MAFS 5030 - Quantitative Modeling of Derivative Securities

Topic 3 – Black-Scholes-Merton framework and Martingale Pricing Theory

3.1 Review of stochastic processes and Ito calculus

3.2 Change of measure – Girsanov's Theorem

3.3 Riskless hedging principle and dynamic replication strategy

3.4 Risk neutral measure

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3.1 Review of stochastic processes and Ito calculus

- A *Markovian process* is a stochastic process that, given the value of X_s, the value of $X_t, t > s$, depends only on X_s but not on the values taken by $X_u, u < s$.
- If the asset price process follows a Markovian process, then only the present asset prices are relevant for predicting their future values.
- This Markovian property of the asset price process is consistent with the *weak form of market efficiency*, which assumes that the present value of an asset price already impounds all information in past prices and the particular path taken by the asset price to reach the present value is irrelevant.

Brownian motion

The *Brownian motion with drift* is a stochastic process $\{X(t); t \geq 0\}$ with the following properties:

(i) Every increment $X(t+s) - X(s)$ is normally distributed with mean μt and variance $\sigma^2 t$; μ and σ are fixed parameters.

(ii) For every $t_1 < t_2 < \cdots < t_n$, the increments $X(t_2) - X(t_1), \cdots, X(t_n) - X(t_{n-1})$ are independent random variables with distributions given in (i). That is, the Brownian motion has stationary increments.

(iii) $X(0) = 0$ and the sample paths of $X(t)$ are continuous.

- Note that the Brownian increment $X(t+s) - X(s)$ is independent of the past history of the random path, that is, the knowledge of $X(\tau)$ for $\tau < s$ has no effect on the probability distribution for $X(t+s) - X(s)$. This is precisely the Markovian character of the Brownian motion.

Standard Brownian motion

For the particular case $\mu = 0$ and $\sigma^2 = 1$, the Brownian motion is called the *standard Brownian motion* (or *standard Wiener process*). By virtue of the normal distribution of the Brownian increment $Z(t) - Z(t_0)$, the conditional probability distribution for the standard Wiener process $\{Z(t); t \geq 0\}$ is given by

$$\begin{aligned} P[Z(t) \leq z | Z(t_0) = z_0] &= P[Z(t) - Z(t_0) \leq z - z_0] \\ &= \frac{1}{\sqrt{2\pi(t - t_0)}} \int_{-\infty}^{z - z_0} \exp\left(-\frac{x^2}{2(t - t_0)}\right) dx \\ &= N\left(\frac{z - z_0}{\sqrt{t - t_0}}\right). \end{aligned}$$

Overlapping Brownian increments

(a) $E[Z(t)^2] = \text{var}(Z(t)) + E[Z(t)]^2 = t$.

(b) $E[Z(t)Z(s)] = \min(t, s)$.

To show the result in (b), we assume $t > s$ and consider

$$\begin{aligned} E[Z(t)Z(s)] &= E[\{Z(t) - Z(s)\}Z(s) + Z(s)^2] \\ &= E[\{Z(t) - Z(s)\}Z(s)] + E[Z(s)^2]. \end{aligned}$$

Since $Z(t) - Z(s)$ and $Z(s)$ are independent and both $Z(t) - Z(s)$ and $Z(s)$ have zero mean, so

$$E[Z(t)Z(s)] = E[Z(s)^2] = s = \min(t, s).$$

When $t > s$, the correlation coefficient ρ between the two overlapping Brownian increments $Z(t)$ and $Z(s)$ is given by

$$\rho = \frac{E[Z(t)Z(s)]}{\sqrt{\text{var}(Z(t))}\sqrt{\text{var}(Z(s))}} = \frac{s}{\sqrt{st}} = \sqrt{\frac{s}{t}}.$$

Joint distribution of $Z(s)$ and $Z(t)$

Since both $Z(t)$ and $Z(s)$ are normally distributed with zero mean and variance t and s, respectively, where $s < t$, the probability distribution of the overlapping Brownian increments is given by the bivariate normal distribution function.

If we define $X_1 = Z(t)/\sqrt{t}$ and $X_2 = Z(s)/\sqrt{s}$, then X_1 and X_2 become standard normal random variables. We then have

$$\begin{aligned} P[Z(t) \leq z_t, Z(s) \leq z_s] &= P[X_1 \leq z_t/\sqrt{t}, X_2 \leq z_s/\sqrt{s}] \\ &= N_2(z_t/\sqrt{t}, z_s/\sqrt{s}; \sqrt{s/t}) \end{aligned}$$

where the bivariate normal distribution function is given by

$$\begin{aligned} N_2(x_1, x_2; \rho) &= \int_{-\infty}^{x_2} \int_{-\infty}^{x_1} \frac{1}{2\pi\sqrt{1-\rho^2}} \\ &\quad \exp\left(-\frac{\xi_1^2 - 2\rho\xi_1\xi_2 + \xi_2^2}{2(1-\rho^2)}\right) d\xi_1 d\xi_2. \end{aligned}$$

Geometric Brownian motion

Let $X(t)$ denote the Brownian motion with drift parameter μ and variance parameter σ^2. The stochastic process defined by

$$Y(t) = e^{X(t)}, \quad t \geq 0,$$

is called the *Geometric Brownian motion*. The value taken by $Y(t)$ is non-negative.

Since $X(t) = \ln Y(t)$ is a Brownian motion, by properties (i) and (ii), we deduce that $\ln Y(t) - \ln Y(0)$ is normally distributed with mean μt and variance $\sigma^2 t$. For common usage, $\dfrac{Y(t)}{Y(0)}$ is said to be lognormally distributed.

The density functions of $X(t)$ and $Y(t)$ are related by

$$f_X(x,t)dx = f_Y(y,t)\frac{dy}{y},$$

where $x = \ln y$ and

$$f_X(x,t) = \frac{1}{\sqrt{2\pi\sigma^2 t}} \exp\left(-\frac{(x-\mu t)^2}{2\sigma^2 t}\right).$$

The density function of $\frac{Y(t)}{Y(0)}$ is deduced to be

$$f_Y(y,t) = \frac{1}{y\sqrt{2\pi\sigma^2 t}} \exp\left(-\frac{(\ln y - \mu t)^2}{2\sigma^2 t}\right).$$

The mean of $Y(t)$ conditional on $Y(0) = y_0$ is found to be

$$\begin{aligned}
&E[Y(t)|Y(0) = y_0] \\
=\quad & y_0 \int_0^\infty y f_Y(y,t)\ dy \\
=\quad & y_0 \int_{-\infty}^\infty \frac{e^x}{\sqrt{2\pi\sigma^2 t}} \exp\left(-\frac{(x-\mu t)^2}{2\sigma^2 t}\right)\ dx, \qquad x = \ln y, \\
=\quad & y_0 \int_{-\infty}^\infty \frac{1}{\sqrt{2\pi\sigma^2 t}} \exp\left(-\frac{[x-(\mu t + \sigma^2 t)]^2 - 2\mu t\sigma^2 t - \sigma^4 t^2}{2\sigma^2 t}\right)\ dx \\
=\quad & y_0 \exp\left(\mu t + \frac{\sigma^2 t}{2}\right).
\end{aligned}$$

The variance of $Y(t)$ conditional on $Y(0) = y_0$ is found to be

$$
\begin{aligned}
& \text{var}(Y(t)|Y(0) = y_0) \\
= \quad & y_0^2 \int_0^\infty y^2 f_Y(y,t)\ dy - \left[y_0 \exp\left(\mu t + \frac{\sigma^2 t}{2} \right) \right]^2 \\
= \quad & y_0^2 \left\{ \int_{-\infty}^\infty \frac{1}{\sqrt{2\pi\sigma^2 t}} \exp\left(-\frac{[x - (\mu t + 2\sigma^2 t)]^2 - 4\mu t \sigma^2 t - 4\sigma^4 t^2}{2\sigma^2 t} \right) dx \right. \\
& \left. \qquad - \left[\exp\left(\mu t + \frac{\sigma^2 t}{2} \right) \right]^2 \right\} \\
= \quad & y_0^2 \exp(2\mu t + \sigma^2 t)[\exp(\sigma^2 t) - 1].
\end{aligned}
$$

Quadratic variation of a Brownian motion

Brownian paths are known to be non-differentiable. The property of non-differentiability property is related to the finiteness of the quadratic variation of a Brownian motion.

Suppose we form a partition π of the time interval $[0, T]$ by the discrete points

$$0 = t_0 < t_1 < \cdots < t_n = T,$$

and let $\delta t_{max} = \max_k (t_k - t_{k-1})$. We write $\Delta t_k = t_k - t_{k-1}$, and define the corresponding quadratic variation of the standard Brownian motion $Z(t)$ by

$$Q_\pi = \sum_{k=1}^{n} [Z(t_k) - Z(t_{k-1})]^2.$$

In the mean square sense, we would like to show that the quadratic variation of $Z(t)$ over $[0, T]$ is given by

$$Q_{[0,T]} = \lim_{\delta t_{max} \to 0} Q_\pi = T.$$

This is equivalent to say

(i) $E[Q_\pi] = T$, and (ii) $\lim_{\delta t_{max} \to 0} \text{var}(Q_\pi - T) = 0$.

Proof

Since $Z(t_k) - Z(t_{k-1})$ has zero mean, we have

$$\begin{aligned} & E[Q_\pi] \\ = & \sum_{k=1}^{n} E[\{Z(t_k) - Z(t_{k-1})\}^2] \\ = & \sum_{k=1}^{n} \text{var}(Z(t_k) - Z(t_{k-1})) \\ = & \sum_{k=1}^{n} (t_k - t_{k-1}) = t_n - t_0 = T. \end{aligned}$$

Consider

$$\mathrm{var}(Q_\pi - T) = E\left[\sum_{k=1}^{n}\sum_{\ell=1}^{n}\left\{[Z(t_k)-Z(t_{k-1})]^2-\Delta t_k\right\}\right.$$
$$\left.\{[Z(t_\ell)-Z(t_{\ell-1})]^2-\Delta t_\ell\}\right].$$

Since the increments $[Z(t_k)-Z(t_{k-1})], k = 1, \cdots, n$ are independent, only those terms corresponding to $k = \ell$ in the above series survive, so we have

$$\begin{aligned}\mathrm{var}(Q_\pi - T) &= E\left[\sum_{k=1}^{n}\left\{[Z(t_k)-Z(t_{k-1})]^2-\Delta t_k\right\}^2\right]\\ &= \sum_{k=1}^{n} E\left[\{Z(t_k)-Z(t_{k-1})\}^4\right]\\ &\quad - 2\Delta t_k \sum_{k=1}^{n} E\left[\{Z(t_k)-Z(t_{k-1})\}^2\right] + \Delta t_k^2.\end{aligned}$$

Since $Z(t_k) - Z(t_{k-1})$ is normally distributed with zero mean and variance Δt_k, its fourth order moment is known to be

$$E[\{Z(t_k) - Z(t_{k-1})\}^4] = 3\Delta t_k^2,$$

so

$$\text{var}(Q_\pi - T) = \sum_{k=1}^{n} [3\Delta t_k^2 - 2\Delta t_k^2 + \Delta t_k^2] = 2\sum_{k=1}^{n} \Delta t_k^2.$$

In taking the limit $\delta t_{max} \to 0$, we observe that $\text{var}(Q_\pi - T) \to 0$.

