but 2 unknowns.

1. The parameters α and M are seen to have opposite sign since the payoff of a call is long stock and short money market account.

2. $u/d < c_u/c_d$ due to the leverage effect inherited in the call option. That is, when a given upside growth/downside drop is experienced in the stock, the corresponding ratio is higher in the call.

- The number of units of asset held is seen to be the ratio of the difference of call values $c_u - c_d$ to the difference of asset values $uS - dS$. This ratio is called the hedge ratio.

- The call option can be replicated by a portfolio of the two basic securities: long risky asset and short riskfree money market account.

Binomial option pricing formula

By no-arbitrage argument, the current value of the call is given by the current value of the replicating portfolio, that is,

$$\begin{aligned} c &= \alpha S + M = \frac{\frac{R-d}{u-d}c_u + \frac{u-R}{u-d}c_d}{R} \\ &= \frac{pc_u + (1-p)c_d}{R} \qquad \text{where} \quad p = \frac{R-d}{u-d}. \end{aligned}$$

This confirms with the risk neutral valuation principle under a risk neutral measure.

- The probability q, which is the subjective probability about upward or downward movement of the asset price, does not appear in the call value. The parameter p can be shown to be $0 < p < 1$ since $u > R > d$ and so p can be interpreted as a probability.

Query Why not perform the simple discounted expectation procedure using the subjective probabilities q and $1-q$, where

$$c = \frac{qc_u + (1-q)c_d}{R}?$$

The relation

$$puS + (1-p)dS = \frac{R-d}{u-d}\, uS + \frac{u-R}{u-d}\, dS = RS$$

shows that the expected rate of returns on the asset with p as the probability of upside move is just equal to the riskless interest rate:

$$S = \frac{1}{R} E^*[S^{\Delta t}|S],$$

where E^* is expectation under this probability measure. This relation reveals that the asset price process is a martingale under the risk neutral measure. We may view p as the *risk neutral probability*.

Treating the binomial model as a one-period securities model

The securities model consists of the riskfree asset and one risky asset with initial price vector: $\boldsymbol{S}^*(0) = (1 \quad S)$ and discounted terminal payoff matrix: $\widehat{S}^*(1) = \begin{pmatrix} 1 & \frac{uS}{R} \\ 1 & \frac{dS}{R} \end{pmatrix}$.

The risk neutral probability measure $\boldsymbol{Q}(\omega) = (Q(\omega_u) \quad Q(\omega_d))$ is obtained by solving

$$(Q(\omega_u) \quad Q(w_d)) \begin{pmatrix} 1 & \frac{uS}{R} \\ 1 & \frac{dS}{R} \end{pmatrix} = (1 \quad S).$$

We obtain

$$Q(\omega_u) = 1 - Q(\omega_d) = \frac{R-d}{u-d}.$$

The securities model is complete since there are two states and two securities. Provided that the securities model admits no arbitrage opportunities, we have uniqueness of the risk neutral measure.