By virtue of $\lim_{\delta t_{max} \to 0} \text{var}(Q_\pi - T) = 0$, we say that T is the *mean square limit* of Q_π.

Remarks

1. In general, the quadratic variation of the Brownian motion with variance rate σ^2 over the time interval $[t_1, t_2]$ is given by

$$Q_{[t_1,t_2]} = \sigma^2(t_2 - t_1).$$

2. If we write $dZ(t) = Z(t) - Z(t - dt)$, where the time interval is dt. Note that the standard Brownian motion has unit variance rate and zero mean. In terms of differentials, we deduce that

$$E[dZ(t)^2] = dt \quad \text{and} \quad \text{var}(dZ(t)^2) = 2\ dt^2.$$

Since dt^2 is an infinitesimally small quantity of higher order, we may claim that $dZ(t)^2$ converges in the mean square sense to the deterministic quantity dt. We then have

$$\int_0^T (\mathrm{d}Z(t))^2 = \int_0^T \mathrm{d}t = T.$$

Definition of stochastic integration

Let $f(Z,t)$ be an arbitrary function of Z and t, where $Z(t)$ be the standard Brownian motion. First, we consider the definition of the stochastic integral $\int_0^T f(Z,t)\, dZ(t)$ as a limit of the following partial sums:

$$\int_0^T f(Z,t)\, dZ(t) = \lim_{n\to\infty} \sum_{k=1}^{n} f(Z(\xi_k),\xi_k)[Z(t_k) - Z(t_{k-1})]$$

where the discrete points $0 < t_0 < t_1 < \cdots < t_n = T$ form a partition of the interval $[0,T]$ and ξ_k is some immediate point between t_{k-1} and t_k. The limit is taken in the *mean square sense*.

Unfortunately, the limit depends on how the immediate points are chosen. For example, suppose we take $f(Z,t) = Z$ and choose $\xi_k = \alpha t_k + (1-\alpha)t_{k-1}, 0 \le \alpha \le 1$, for all k. We consider

$$
\begin{aligned}
& E\left[\sum_{k=1}^{n} Z(\xi_k)[Z(t_k) - Z(t_{k-1})\right] \\
= & \sum_{k=1}^{n} E\left[Z(\xi_k)Z(t_k) - Z(\xi_k)Z(t_{k-1})\right] \\
= & \sum_{k=1}^{n} [\min(\xi_k, t_k) - \min(\xi_k, t_{k-1})] \\
= & \sum_{k=1}^{n} (\xi_k - t_{k-1}) = \alpha \sum_{k=1}^{n} (t_k - t_{k-1}) = \alpha T,
\end{aligned}
$$

so that the expected value of the stochastic integral depends on the choice of the immediate points.

Non-anticipative function

A function $f(Z,t)$ is said to be *non-anticipative* (非預見) with respect to the Brownian motion $Z(t)$ if the value of the function at time t is determined by the path history of $Z(t)$ up to time t. We may write $f(Z,t) \in \mathcal{F}_t^Z$ (or say $f(Z,t)$ is measurable with respect to $\mathcal{F}_t^Z$), where $\mathcal{F}_t^Z$ is the natural filtration generated by $Z(t)$.

Examples

1. $f_1(Z,t) = \begin{cases} 0 & \text{if } \max\limits_{0 \le s \le t} Z(s) < 5 \\ 1 & \text{if } \max\limits_{0 \le s \le t} Z(s) \ge 5 \end{cases}$ is non-anticipative.

2. $f_2(Z,t) = \begin{cases} 0 & \text{if } \max\limits_{0 \le s \le 1} Z(s) < 5 \\ 1 & \text{if } \max\limits_{0 \le s \le 1} Z(s) \ge 5 \end{cases}$ is not non-anticipative.

   For $t < 1$, the value of $f_2(Z,t)$ cannot be determined since it depends on the realization of the path of $Z(t)$ over $[0,1]$.

- In finance, the investor's portfolio choice is non-anticipative in nature since he makes the investment decision at time t based on the path of the asset price up to time t.

Stochastic integral (Ito version)

Define the stochastic integration by taking $\xi_k = t_{k-1}$ (**left-hand point in each sub-interval**). The *Ito definition of stochastic integral* is given by

$$\int_0^T f(Z,t)\ dZ(t) = \lim_{n\to\infty} \sum_{k=1}^{n} f(Z(t_{k-1}),t_{k-1})[Z(t_k) - Z(t_{k-1})],$$

where $f(Z,t)$ is non-anticipative with respect to $Z(t)$. A Brownian path is "sliced" into consecutive Brownian (Gaussian) increments, and the Brownian increment $Z(t_k) - Z(t_{k-1})$ is multiplied by $f(Z(t_{k-1}),t_{k-1})$, and these numbers are added together to give the stochastic integral.

Consider the k^{th} term:

$$f(Z(t_{k-1}), t_{k-1})\Delta Z_k = f(Z(t_{k-1}), t_{k-1})[Z(t_k) - Z(t_{k-1})],$$

once the history of the path up to time t_{k-1} is revealed, the value of $f(Z(t_{k-1}), t_{k-1})$ is known since $f(Z(t), t)$ is non-anticipative with respect to $Z(t)$. The increment of $f(Z(t_{k-1}), t_{k-1})[Z(t_k) - Z(t_{k-1})]$ over (t_{k-1}, t_k) conditional on the path history up to t_{k-1} is Gaussian with mean zero and variance $[f(Z(t_{k-1}), t_{k-1})]^2(t_k - t_{k-1})$.

Since $Z(t_k) - Z(t_{k-1})$ is the forward Brownian increment beyond t_{k-1}, so it is independent of $f(Z(t_{k-1}), t_{k-1})$. The stochastic integral has zero expectation at $t = 0$ since the mean of individual term over each differential time interval is zero as shown below:

$$\begin{aligned} & E_0[f(Z(t_{k-1}), t_{k-1})[Z(t_k) - Z(t_{k-1})]] \\ = \; & E_0[f(Z(t_{k-1}), t_{k-1})]E_0[Z(t_k) - Z(t_{k-1})] = 0. \end{aligned}$$

That is, the **expectation of an Ito integral is zero**.

Example

Consider the evaluation of the Ito stochastic integral $\int_0^T Z(t)\, dZ(t)$. A naive evaluation according to the usual integration rule gives

$$\int_0^T Z(t)\, dZ(t) = \frac{1}{2}\int_0^T \frac{d}{dt}[Z(t)]^2\, dt = \frac{Z(T)^2 - Z(0)^2}{2},$$

which gives a wrong result. The correct approach is given by

$$\begin{aligned}
\int_0^T Z(t)\, dZ(t) &= \lim_{n\to\infty} \sum_{k=1}^{n} Z(t_{k-1})[Z(t_k) - Z(t_{k-1})] \\
&= \lim_{n\to\infty} \frac{1}{2} \sum_{k=1}^{n} (\{Z(t_{k-1}) + [Z(t_k) - Z(t_{k-1})]\}^2 \\
&\qquad - Z(t_{k-1})^2 - [Z(t_k) - Z(t_{k-1})]^2) \\
&= \frac{1}{2} \lim_{n\to\infty} [Z(t_n)^2 - Z(t_0)^2] \\
&\qquad - \frac{1}{2} \lim_{n\to\infty} \sum_{k=1}^{n} [Z(t_k) - Z(t_{k-1})]^2 \\
&= \frac{Z(T)^2 - Z(0)^2}{2} - \frac{T}{2}.
\end{aligned}$$

Rearranging the terms,

$$2\int_0^T Z(t)\ dZ(t) + \int_0^T\ dt = \int_0^T \frac{d}{dt}[Z(t)]^2\ dt,$$

or in differential form,

$$2Z(t)\ dZ(t) + dt = d[Z(t)]^2.$$

Unlike the usual differential rule, we have the extra term dt.

This comes from the finiteness of the quadratic variation of the Brownian motion. This is because $|Z(t_k) - Z(t_{k-1})|^2$ is of order Δt_k and
$\lim_{n\to\infty} \sum_{k=1}^{n} [Z(t_k) - Z(t_{k-1})]^2$ remains finite on taking the limit.

Stochastic representation of an Ito process

Let $\mathcal{F}_t$ be the natural filtration generated by the standard Brownian motion $Z(t)$ through the observation of the trajectory of $Z(t)$. Let $\mu(t)$ and $\sigma(t)$ be non-anticipative with respect to $Z(t)$ with $\int_0^T |\mu(t)|\ dt < \infty$ and $\int_0^T \sigma^2(t)\ dt < \infty$ (almost surely) for all T. The process $X(t)$ defined by

$$X(t) = X(0) + \int_0^t \mu(s)\ ds + \int_0^t \sigma(s)\ dZ(s),$$

is called an *Ito process*. The integral form is more formal since stochastic integrals are well defined. The differential form of the above equation is given as

$$dX(t) = \mu(t)\ dt + \sigma(t)\ dZ(t).$$

For notational convenience, we write the drift rate as $\mu(t)$ and volatility as $\sigma(t)$ showing time dependence only, though they usually have dependence on $Z(t)$ as well.

Ito's Lemma

Suppose $f(x,t)$ is a twice continuously differentiable function and the stochastic process Y is defined by $Y = f(X,t)$, where

$$dX(t) = \mu(t)\ dt + \sigma(t)\ dZ(t).$$

Since $dZ(t)^2$ converges in the mean square sense to dt, the second order term dX^2 also contributes to the differential dY.

The Ito formula of computing the differential of the stochastic function $f(X,t)$ is given by

$$\begin{aligned} dY &= \left[\frac{\partial f}{\partial t}(X,t) + \mu(t)\frac{\partial f}{\partial x}(X,t) + \frac{\sigma^2(t)}{2}\frac{\partial^2 f}{\partial x^2}(X,t)\right] dt \\ &\quad + \sigma(t)\frac{\partial f}{\partial x}(X,t)\ dZ. \end{aligned}$$

Sketchy proof

Expand ΔY by the Taylor series up to the second order terms:

$$\begin{aligned}\Delta Y &= \frac{\partial f}{\partial t}\Delta t + \frac{\partial f}{\partial x}\Delta X \\ &+ \frac{1}{2}\left(\frac{\partial^2 f}{\partial t^2}\Delta t^2 + 2\frac{\partial^2 f}{\partial x \partial t}\Delta X \Delta t + \frac{\partial^2 f}{\partial x^2}\Delta X^2\right) + O(\Delta X^3, \Delta t^3).\end{aligned}$$

In the limit $\Delta X \to 0$ and $\Delta t \to 0$, we apply the multiplication rules where $dZ^2 = dt, dZdt = 0$, $dt^2 = 0$ and $dX(t)^2 = \sigma^2(t)\ dt$ so that

$$dY = \frac{\partial f}{\partial t}\ dt + \frac{\partial f}{\partial x}\ dX + \frac{\sigma^2(t)}{2}\frac{\partial^2 f}{\partial x^2}\ dt.$$

Writing out in full in terms of dZ and dt, we obtain the Ito formula.

Stochastic differential equation of an exponential Brownian

Consider the exponential Brownian defined by

$$S(t) = S_0 e^{\left(r - \frac{\sigma^2}{2}\right)t + \sigma Z(t)}.$$

Suppose we write

$$X(t) = \left(r - \frac{\sigma^2}{2}\right) t + \sigma Z(t)$$

so that $S(t) = S_0 e^{X(t)}$ and

$$dX(t) = \left(r - \frac{\sigma^2}{2}\right) dt + \sigma\, dZ(t).$$

Treating e^X as a function of the state variable X, the respective partial derivatives of $S = S_0 e^X$ are

$$\frac{\partial S}{\partial t} = 0, \quad \frac{\partial S}{\partial X} = S \quad \text{and} \quad \frac{\partial^2 S}{\partial X^2} = S.$$

Note that $\dfrac{\partial S}{\partial t} = 0$ since $S = S_0 e^X$, where S contains no explicit dependence on the time variable t. The dependence of S on t is via the dependence of X on t.

By the Ito lemma, we obtain

$$\begin{aligned} dS &= \frac{\partial S}{\partial X}\, dX + \frac{\sigma^2}{2}\frac{\partial^2 S}{\partial X^2}\, dt \\ &= \left(r - \frac{\sigma^2}{2} + \frac{\sigma^2}{2}\right) S\ dt + \sigma S\ dZ \end{aligned}$$

or

$$\frac{dS}{S} = r\ dt + \sigma\ dZ.$$

Since $E[X(t)] = \left(r - \dfrac{\sigma^2}{2}\right)t$ and $\mathrm{var}(X(t)) = \sigma^2 t$, the mean and variance of $\ln \dfrac{S(t)}{S_0}$ are found to be $\left(r - \dfrac{\sigma^2}{2}\right)t$ and $\sigma^2 t$, respectively.

Multi-dimensional version of Ito's lemma

Suppose $f(x_1, \cdots, x_n, t)$ is a multi-variate twice continuously differentiable function and the stochastic process Y_n is defined by

$$Y_n = f(X_1, \cdots, X_n, t),$$

where the underlying stochastic process $X_j(t)$ follows the Ito process

$$dX_j(t) = \mu_j(t)\ dt + \sigma_j(t)\ dZ_j(t), \quad j = 1, 2, \cdots, n.$$

The Brownian motions $Z_j(t)$ and $Z_k(t)$ are assumed to be correlated with correlation coefficient ρ_{jk}, where $dZ_j\ dZ_k = \rho_{jk}\ dt$.

In a similar manner, we expand ΔY_n up to the second order terms in ΔX_j:

$$\begin{aligned}\Delta Y_n \;&=\; \frac{\partial f}{\partial t}(X_1, \cdots, X_n, t)\ \Delta t + \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}(X_1, \cdots, X_n, t)\ \Delta X_j \\ &\quad + \frac{1}{2} \sum_{j=1}^{n} \sum_{k=1}^{n} \frac{\partial^2 f}{\partial x_j \partial x_k}(X_1, \cdots, X_n, t)\ \Delta X_j\ \Delta X_k \\ &\qquad + O(\Delta t \Delta X_j) + O(\Delta t^2).\end{aligned}$$

In the limits $\Delta X_j \to 0, j = 1, 2, \cdots, n$, and $\Delta t \to 0$, we neglect the higher order terms in $O(\Delta t \Delta X_j)$ and $O(\Delta t^2)$ and observe $dX_j \, dX_k = \sigma_j(t)\sigma_k(t)\rho_{jk} \; dt$. We then obtain the following multi-dimensional version of the Ito lemma:

$$
\begin{aligned}
dY_n \;=\; & \left[\frac{\partial f}{\partial t}(X_1, \cdots, X_n, t) + \sum_{j=1}^{n} \mu_j(t) \frac{\partial f}{\partial x_j}(X_1, \cdots, X_n, t) \right. \\
& \left. + \frac{1}{2} \sum_{j=1}^{n} \sum_{k=1}^{n} \sigma_j(t)\sigma_k(t)\rho_{jk} \frac{\partial^2 f}{\partial x_j \partial x_k}(X_1, \cdots, X_n, t) \right] dt \\
& + \sum_{j=1}^{n} \sigma_j(t) \frac{\partial f}{\partial x_j}(X_1, \cdots, X_n, t) \; dZ_j.
\end{aligned}
$$

Note that the second order cross-derivative terms arise from the correlation between Z_j and Z_k, quantified by $\rho_{jk} \; dt = dZ_j dZ_k$, $j = 1, 2, \ldots, n$, $k = 1, 2, \ldots, n$.

Example

Suppose S_1 and S_2 follow the geometric Brownian motion, where

$$\frac{dS_i}{S_i} = \mu_i\ dt + \sigma_i\ dZ_i, \quad i = 1, 2,$$

then the product $f = S_1 S_2$ and quotient $g = S_1/S_2$ remain to be geometric Brownian motion (see Qn 3 in HW 3). More specifically, we have

$$\begin{aligned}\frac{df}{f} &= (\mu_1 + \mu_2 + \rho_{12}\sigma_1\sigma_2)dt + \sqrt{\sigma_1^2 + \sigma_2^2 + 2\rho_{12}\sigma_1\sigma_2}\ dZ_f \\ \frac{dg}{g} &= (\mu_1 - \mu_2 - \rho_{12}\sigma_1\sigma_2 + \sigma_2^2)dt + \sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho_{12}\sigma_1\sigma_2}\ dZ_g,\end{aligned}$$

where $dZ_1\ dZ_2 = \rho_{12}\ dt$. These results are important when we discuss pricing of quanto options in Topic 4.

When S_1 and S_2 are both geometric Brownian motions, their product and quotient remain to be geometric Brownian motions. This property is consistent with the observation that sum and difference of normal distributions remain to be normal.

Martingale property of a zero-drift Ito process

Consider an Ito process defined in an integral form

$$X(t) = X(0) + \int_0^t \mu(s)\ ds + \int_0^t \sigma(s)\ dZ(s)$$

with non-zero drift term $\mu(t)$. Let $\mathcal{M}(t) = \int_0^t \sigma(s)\ dZ(s)$, so

$$\mathcal{M}(T) = \mathcal{M}(t) + \int_t^T \sigma(s)\ dZ(s), \qquad T > t.$$

Suppose we take the conditional expectation of $\mathcal{M}(T)$ given the history of the Brownian path up to the time t (denoted by the operator E_t), we obtain

$$E_t[\mathcal{M}(T)] = \mathcal{M}(t) + E_t\left[\int_t^T \sigma(s)\ dZ(s)\right] = \mathcal{M}(t)$$

since the second stochastic integral has zero expectation conditional on $\mathcal{F}_t^Z$ (see p.19). Hence, $\mathcal{M}(t)$ is a martingale. However, $X(t)$ is not a martingale if $\mu(t)$ is non-zero. If we set $r = 0$, the exponential Brownian $S(t) = S_0 e^{-\frac{\sigma^2}{2}t + \sigma Z(t)}$ (see p.26), is a martingale.

3.2 Change of measure – Girsanov's Theorem

Transition density function of a Brownian motion

Let X_t be the unrestricted zero-drift Brownian motion with variance rate σ^2. Write $u(x,t)$ as the density function such that X_t falls within the interval $\left(x - \frac{dx}{2}, x + \frac{dx}{2}\right)$ with probability $u(x,t)\,dx$.

Assume that $X_0 = \xi$, that is, the Brownian path starts at the position ξ at $t = 0$. The governing equation for $u(x,t)$ is given by

$$\frac{\partial u}{\partial t} = \frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x^2}, \qquad -\infty < x < \infty, t > 0,$$

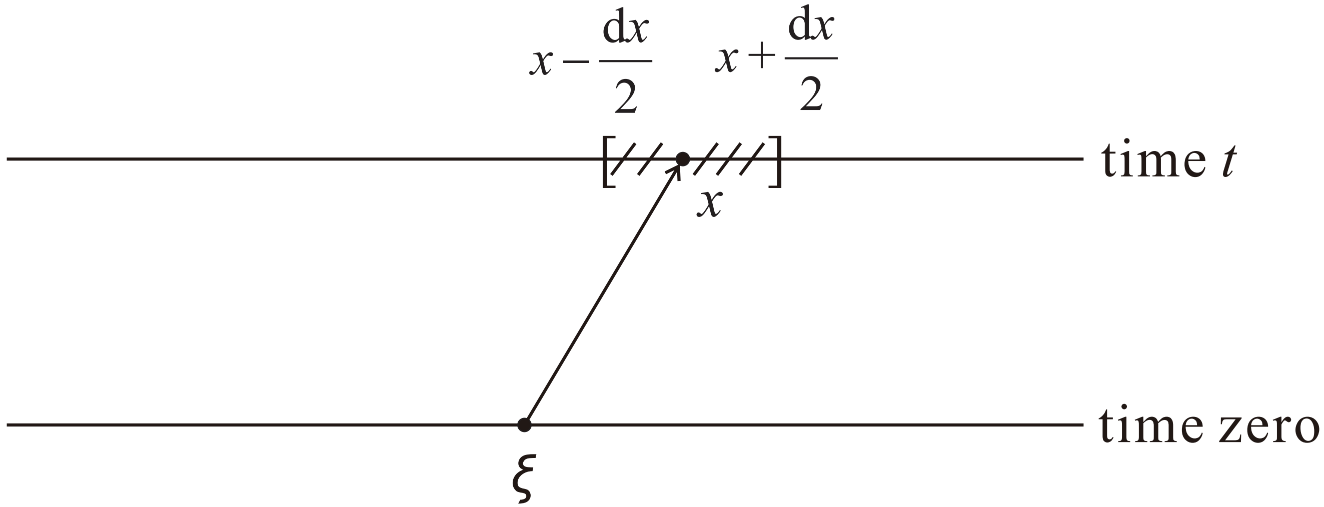
with the initial condition: $u(x,0) = \delta(x - \xi)$.