Condition on u, d and R for absence of arbitrage

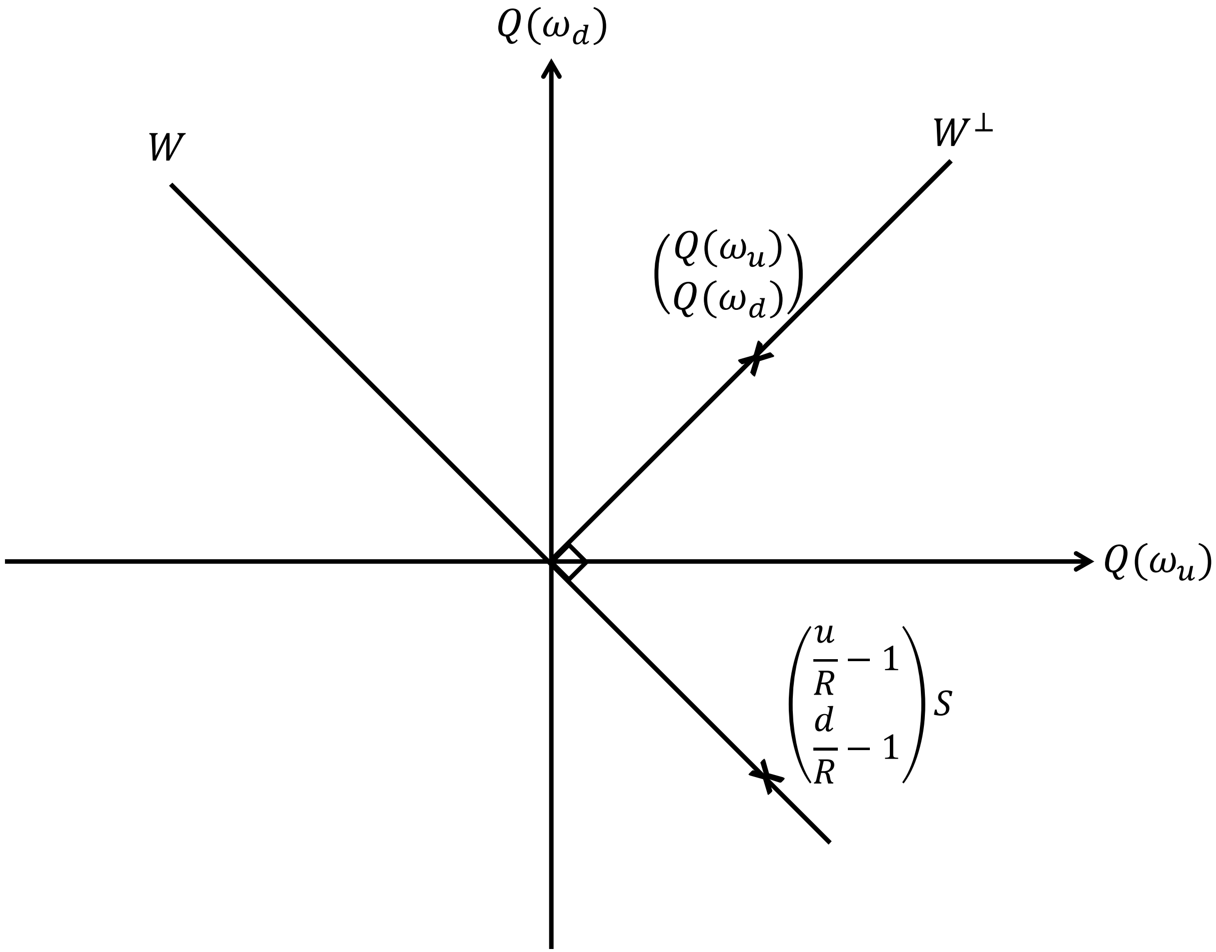
The set of risk neutral measures is given by $R = P^+ \cap W^\perp$, where W is the subspace of discounted gains. In the binomial random walk model, W is spanned by the single vector $\begin{pmatrix} \frac{u}{R} - 1 \\ \frac{d}{R} - 1 \end{pmatrix} S$ since there is only one risky asset. Knowing that $u > d$, the no arbitrage condition: $u > R > d$ is equivalent to

$$\frac{u}{R} - 1 > 0 \quad \text{and} \quad \frac{d}{R} - 1 < 0 \quad \Leftrightarrow \quad u > R > d$$

To derive the above "no-arbitrage" condition using geometrical intuition, a vector normal to $\begin{pmatrix} \frac{u}{R} - 1 \\ \frac{d}{R} - 1 \end{pmatrix} S$ lies in the first quadrant of $Q(\omega_u)$-$Q(\omega_d)$ plane if and only if $u > R > d$.