Note that the Dirac function (impulse at $x = \xi$) observes

$$\delta(x-\xi) = \begin{cases} \infty & x = \xi \\ 0 & x \neq \xi \end{cases}$$

and

$$\int_{-\infty}^{\infty} \delta(x-\xi)\ dx = 1.$$



$$P[X_t \in \mathrm{d}x] = u(x,t)\mathrm{d}x$$

Why the initial condition is given as $u(x,0) = \delta(x-\xi)$? This is because $X_0 = \xi$ for sure at time 0 so that the density function reduces to a probability mass function of a discrete random variable that assumes single value ξ with probability 100%.

Probability mass function of a discrete random varaible

When a discrete random variable X assumes discrete values $x_1, x_2, ..., x_n$, its distribution function is

$$F_X(x) = P[X \leq x] = \sum_{i=1}^{n} P[X = x_i]H(x - x_i).$$

where the step function $H(x - x_i) = \begin{cases} 1 & \text{if } x \geq x_i \\ 0 & \text{otherwise} \end{cases}$.

The derivative of $H(x - x_i)$ becomes infinite at $x = x_i$; otherwise, it is zero. Mathematically, we can establish

$$\delta(x - x_i) = \frac{\mathrm{d}}{\mathrm{d}x} H(x - x_i).$$

When the discrete random variable X assumes single value ξ for sure, then the distribution function $F_X(x)$ and density function $f_X(x)$ both reduce to one term:

$$F_X(x) = H(x - \xi) \quad \text{and} \quad f_X(x) = \delta(x - \xi).$$

The solution to $u(x,t)$ is known to be

$$u(x,t) = \frac{1}{\sigma\sqrt{2\pi t}} \exp\left(-\frac{(x-\xi)^2}{2\sigma^2 t}\right).$$

This is the same as the density function of a normal random variable with mean ξ and variance $\sigma^2 t$. This is not surprising since Brownian increments are normally distributed.

One can verify that $u(x,t)$ does satisfy the differential equation by performing the very tedious calculus calculations of computing $\frac{\partial u}{\partial t}$ and $\frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x^2}$, and check that they are equal.

To check the initial condition, we note that the Gaussian density function tends to the Dirac function as an impulse at the mean ξ when the variance $\sigma^2 t$ tends to zero as $t \to 0$.

Brownian motion with drift

For a Brownian motion with variance rate σ^2 and drift rate μ, the density function is

$$u(x,t) = \frac{1}{\sigma\sqrt{2\pi t}} \exp\left(-\frac{(x-\mu t-\xi)^2}{2\sigma^2 t}\right)$$

since the mean position at time t is $\xi + \mu t$. The corresponding governing equation becomes [see eq.(2.3.11) on P.75 in Kwok's text]

$$\frac{\partial u}{\partial t} = -\mu\frac{\partial u}{\partial x} + \frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x^2}, \qquad u(x,0) = \delta(x-\xi).$$

Moving frame of reference

To the observer in the y-frame moving at the speed μ, the position of the particle moving at the speed μ in the original x-frame appears to be stationary at the position ξ to the moving observer. A position at $\xi + \mu t$ in the x-frame becomes ξ in the y-frame.

The observer at position y in the y-frame is equivalent to the position $x = y + \mu t$ in the x-frame.

In terms of y, the density function becomes

$$u(y,t) = \frac{1}{\sigma\sqrt{2\pi t}} \exp\left(-\frac{(y-\xi)^2}{2\sigma^2 t}\right),$$

which gives the density function of a zero-drift Brownian motion with variance rate σ^2 and starting position ξ under the y-frame.

In our subsequent discussion, for simplicity of presentation, we consider Brownian motion with unit variance rate so that $\sigma^2 = 1$. Also, the starting position ξ is taken to be zero.

We consider the ratio of the two density functions:

$$\frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{(y+\mu t)^2}{2t}\right) \bigg/ \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{y^2}{2t}\right) = \exp\left(-\mu y - \frac{\mu^2 t}{2}\right),$$

which is called the likelihood ratio. Recall that $x = y + \mu t$, the numerator is the zero-drift density function in the x-frame, while the denominator is the zero-drift density function in the y-frame. The likelihood ratio is commonly used in importance sampling algorithm that effects the change of probability measure.

Radon-Nikodym derivatives

Consider a random variable X under two different probability measures P and $\widetilde{P}$, we define symbolically

$$\begin{aligned}
\mathsf{d}P_X(x) &= P\left[X \in \left(x - \frac{\mathsf{d}x}{2}, x + \frac{\mathsf{d}x}{2}\right)\right] = f_X^P(x)\ \mathsf{d}x \\
\mathsf{d}\widetilde{P}_X(x) &= \widetilde{P}\left[X \in \left(x - \frac{\mathsf{d}x}{2}, x + \frac{\mathsf{d}x}{2}\right)\right] = f_X^{\widetilde{P}}(x)\ \mathsf{d}x.
\end{aligned}$$

The expectation calculations of X under P and $\widetilde{P}$ are related by

$$\begin{aligned}
E_{\widetilde{P}}[X] &= \int x\ \mathsf{d}\widetilde{P}_X(x) = \int x\ f_X^{\widetilde{P}}(x)\ \mathsf{d}x = \int x \left[\frac{f_X^{\widetilde{P}}(x)}{f_X^P(x)}\right] f_X^P(x)\ \mathsf{d}x \\
&= \int x \left[\frac{f_X^{\widetilde{P}}(x)}{f_X^P(x)}\right] \mathsf{d}P_X(x) = E_P\left[X \frac{\mathsf{d}\widetilde{P}_X}{\mathsf{d}P_X}\right],
\end{aligned}$$

where $\left.\dfrac{\mathsf{d}\widetilde{P}_X}{\mathsf{d}P_X}\right|_x$ is the likelihood ratio $f_X^{\widetilde{P}}(x)/f_X^P(x)$ of the density functions of X under $\widetilde{P}$ and P. It is coined as the Radon-Nikodym derivative.

Change of measure

Consider the standard P-Brownian motion $Z_P(t)$, which is known to have zero drift and unit variance rate under the measure P. Adding the drift term μt to $Z_P(t)$ (here μ is taken to be constant), then $Z_P^\mu(t) = Z_P(t) + \mu t$ is a Brownian motion with drift rate μ under the measure P.

Can we modify the probability density through the multiplication of a factor such that $Z_P^\mu(t)$ becomes a Brownian motion (zero drift) under the modified measure $\widetilde{P}$? The factor is the Radon-Nikodym derivative $\dfrac{d\widetilde{P}}{dP}$. This procedure is called the change of measure from the original measure P to the new measure $\widetilde{P}$.

For a given value of T, as deduced by the earlier likelihood ratio, the corresponding Radon-Nikodym derivative is seen to be the random variable defined by an exponential Brownian, where

$$\frac{d\widetilde{P}}{dP} = \exp\left(-\mu Z_P(T) - \frac{\mu^2}{2}T\right).$$

To verify the claim, it suffices to show that $Z_{\widetilde{P}}(T)$ visualized as a random variable is normal with zero mean and variance T under the measure $\widetilde{P}$ by looking at the corresponding moment generating function. A random variable X is normal with mean m and variance σ^2 under a measure P if and only if

$$E_P[\exp(\alpha X)] = \exp\left(\alpha m + \frac{\alpha^2}{2}\sigma^2\right), \quad \text{for any real } \alpha.$$

Now, we consider

$$\begin{aligned}
& E_{\widetilde{P}}\left[\exp(\alpha Z_{\widetilde{P}}(T))\right] \\
= \; & E_P\left[\frac{d\widetilde{P}}{dP}\exp(\alpha Z_P(T) + \alpha\mu T)\right] \\
= \; & E_P\left[\exp\left((\alpha - \mu)Z_P(T)\right)\exp\left(\alpha\mu T - \frac{\mu^2}{2}T\right)\right] \\
= \; & \exp\left(\frac{(\alpha-\mu)^2}{2}T + \alpha\mu T - \frac{\mu^2}{2}T\right) = \exp\left(\frac{\alpha^2}{2}T\right), \quad \text{for any real } \alpha,
\end{aligned}$$

hence $Z_{\widetilde{P}}(T)$ is normal with zero mean and variance T under $\widetilde{P}$.

Girsanov Theorem

Consider a non-anticipative function $\gamma(t)$ with respect to $Z_P(t)$ that satisfies the Novikov condition:

$$E[e^{\int_0^t \frac{1}{2}\gamma(s)^2\ ds}] < \infty,$$

and consider the Radon-Nikodym derivative:

$$\frac{d\widetilde{P}}{dP} = \rho(t) = \exp\left(\int_0^t -\gamma(s)\ dZ_P(s) - \frac{1}{2}\int_0^t \gamma(s)^2\ ds\right).$$

Here, $Z_P(t)$ is a Brownian motion under the measure P (called P-Brownian motion). Under the measure $\widetilde{P}$, the stochastic process

$$Z_{\widetilde{P}}(t) = Z_P(t) + \int_0^t \gamma(s)\ ds$$

is $\widetilde{P}$-Brownian. As a remark, when $\gamma(t) = \mu$, a constant, then

$$\rho(t) = \exp\left(-\gamma Z_P(t) - \frac{\gamma^2}{2}t\right).$$

Measure change from Q to P

Under the actual probability measure P, the asset price process follows

$$\frac{dS_t}{S_t} = \rho\, dt + \sigma\, dZ_t^P$$

where Z_t^P is standard P-Brownian (zero drift rate and unit variance rate). Let $S_t^* = S_t/M_t$ be the discounted asset price process, where $M_t = e^{rt}$ [M_t is the solution to $dM_t = rM_t\, dt$, with $M_0 = 1$] and r is the riskfree interest rate.

Under a risk neutral measure Q, the discounted asset price process S_t^* is Q-martingale. To satisfy the martingale property, S_t^* has to be a zero-drift Ito process. This dictates the dynamics to be governed by

$$\frac{dS_t^*}{S_t^*} = \sigma\, dZ_t^Q \quad \text{or} \quad \frac{dS_t}{S_t} = r\, dt + \sigma\, dZ_t^Q,$$

where Z_t^Q is standard Q-Brownian.

On the other hand, we have

$$\frac{dS_t^*}{S_t^*} = (\rho - r)dt + \sigma\ dZ_t^P.$$

Comparing the two dynamical equations for $\frac{\mathrm{d}S_t^*}{S_t^*}$, Z_t^P and Z_t^Q are related by

$$dZ_t^Q = dZ_t^P + \frac{\rho - r}{\sigma}dt.$$

Note that the drift rate $\frac{\rho - r}{\sigma}$ is the market price of risk. By the Girsanov Theorem, the corresponding Radon-Nikodym derivative is given by

$$\frac{dQ}{dP} = \exp\left(-\frac{\rho - r}{\sigma}Z_t^P - \left(\frac{\rho - r}{\sigma}\right)^2 \frac{t}{2}\right).$$

Remark under the continuous time framework, we can also establish: existence of $Q \Rightarrow$ absence of arbitrage. The proof follows similarly to that of the discrete time model.

Feynman-Kac representation formula

Suppose the Ito process $X(t)$ is governed by the stochastic differential equation

$$dX(s) = \mu(X(s), s)\ ds + \sigma(X(s), s)\ dZ(s), \quad t \le s \le T,$$

with initial condition: $X(t) = x$. Here, $\sigma(X(t), t) \in \mathcal{F}_t^Z$.

Consider a smooth function $F(X(t), t)$, by virtue of the Ito lemma, the differential of which is given by

$$dF = \left[\frac{\partial F}{\partial t} + \mu(X,t)\frac{\partial F}{\partial X} + \frac{\sigma^2(X,t)}{2}\frac{\partial^2 F}{\partial X^2}\right] dt + \sigma\frac{\partial F}{\partial X}\ dZ.$$

Suppose F satisfies the partial differential equation

$$\frac{\partial F}{\partial t} + \mu(X,t)\frac{\partial F}{\partial X} + \frac{\sigma^2(X,t)}{2}\frac{\partial^2 F}{\partial X^2} = 0$$

with terminal condition: $F(X(T), T) = h(X(T))$, then the drift term disappears. Therefore, dF becomes

$$dF = \sigma\frac{\partial F}{\partial X}\ dZ.$$

Remark

We deduce the governing partial differential equation for F such that $F(X(t), t)$ becomes a zero-drift Ito process. Recall that a zero-drift Ito process is a martingale. The Feynman-Kac representation formula is derived from this martingale property.

By the martingale property of $F(x(t), t)$, observing $X(t) = x$ and $F(X(T), T) = h(X(T))$, we then obtain the following Feynman-Kac representation formula

$$F(x, t) = E_{x,t}[h(X(T))], \quad t < T,$$

where $F(x, t)$ satisfies the partial differential equation and $E_{x,t}$ refers to expectation taken conditional on $X(t) = x$.

3.3 Riskless hedging principle and dynamic replicating strategy

Riskless hedging principle of Black and Scholes

Writer of a call option – hedges his exposure by holding certain units of the underlying asset in order to create a riskless hedged portfolio.

In an efficient market with no riskless arbitrage opportunity, a riskless hedged portfolio must earn its rate of return equals the riskless interest rate.

Let $\Pi(t)$ be the value of a riskless hedged portfolio. By invoking no-arbitrage argument, we must have

$$d\Pi(t) = r\Pi(t)\,dt,$$

where r is the riskfree interest rate.

Merton's construction of hedged portfolio of option, underlying asset and money market account

Include the money market account in the hedging procedure.

Derive the replication formula: $V = \Delta S + M$, where $\Delta = \frac{\partial V}{\partial S}$.

More importantly, show the *equality of market price of risk* among hedgeable securities. The market price of risk (or called the Sharpe ratio in financial industry) is defined as the excess expected rate of return above the riskfree interest rate r normalized by volatility (taken as proxy of risk).

Recall the formulas for the market prices of risk:

$$\lambda_S = \frac{\rho_S - r}{\sigma_S} \quad \text{and} \quad \lambda_V = \frac{\rho_V - r}{\sigma_V},$$

we obtain $\lambda_S = \lambda_V$ if the asset and option (*both tradeable*) are hedgeable with each other.

Black-Scholes' assumptions on the financial market

(i) Trading takes place continuously in time.

(ii) The riskless interest rate r is known and constant over time.

(iii) The asset pays no dividend.

(iv) There are no transaction costs in buying or selling the asset or the option, and no taxes.

(v) The assets are perfectly divisible.

(vi) There are no penalties to short selling and the full use of proceeds is permitted.

(vii) There are no arbitrage opportunities.

The stochastic process of the asset price S_t is assumed to follow the Geometric Brownian motion

$$\frac{dS_t}{S_t} = \rho \ dt + \sigma \ dZ_t.$$

Consider a portfolio which involves short selling of one unit of a European call option and long holding of Δ_t units of the underlying asset. The portfolio value $\Pi_t = \Pi(S_t, t)$ at time t is given by

$$\Pi_t = -c_t + \Delta_t S_t,$$

where $c_t = c(S_t, t)$ denotes the call price as a function of the state variable S_t and time t.

Note that Δ_t changes with time t, reflecting the dynamic nature of hedging. Since both c and Π are functions of the stochastic state variable S_t, we apply the Ito Lemma to give

$$dc = \frac{\partial c}{\partial t} \ dt + \frac{\partial c}{\partial S} \ dS + \frac{\sigma^2}{2} S^2 \frac{\partial^2 c}{\partial S^2} \ dt.$$

Black and Scholes assume that Δ_t is held fixed from t to $t + dt$, so that the differential change in the portfolio value Π is given by

$$\begin{aligned} & -dc + \Delta\ dS \\ = & \left(-\frac{\partial c}{\partial t} - \frac{\sigma^2}{2} S^2 \frac{\partial^2 c}{\partial S^2}\right) dt + \left(\Delta - \frac{\partial c}{\partial S}\right) dS \\ = & \left[-\frac{\partial c}{\partial t} - \frac{\sigma^2}{2} S^2 \frac{\partial^2 c}{\partial S^2} + \left(\Delta - \frac{\partial c}{\partial S}\right) \rho S\right] dt + \left(\Delta - \frac{\partial c}{\partial S}\right) \sigma S\ dZ. \end{aligned}$$

By taking $\Delta = \frac{\partial c}{\partial S}$, the stochastic term associated with dZ vanishes. Also, the term involving ρ also vanishes. The riskless hedged portfolio should earn the riskless rate of return. We then have

$$d\Pi = r\Pi\, dt$$

so that

$$\begin{aligned} & -\frac{\partial c}{\partial t} - \frac{\sigma^2}{2} S^2 \frac{\partial^2 c}{\partial S^2} = r\left(-c + S\frac{\partial c}{\partial S}\right) \\ \Leftrightarrow\ & \frac{\partial c}{\partial t} + rS\frac{\partial c}{\partial S} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 c}{\partial S^2} - rc = 0, \quad \text{where } c = c(S, t). \end{aligned}$$

- The negligence of the product rule: $d(\Delta_t S_t) = \Delta_t \, dS_t + S_t \, d\Delta_t$ is justified since the investor's trading strategy of holding Δ_t units of asset is made at the beginning of the time period $(t, t + dt)$.

- The above parabolic partial differential equation is called the *Black-Scholes equation*. Note that the parameter ρ, which is the expected rate of return of the asset, does not appear in the equation. The independence of the pricing model on ρ is related to the concept of *risk neutrality*.

- The terminal payoff at time T of the European call with strike price X is translated into the following terminal condition:

$$c(S, T) = \max(S - X, 0).$$

- The option pricing model involves five parameters: S, T, X, r and σ, all except the volatility σ are directly observable parameters.

Deficiencies in the model

1. Geometric Brownian motion assumption of the asset price process is always debatable. Actual asset price dynamics is much more complicated. Later models allow the asset price process to follow the jump-diffusion process and exhibit stochastic volatility.

2. Continuous hedging at all times

   Since trading usually involves transaction costs, continuous hedging would incur infinite transaction costs.

3. Interest rate should be stochastic instead of deterministic.

Black and Scholes use the differential formulation of $d\Pi$ and follow the "pragmatic" approach of keeping the hedge ratio Δ_t to be instantaneously "frozen" in the next differential time interval dt.

Merton's construction of hedged portfolio of three securities

$Q_S(t) =$ number of units of asset

$Q_V(t) =$ number of units of option

$M_S(t) =$ dollar value of $Q_S(t)$ units of asset

$M_V(t) =$ dollar value of $Q_V(t)$ units of option

$M(t) =$ value of riskless asset invested in money market account

- Construction of a self-financing and dynamically hedged portfolio containing risky asset, option and money market account.

- Dynamic replication: Composition is allowed to change at all times in the replication process.
- The self-financing portfolio is set up with zero initial net investment cost and no additional funds added or withdrawn afterwards.

The assumption of zero net investment at time t gives

$$\begin{aligned}\Pi(t) &= M_S(t) + M_V(t) + M(t) \\ &= Q_S(t)S + Q_V(t)V + M(t) = 0.\end{aligned}$$

Using Ito's lemma, we compute the differential of option value V as follows:

$$\begin{aligned}dV &= \frac{\partial V}{\partial t}\,dt + \frac{\partial V}{\partial S}\,dS + \frac{\sigma^2}{2}S^2\frac{\partial^2 V}{\partial S^2}\,dt \\ &= \left(\frac{\partial V}{\partial t} + \rho S\frac{\partial V}{\partial S} + \frac{\sigma^2}{2}S^2\frac{\partial^2 V}{\partial S^2}\right)dt + \sigma S\frac{\partial V}{\partial S}\,dZ.\end{aligned}$$

Formally, we write the stochastic dynamics of V as

$$\frac{dV}{V} = \rho_V \, dt + \sigma_V \, dZ$$

where

$$\rho_V = \frac{\frac{\partial V}{\partial t} + \rho S \frac{\partial V}{\partial S} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2}}{V} \quad \text{and} \quad \sigma_V = \frac{\sigma S \frac{\partial V}{\partial S}}{V}.$$

The differential change in portfolio value is given by

$$\begin{aligned} d\Pi(t) &= [Q_S(t)\, dS + Q_V(t)\, dV + rM(t)\, dt] \\ &\quad + \underbrace{[S\, dQ_S(t) + V\, dQ_V(t) + dM(t)]}_{\text{zero due to self-financing trading strategy}} \end{aligned}$$

- We apply the same argument that $Q_S(t)$ and $Q_V(t)$ are kept the same value over $[t, t + \mathrm{d}t]$, so the contributions to $\mathrm{d}\Pi(t)$ arise from the differential changes $\mathrm{d}S$ and $\mathrm{d}V$.

- The term $rM(t)\, dt$ arises from the interest amount earned from the money market account over dt.