By risk neutral valuation formula, we have

$$c = \frac{Q(\omega_u)c_u + Q(\omega_d)c_d}{R} = \frac{1}{R} E^*[c^{\Delta t} | S].$$

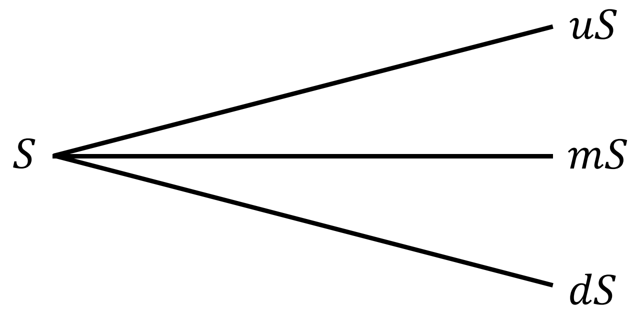


Two equations for the determination of $Q(\omega_u)$ *and* $Q(\omega_d)$

$$Q(\omega_u)\left(\frac{u}{R} - 1\right)S + Q(\omega_d)\left(\frac{d}{R} - 1\right)S = 0$$
$$Q(\omega_u) + Q(\omega_d) = 1.$$

Extension to the trinomial model with 3 states of the world

We extend the two-jump assumption to the three-jump model:



with $u > m > d$

We lose market completeness if we only have the money market account and the underlying risky asset in the securities model. We expect non-uniqueness of risk neutral measures, if they do exist. The system of equations for the determination of the set of risk neutral measures is given by

$$(Q(\omega_u) \quad Q(\omega_m) \quad Q(\omega_d)) \begin{pmatrix} 1 & \frac{uS}{R} \\ 1 & \frac{mS}{R} \\ 1 & \frac{dS}{R} \end{pmatrix} = (1 \quad S).$$

Summary

- The binomial call value formula can be expressed by the following risk neutral valuation formulation:

$$c = \frac{1}{R} E^*[c^{\Delta t}|S],$$

  where c denotes the call value at the current time, and $c^{\Delta t}$ denotes the random variable representing the call value one period later. The call price can be interpreted as the expectation of the payoff of the call option at expiry under the risk neutral probability measure E^* discounted at the riskless interest rate.

- Since there are 3 states of the world in a trinomial model, the application of the principle of replication of claims fails to derive the trinomial option pricing formula. Alternatively, one may use the risk neutral valuation approach via the determination of the risk neutral measures.

Multiperiod binomial models

Let c_{uu} denote the call value at two periods beyond the current time with two consecutive upward moves of the asset price and similar notational interpretation for c_{ud} and c_{dd}. The call values c_u and c_d are related to c_{uu}, c_{ud} and c_{dd} as follows:

$$c_u = \frac{pc_{uu} + (1-p)c_{ud}}{R} \text{ and } c_d = \frac{pc_{ud} + (1-p)c_{dd}}{R}.$$

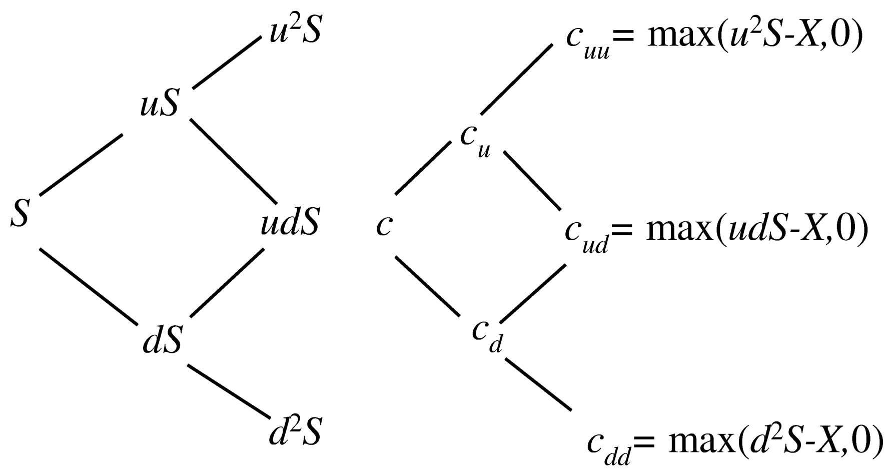
The call value at the current time which is two periods from expiry is found to be

$$c = \frac{p^2 c_{uu} + 2p(1-p)c_{ud} + (1-p)^2 c_{dd}}{R^2},$$

where the corresponding terminal payoff values are given by

$$c_{uu} = \max(u^2 S - X, 0), c_{ud} = \max(udS - X, 0), c_{dd} = \max(d^2 S - X, 0).$$