Recall $\Pi(t) = 0$, the instantaneous portfolio return $d\Pi(t)$ can be expressed in terms of $M_S(t)$ and $M_V(t)$ as follows:

$$\begin{aligned} d\Pi(t) &= Q_S(t)\,dS + Q_V(t)\,dV + rM(t)\,dt \\ &= M_S(t)\frac{dS}{S} + M_V(t)\frac{dV}{V} + rM(t)\,dt \\ &= [(\rho - r)M_S(t) + (\rho_V - r)M_V(t)]\,dt \\ &\quad + \ [\sigma M_S(t) + \sigma_V M_V(t)]\,dZ. \end{aligned}$$

We make the self-financing portfolio to be instantaneously riskless by choosing $M_S(t)$ and $M_V(t)$ such that the stochastic term becomes zero.

From the relation:

$$\sigma M_S(t) + \sigma_V M_V(t) = \sigma S Q_S(t) + \frac{\sigma S \frac{\partial V}{\partial S}}{V} V Q_V(t) = 0,$$

we obtain the following ratio of the units of asset and derivative to be held

$$\frac{Q_S(t)}{Q_V(t)} = -\frac{\partial V}{\partial S}.$$

Taking $Q_V(t) = -1$, and knowing

$$0 = \Pi(t) = -V + \Delta S + M(t)$$

we obtain

$$V = \Delta S + M(t), \text{ where } \Delta = \frac{\partial V}{\partial S}.$$

- In the case of shorting one unit of the option, $Q_V(t) = -1$, the above equation implies that the position of one unit of option can be replicated by a self-financing trading strategy using Δ units of S and $M(t)$, where $\Delta = \dfrac{\partial V}{\partial S}$.

Numerical example

Suppose the call option value increases by \$0.3 when the underlying asset increases \$1 in value, then $\partial V/\partial S \approx 0.3$. To hedge the sale of one unit of the call, the hedger holds 0.3 units of the underlying asset so that

$$\$1 \times 0.3 + \$0.3 \times (-1) = 0.$$

The dynamic replicating portfolio is riskless and requires no net investment, so $d\Pi(t) = 0$. Putting all these relations together, we obtain

$$0 = [(\rho - r)M_S(t) + (\rho_V - r)M_V(t)]\,dt.$$

Putting $\dfrac{Q_S(t)}{Q_V(T)} = -\dfrac{\partial V}{\partial S}$, we obtain

$$(\rho - r)S\frac{\partial V}{\partial S} = (\rho_V - r)V.$$

Substituting ρ_V by $\left[\dfrac{\partial V}{\partial t} + \rho S\dfrac{\partial V}{\partial S} + \dfrac{\sigma^2}{2}S^2\dfrac{\partial^2 V}{\partial S^2}\right]\Big/ V$, we obtain the Black-Scholes equation

$$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2}S^2\frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0.$$

Pricing equation without financial economics concepts

From $\rho_V = \dfrac{\frac{\partial V}{\partial t} + \rho S \frac{\partial V}{\partial S} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2}}{V}$, we obtain

$$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} + \rho S \frac{\partial V}{\partial S} - \rho_V V = 0.$$

This equation was derived in 1960s based on mathematics only, without hedging and/or replication concepts and the application of no-arbitrage principle.

To use the differential equation for pricing an option, one needs to calibrate the parameters ρ and ρ_V, or find some other means to avoid such nuisance.

Equality of market prices of risk

From $(\rho - r)S\frac{\partial V}{\partial S} = (\rho_V - r)V$ (derived from hedging and no-arbitrage argument), by combining with the relation: $S\frac{\partial V}{\partial S} = \frac{\sigma_V}{\sigma}V$ (from Ito's lemma), we obtain

$$\underbrace{\frac{\rho_V - r}{\sigma_V}}_{\lambda_V} = \underbrace{\frac{\rho - r}{\sigma}}_{\lambda_S} \quad \Rightarrow \text{ Black-Scholes equation.}$$

Here, λ_V and λ_S are the market price of risk of V and S, respectively. For risk aversion (risk neutral) investors, they demand positive (zero) market price of risk.

The analytic form of the market price of risk is specific to the Geometric Brownian motion assumption in the asset price process, where risk is proxied by volatility σ. The proxy of risk becomes more complicated when the asset price process has different sources of risks.

Relation between the hedge ratio and ratio of volatilities of underlying asset and its derivative

From Ito's lemma, which is a pure mathematical result, we obtain

$$\frac{V\sigma_V}{S\sigma_S} = \frac{\partial V}{\partial S}.$$

The contribution to the random component in dV_t due to the random term dS_t is given by $\frac{\partial V}{\partial S}$ dS_t as revealed in Ito's lemma.

Note that $\frac{\partial V}{\partial S}$ can be negative. In this case, $V\sigma_V$ and $S\sigma_S$ have opposite sign. The randomness of dZ_t leads to different directional moves of dV_t and dS_t. For example, the hedge ratio of a put option is negative. When S increases, the put price decreases.

Since V and S share the source of randomness from the same Brownian motion, this explains why the two instruments are hedgeable. The level of randomness in V is $\frac{\partial V}{\partial S}$ times that in S. Therefore, the ratio of their total volatilities is given by the above relation.

Suppose the writer shorts one unit of the derivative, according to the above relation, he needs to long $\Delta = \frac{\partial V}{\partial S}$ units of the underlying asset. This is because $\frac{\partial V}{\partial S}$ units of the underlying assets is sufficient to offset the randomness generated from one unit of the derivative.

Remarks on risk neutral valuation

- Normally, risk aversion dictates the expected rate of return demanded in investing in an asset. This then affects the price paid by an investor to buy the asset.
- However, if a riskless hedging procedure can be set up with the derivative, it should be priced with no risk premium since hedger's portfolio becomes riskfree.
- When a derivative's payoff can be replicated by a portfolio of instruments available today (underlying asset plus riskless borrowing), then no-arbitrage enforces the derivative price to the same under risk aversion as in a world of risk neutral investors.

Notion of risk neutrality

- The market price of risk is the excess expected rate of return above r per unit risk. The two hedgeable securities (option and asset) should have the same market price of risk. Apparently, the Black-Scholes equation can be obtained by setting $\rho = \rho_V = r$ (implying zero market price of risk). This is why the term "risk neutrality" is commonly adopted in option pricing theory.

- We find the price of a derivative *relative* to that of the underlying asset. The mathematical relationship between the prices is invariant to the risk preference of the investor (independent of ρ). We do not need to assume investors to be risk neutral. If risk neutral behavior of investors is assumed, then the option pricing theory has very limited scope. We use the convenience of risk neutrality to arrive at the mathematical relationship. However, the actual dynamics of S_t does depend on ρ, and thus indirectly affect the option price.

- In the proof of the Black-Scholes equation, ρ disappears when hedging is executed by setting $\Delta = \frac{\partial V}{\partial S}$. Independence of risk aversion prevails as an inherited property of hedging (a gift in the simplication of the pricing procedure). By no-arbitrage argument, $\mathrm{d}\Pi_t = r\Pi_t \ \mathrm{d}t$. This is how the riskless interest rate comes into the Black-Scholes equation.

- Black-Scholes' conceptual breakthrough is to derive no-arbitrage pricing approach such that neither risk aversion nor expected stock returns enter into the Black-Scholes model. Risk averse investors would not require a risk premium in option prices (relative to the stock price). Note that real world limitations to hedging procedures allow mispricing to arise.

“How we came up with the option formula?” —— Black (1989)

- It started with tinkering （笨拙的修補） and ended with delayed recognition.

- The expected return on a warrant should depend on the risk of the warrant in the same way that a common stock’s expected return depends on its risk. In simple language, this statement means equality of market prices of risk.

- I spent many, many days trying to find the solution to that (differential) equation. I have a PhD in applied mathematics, but had never spent much time on differential equations, so I didn’t know the standard methods used to solve problems like that. I have an A.B. in physics, but I didn’t recognize the equation as a version of the heat equation, which has well-known solutions.

Pricing of derivative whose underlying is a non-tradable index

What happens when the underlying is not a tradeable security?

Suppose the derivative price $V(Q, t; T)$ is dependent on some price index Q whose dynamics is

$$dQ_t = \mu(Q_t, t)\, dt + \sigma_Q(Q_t, t)\, dZ_t.$$

Now, Q is not the price of a traded security. We can only hedge two derivatives with respective maturity T_1 and T_2, whose values are dependent on Q. They are hedgeable since their prices are dependent on the same random term dZ_t.

The portfolio value Π of longing T_1-maturity derivative on Q and shorting T_2-maturity derivative on the same Q is given by

$$\Pi = V_1(Q, t; T_1) - V_2(Q, t; T_2),$$

where

$$\frac{dV_i}{V_i} = \mu_V(Q, t; T_i)\, dt + \sigma_V(Q, t; T_i)\, dZ_t, \quad i = 1, 2.$$

By Ito's lemma:

$$\mu_V(Q,t;T_i) = \frac{1}{V_i}\left(\frac{\partial V_i}{\partial t} + \mu\frac{\partial V_i}{\partial Q} + \frac{\sigma_Q^2}{2}\frac{\partial^2 V_i}{\partial Q^2}\right)$$
$$\sigma_V(Q,t;T_i) = \frac{\sigma_Q}{V_i}\frac{\partial V_i}{\partial Q}, \quad i = 1, 2.$$

The change in portfolio value is

$$d\Pi = [V_1\mu_V(Q,t;T_1) - V_2\mu_V(Q,t;T_2)]\, dt + [V_1\sigma_V(Q,t;T_1) - V_2\sigma_V(Q,t;T_2)]\, dZ_t.$$

Suppose V_1 and V_2 are chosen such that

$$V_1 = \frac{\sigma_V(T_2)}{\sigma_V(T_2) - \sigma_V(T_1)}\Pi \quad \text{and} \quad V_2 = \frac{\sigma_V(T_1)}{\sigma_V(T_2) - \sigma_V(T_1)}\Pi,$$

then the stochastic term vanishes in $d\Pi$ and $\Pi = V_1 - V_2$ is satisfied.

We are lucky to have two equations for V_1 and V_2: $V_1\sigma_V(T_1) = V_2\sigma_V(T_2)$ and $\Pi = V_1 - V_2$ and two unknowns: V_1 and $V_:2$, so unique solution can be found.

Once randomness is eliminated, the riskless hedged portfolio value observes: $d\Pi = r\Pi\ dt$ so that

$$\frac{d\Pi}{\Pi} = \frac{\mu_V(T_1)\sigma_V(T_2) - \mu_V(T_2)\sigma_V(T_1)}{\sigma_V(T_2) - \sigma_V(T_1)}\,dt = r\,dt.$$

Rearranging the last two terms, we obtain

$$\frac{\mu_V(T_1) - r}{\sigma_V(T_1)} = \frac{\mu_V(T_2) - r}{\sigma_V(T_2)}.$$

The relation is valid for arbitrary maturity dates T_1 and T_2. Hence,

$$\frac{\mu_V(Q,t) - r}{\sigma_V(Q,t)} = \lambda(Q,t) = \text{ market price of risk of } V,$$

where $\lambda(Q,t)$ has no dependence on T. Substituting the expressions for μ_V and σ_V, we obtain

$$\frac{\partial V}{\partial t} + \mu\frac{\partial V}{\partial Q} + \frac{\sigma_Q^2}{2}\frac{\partial^2 V}{\partial Q^2} - rV = \lambda\sigma_Q\frac{\partial V}{\partial Q}.$$

The governing equation for the derivative value, $V = V(Q, t; T)$, becomes

$$\frac{\partial V}{\partial t} + (\mu - \lambda\sigma_Q)\frac{\partial V}{\partial Q} + \frac{\sigma_Q^2}{2}\frac{\partial^2 V}{\partial Q^2} - rV = 0,$$

where the market price of risk of V is involved. When the index Q is non-tradeable, the drift rate in the option pricing equation is reduced by $\lambda\sigma_Q$ with respect to the actual drift rate μ.

What happens when Q becomes the price of a tradeable security so that the "price of the index" has sensible meaning? We expect that $V = Q$ satisfies the above equation under such scenario. This gives

$$\mu - \lambda \sigma_Q = rQ.$$

Furthermore, we set $\sigma_Q = \sigma Q$, where σ is a constant. We recover the Black-Scholes equation

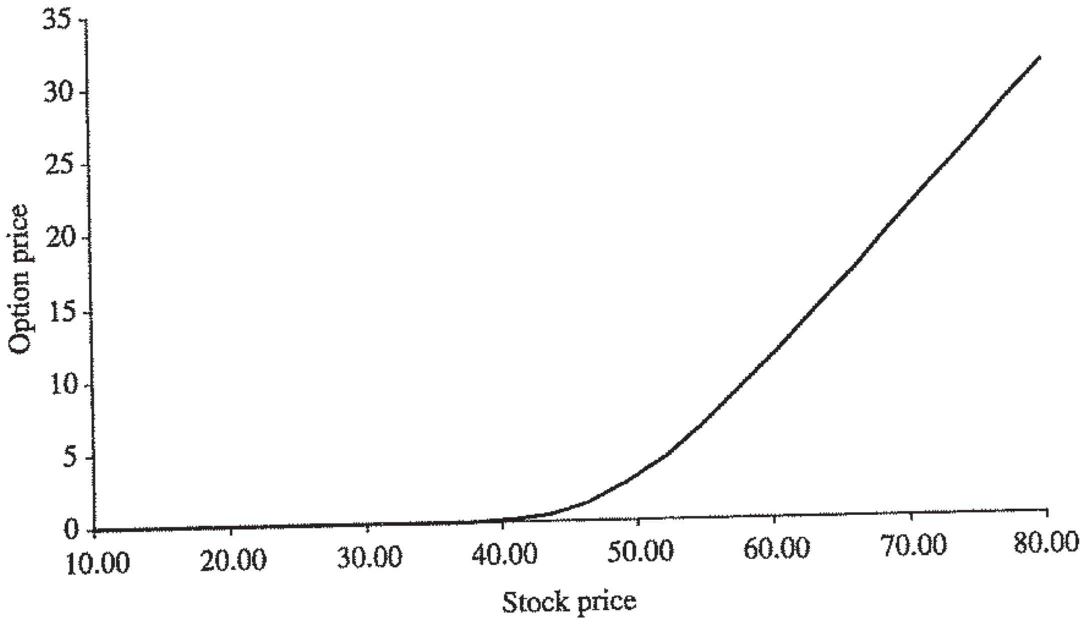
$$\frac{\partial V}{\partial t} + rQ \frac{\partial V}{\partial Q} + \frac{\sigma^2}{2} Q^2 \frac{\partial^2 V}{\partial Q^2} - rV = 0.$$

Note that non-tradable Q would *not* allow us to set $V = Q$.

Dynamic hedging of a call option

A trader sells 100,000 European call options on a non-dividend-paying stock: $S = \$49$, $X = \$50$, $r = 5\%$, $\sigma = 20\%$, $T = 20$ weeks.

Terminal payoff of a call option $= \max(S_T - X, 0)$.



Value of Call Option as a Function of Stock Price

Value $V(S,t)$ and delta $\Delta = \frac{\partial V}{\partial S}$ of the call are calculated based on an option pricing model (potential exposure to model risk since the hedger needs to specify the volatility of the underlying asset price).

Dynamic hedging of an European call position at work

At the time of the trade, the call option fair value is \$2.40 and the delta is 0.522. Suppose the amount received by the seller for the options is $\$300,000$ (good for the seller). Since the seller is short $100,000$ options, the value of the seller's portfolio is $-\$240,000$.

Immediately after the trade, the seller's portfolio can be made delta neutral by buying $52,200$ shares of the underlying stock. The cost of shares purchased $= 52,200 \times \$49 = 2,557.8$ thousand.

Since the delta changes when stock price changes over the life of the option, the trader has to adjust the stock holding amount via re-balancing in order to maintain delta-neutral. This is called *dynamic hedging*.

Scenario One: call option expires in-the-money

Simulation of Delta Hedging (option closes in-the-money and cost of hedging is $263,300)

Week	Stock Price	Delta	Shares Purchased	Cost of Shares Purchased ($000)	Cumulative Cash Outflow ($000)	Interest Cost ($000)
0	49.00	0.522	52,200	2,557.8	2,557.8	2.5
1	48.12	0.458	(6,400)	(308.0)	2,252.3	2.2
2	47.37	0.400	(5,800)	(274.7)	1,979.8	1.9
3	50.25	0.596	19,600	984.9	2,966.6	2.9
4	51.75	0.693	9,700	502.0	3,471.5	3.3
5	53.12	0.774	8,100	430.3	3,905.1	3.8
6	53.00	0.771	(300)	(15.9)	3,893.0	3.7
7	51.87	0.706	(6,500)	(337.2)	3,559.5	3.4
8	51.38	0.674	(3,200)	(164.4)	3,398.5	3.3
9	53.00	0.787	11,300	598.9	4,000.7	3.8
10	49.88	0.550	(23,700)	(1,182.2)	2,822.3	2.7
11	48.50	0.413	(13,700)	(664.4)	2,160.6	2.1
12	49.88	0.542	12,900	643.5	2,806.2	2.7
13	50.37	0.591	4,900	246.8	3,055.7	2.9
14	52.13	0.768	17,700	922.7	3,981.3	3.8
15	51.88	0.759	(900)	(46.7)	3,938.4	3.8
16	52.87	0.865	10,600	560.4	4,502.6	4.3
17	54.87	0.978	11,300	620.0	5,126.9	4.9
18	54.62	0.990	1,200	65.5	5,197.3	5.0
19	55.87	1.000	1,000	55.9	5,258.2	5.1
20	57.25	1.000	0	0.0	5,263.3	

Cash flows arising from rehedging (dynamic rebalancing) and interest costs

The stock price falls by the end of the first week to \$48.12. The delta declines to 0.458. A long position in $45,800$ shares is now required to hedge the option position. A total of $6,400$ ($= 52,200 - 45,800$) shares are therefore sold to maintain the delta neutrality of the hedge.

The strategy realizes \$308,000 in cash, and the cumulative borrowings at the end of week 1 are reduced to \$2,252,300. Note that interest rate cost of one week, calculated by

$$2,557.8 \text{ thousand} \times 0.05/52 \approx 2.5 \text{ thousand}$$

has to be added. This comes out to be (in thousands)

$$2,557.8 - 308 + 2.5 = 2,252.3.$$

- During the second week, the stock price reduces to \$47.37 and delta declines again. This leads to $5,800$ shares being sold at the end of the second week.

- During the third week, the stock price increases to over \$50 and delta increases. This leads to $19,600$ shares being purchased at the end of the third week!

Toward the maturity date of the option, it becomes apparent that the option will be exercised and delta approaches 1.0. By week 20, therefore, the hedger owns $100,000$ shares.

Since the strike price is \$50, the hedger receives \$5 million ($= 100,000 \times \$50$) for these shares when the option is exercised so that the total cost of hedging it is $\$5,263,300 - \$5,000,000 = \$263,300$.

How would you compare the fair value of the call, which is \$2.4 at initiation of the trade, with the total cost of hedging per unit of the call option, which is \$2.633? It is necessary to adjust the time value, where the value of the call 20 weeks after the trade is $\$2.4 \times (1+0.05\times 20/52) \approx 2.446$. The seller loses if he charges the price of the call option at the "fair value". The higher cost of hedging when compared with the fair value may be attributed to the overhedging due to delay in rebalancing (weekly adjustment of hedging position). However, more frequent rebalancing means higher transaction costs. Luckily, the seller received \$3 per call option, so he maintains a gain of $\$3 \times (1 + 0.05 \times 20/52) = \$3.058 - \$2.633 = \$0.425$ per option at maturity.

The delta-hedging procedure in effect creates a long position in the option synthetically to neutralize the seller's short option position. The seller is forced into the buy-high and sell-low trading strategy since the hedging procedure involves selling stock just after the price has gone down and buying stock just after the price has gone up. Note that transaction costs have not been included.

Remarks

- As the call option expires in-the-money ($S_T = \$57.25$ and $X = \$50$), the total sum of the stock units purchased over the 20 weeks must be $100,000$ shares. These shares can be delivered to honor the obligation since the option buyer chooses to exercise the call.

- The fair call option premium received upfront by the writer is the present value of the total costs of setting up the hedging procedure.

One may query whether the cost of buying $100,000$ shares over time can be covered by the option premium of \$3 per each unit. For example, can the hedger cover the high cost if the stock price increases sharply (say, up to \$150 which is well above $X = \$50$)?

It is not necessary to worry if one follows the dynamic hedging procedure throughout the whole life of the option (not to start buying more shares only when the stock price increases sharply). This is quite a miracle. Indeed, the hedger almost holds the full amount of $100,000$ units well before the stock price rises to \$150.