Note that the coefficients $p^2, 2p(1-p)$ and $(1-p)^2$ represent the respective risk neutral probability of having two up jumps, one up jump and one down jump, and two down jumps in two moves of the binomial process.



Dynamics of asset price and call price in a two-period binomial model.

- With n binomial steps, the risk neutral probability of having j up jumps and $n-j$ down jumps is given by $\begin{pmatrix} n \\ j \end{pmatrix} p^j(1-p)^{n-j}$, where $\begin{pmatrix} n \\ j \end{pmatrix} = \frac{n!}{j!(n-j)!}$ is the binomial coefficient.

- The corresponding terminal payoff when j up jumps and $n-j$ down jumps occur is seen to be $\max(u^j d^{n-j}S - X, 0)$.

- The call value obtained from the n-period binomial model is given by

$$c = \frac{\sum_{j=0}^{n} \begin{pmatrix} n \\ j \end{pmatrix} p^j(1-p)^{n-j} \max(u^j d^{n-j}S - X, 0)}{R^n}.$$

Minimum number of upward moves required for the call being in-the-money at expiry

We define k to be the smallest non-negative integer such that $u^k d^{n-k} S \geq X$, that is, $k \geq \frac{\ln \frac{X}{Sd^n}}{\ln \frac{u}{d}}$. It is seen that

$$\max(u^j d^{n-j} S - X, 0) = \begin{cases} 0 & \text{when} \quad j < k \\ u^j d^{n-j} S - X & \text{when} \quad j \geq k \end{cases}.$$

The integer k gives the minimum number of upward moves required for the asset price in the multiplicative binomial process in order that the call expires in-the-money.

The call price formula is simplified as

$$c = S \sum_{j=k}^{n} \binom{n}{j} p^j (1-p)^{n-j} \frac{u^j d^{n-j}}{R^n} - XR^{-n} \sum_{j=k}^{n} \binom{n}{j} p^j (1-p)^{n-j}.$$

Interpretation of the call price formula

The last term in above equation can be interpreted as the expectation value of the payment made by the holder at expiration discounted by the factor R^{-n}, and $\sum_{j=k}^{n} \binom{n}{j} p^j (1-p)^{n-j}$ is seen to be the probability (under the risk neutral measure) that the call will expire in-the-money.

The above probability is related to the *complementary binomial distribution function* defined by

$$\Phi(n,k,p) = \sum_{j=k}^{n} \binom{n}{j} p^j (1-p)^{n-j}.$$

Note that $\Phi(n,k,p)$ gives the probability for at least k successes in n trials of a binomial experiment, where p is the probability of success in each trial.

Further, if we write $p' = \frac{up}{R}$ so that $1 - p' = \frac{d(1-p)}{R}$, then the call price formula for the n-period binomial model can be expressed as

$$c = S\Phi(n,k,p') - XR^{-n}\Phi(n,k,p).$$

Alternatively, from the risk neutral valuation principle, we have

$$c = \frac{1}{R^n} E_Q\left[S_T \mathbf{1}_{\{S_T > X\}}\right] - \frac{X}{R^n} E_Q\left[\mathbf{1}_{\{S_T > X\}}\right].$$

- The first term gives the discounted expectation of the asset price at expiration given that the call expires in-the-money.

- The second term gives the present value of the risk neutral expectation of payment incurred conditional on the call being in-the-money at expiry.