Scenario Two: Call option expires out-of-the-money (no exercise of call)

Simulation of Delta Hedging (option closes out-of-the-money and cost of hedging = $256,600)

Week	Stock Price	Delta	Shares Purchased	Cost of Shares Purchased ($000)	Cumulative Cash Outflow ($000)	Interest Cost ($000)
0	49.00	0.522	52,200	2,557.8	2,557.8	2.5
1	49.75	0.568	4,600	228.9	2,789.2	2.7
2	52.00	0.705	13,700	712.4	3,504.3	3.4
3	50.00	0.579	(12,600)	(630.0)	2,877.7	2.8
4	48.38	0.459	(12,000)	(580.6)	2,299.9	2.2
5	48.25	0.443	(1,600)	(77.2)	2,224.9	2.1
6	48.75	0.475	3,200	156.0	2,383.0	2.3
7	49.63	0.540	6,500	322.6	2,707.9	2.6
8	48.25	0.420	(12,000)	(579.0)	2,131.5	2.1
9	48.25	0.410	(1,000)	(48.2)	2,085.4	2.0
10	51.12	0.658	24,800	1,267.8	3,355.2	3.2
11	51.50	0.692	3,400	175.1	3,533.5	3.4
12	49.88	0.542	(15,000)	(748.2)	2,788.7	2.7
13	49.88	0.538	(400)	(20.0)	2,771.4	2.7
14	48.75	0.400	(13,800)	(672.7)	2,101.4	2.0
15	47.50	0.236	(16,400)	(779.0)	1,324.4	1.3
16	48.00	0.261	2,500	120.0	1,445.7	1.4
17	46.25	0.062	(19,900)	(920.4)	526.7	0.5
18	48.13	0.183	12,100	582.4	1,109.6	1.1
19	46.63	0.007	(17,600)	(820.7)	290.0	0.3
20	48.12	0.000	(700)	(33.7)	256.6	

Remarks

- In case the call option expires out-of-the-money, the net number of shares bought throughout the hedging procedure would be zero.

- Though the hedged portfolio ends up with zero number of shares held at maturity, there is cost incurred in performing the dynamic hedging procedure. This hedging cost is compensated by the option premium collected at initiation.

- Major part of the hedging cost arises from the buy-high and sell-low strategy, seems to be senseless as a trading strategy. However, the hedger is forced to follow such strategy in hedging procedure. Another source of cost is the interest rate cost.

Understanding and implementation of the dynamic hedging strategy enhance the growth of trading of options (like the strong warrant markets in Hong Kong).

3.4 Risk neutral measure

Under the probability measure P, the price processes of S_t and M_t are

$$\begin{aligned} \frac{\mathrm{d}S_t}{S_t} &= \rho\ dt + \sigma\ \mathrm{d}Z_t^P, \\ \mathrm{d}M_t &= rM_t\ \mathrm{d}t. \end{aligned}$$

Recall that $S_t = S_0 e^{\left(\rho - \frac{\sigma^2}{2}\right)t + \sigma Z^P(t)}$ so that the price process of $S_t^* = S_t/M_t$ becomes $S_0 e^{\left(\rho - r - \frac{\sigma^2}{2}\right)t + \sigma Z^P(t)}$. Therefore, the dynamics of S_t^* is given by

$$\frac{\mathrm{d}S_t^*}{S_t^*} = (\rho - r)\mathrm{d}t + \sigma\ \mathrm{d}Z_t^P.$$

Based on the formula on the mean of Geometric Brownian motion (see p.8), we obtain

$$E_0\left[\frac{S_t^*}{S_0^*}\right] = \exp\left(\left(\rho - r - \frac{\sigma^2}{2}\right)t + \frac{\sigma^2}{2}t\right) = e^{(\rho - r)t}.$$

We would like to find the equivalent martingale measure Q such that the discounted asset price S_t^* is Q-martingale. By the Girsanov Theorem, suppose we choose $\gamma(t)$ in the Radon-Nikodym derivative such that

$$\gamma(t) = \frac{\rho - r}{\sigma},$$

then Z_t^Q is the standard Brownian motion under the probability measure Q, where

$$dZ_t^Q = dZ^P(t) + \frac{\rho - r}{\sigma}\, dt.$$

Note that Z_t^Q is a Brownian motion with drift rate $\dfrac{\rho - r}{\sigma}$ under P.

The corresponding Radon-Nikodym derivative is given by

$$\frac{dQ}{dP} = \exp\left(-\frac{\rho - r}{\sigma} Z_t^P - \left(\frac{\rho - r}{\sigma}\right)^2 \frac{t}{2}\right).$$

Under the Q-measure, the discounted price process S_t^* now becomes

$$\frac{\mathrm{d}S_t^*}{S_t^*} = \sigma \ \mathrm{d}Z_t^Q,$$

Since S_t^* is a zero drift Ito process under Q, so S_t^* is Q-martingale. The asset price S_t under the Q-measure is governed by

$$\frac{\mathrm{d}S_t}{S_t} = r \ dt + \sigma \ \mathrm{d}Z_t^Q.$$

When the money market account is used as the numeraire, the corresponding equivalent martingale measure is called the *risk neutral measure* and the drift rate of S_t under the Q-measure is called the *risk neutral drift rate*.

From Black-Scholes equation to risk neutral valuation formula

We have identified the measure Q such that S_t^* is Q-martingale. The dynamics of S_t under a measure Q is governed by

$$dS_t = rS_t\ dt + \sigma S_t\ dZ_t^Q.$$

Let $V(S_t, t)$ be the price function of a financial derivative with the underlying asset price S_t and M_t be the money market account process. Suppose the equation for $V(S, t)$ is governed by

$$\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{\sigma^2}{2}S^2\frac{\partial^2 V}{\partial S^2} - rV = 0.$$

Recall that we have used riskless hedging and no-arbitrage arguments to derive the Black-Scholes equation. We use the Feynman-Kac representation theorem to establish the risk neutral valuation formula under Q measure.

How to eliminate the discount term $-rV$ so that the Feynman-Kac formula can be applied? Define the discounted price function $V^*(S_t, t) = V(S_t, t)/M_t$, where $M_t = e^{rt}$, we deduce that $V^*(S, t)$ is governed by

$$\frac{\partial V^*}{\partial t} + rS\frac{\partial V^*}{\partial S} + \frac{\sigma^2}{2}S^2\frac{\partial^2 V^*}{\partial S^2} = 0.$$

From the Feynman-Kac representation theorem, we have

$$V^*(S, t) = E_t^S[V^*(S_T, T)] \quad \text{or} \quad \frac{V(S, t)}{e^{rt}} = E_t^S\left[\frac{V(S_T, T)}{e^{rT}}\right].$$

Rearranging the terms, we obtain the following *risk neutral valuation formula*:

$$V(S, t) = e^{-r(T-t)}E_t^S[V_T(S)].$$

After identifying Q such that S_t^* is a Q-martingale, the use of the Black-Scholes equation implies the risk neutral valuation formula.

Expectation representation of derivative price

Under the actual probability measure P, the dynamics of the underlying asset price process is

$$\frac{dS_t}{S_t} = \rho \ dt + \sigma \ dZ_t^P.$$

The governing pde is

$$\frac{\partial V}{\partial t} + \rho S \frac{\partial V}{\partial S} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} - \rho_V V = 0, \quad V(S,T) = h(S).$$

By the Feynman-Kac representation, $V(S,t)$ admits the expectation representation

$$V(S,t) = e^{-\rho_V (T-t)} E_P^t[h(S_T)],$$

when E_P^t denotes the expectation under P conditional on filtration $\mathcal{F}_t$. Note that the risky discount factor is $e^{-\rho_V(T-t)}$ and this arises from the discount term: $-\rho_V V$.

- Option valuation can be performed using the risk neutral measure by *artificially* taking the expected rate of returns of the asset and option to be r. We choose a pricing measure (called risk neutral measure or martingale measure) such that the expected rate of return of any risky instrument is r or the discounted value has zero expected rate of return.

- The above transformation requires the financial economics concepts of dynamic hedging/replication and no-arbitrage principle. Both the derivative and it's underlying asset are tradeable and they can be used to hedge risks among themselves.

Suppose the governing pde is the Black-Scholes equation, where

$$\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{\sigma^2}{2}S^2\frac{\partial^2 V}{\partial S^2} - rV = 0,$$

then the derivative price function admits the expectation representation

$$V(S,t) = e^{-r(T-t)}E_Q^t[h(S_T)] = e^{-r(T-t)}\int_{-\infty}^{\infty} h(S_T)\Psi(S_T,T;S_t,t)\,\mathrm{d}S_T,$$

where $\Psi(S_T,T;S_t,t)$ is the transition density function that the asset price starts from S_t at time t and ends up falling within $\left[S_T - \frac{\mathrm{d}u}{2}, S_T + \frac{\mathrm{d}u}{2}\right]$ with probability $\Psi(S_T,T;S_t,t)\ \mathrm{d}u$. This is simply the risk neutral valuation principle. Under the pricing (risk neutral) measure Q, the dynamics of S_t is governed by

$$\frac{dS_t}{S_t} = r\,dt + \sigma\,dZ_t^Q, \qquad Z_t^Q \text{ is } Q\text{-Brownian.}$$

We observe that the discounted price process $S_t^* = S_t/M_t$ is a martingale under Q since S_t^* becomes a zero-drift Ito process, where

$$\frac{dS_t^*}{S_t^*} = \sigma\ dZ_t^Q.$$

The call option value can be either given by

$$e^{-\rho_V(T-t)}E_P^t[(S_T - X)\mathbf{1}_{\{S_T>X\}}]$$

or

$$e^{-r(T-t)}E_Q^t[(S_T - X)\mathbf{1}_{\{S_T>X\}}].$$

Both would give the same value, by virtue of the equality of market price of risk of asset and option.

Suppose the investor is risk averse, a higher expected rate of asset return ρ is counterbalanced by a higher value of ρ_V in the risky discount factor for the derivative.

Equality of market price of risk of hedgeable securities

Suppose two tradeable risky securities $\{S_t^1\}_{t\geq 0}$ and $\{S_t^2\}_{t\geq 0}$ are dependent on the same Brownian motion Z_t^P under P, where

$$\frac{\mathrm{d}S_t^i}{S_t^i} = \rho_i \, \mathrm{d}t + \sigma_i \, \mathrm{d}Z_t^P, \; i = 1, 2.$$

They are hedgeable since their risks arise from the same Brownian motion Z_t^P. Assuming existence of the risk neutral measure Q, under which both discounted price processes of the tradeable assets are Q-martingales. Recall that

$$Z_t^Q = Z_t^P + \frac{\rho_i - r}{\sigma_i} t, \; i = 1, 2,$$

is Q-Brownian motion. Consistency between the two risky assets in defining Z_t^Q in terms of Z_t^P is satisfied if and only if

$$\frac{\rho_1 - r}{\sigma_1} = \frac{\rho_2 - r}{\sigma_2}.$$

We conclude that two hedgeable securities should observe equality of market price of risk.

Self-financing replicating strategy under Q-measure

We assume the existence of a risk neutral measure Q under which S_t^* is Q-martingale. We would like to find a self-financing strategy under Q-measure that replicates a call option. The strategy is characterized by holding Δ_t units of the underlying asset and M_t amount of market money account. Let V_t denote the time-t value of the portfolio and $c(S_t, t)$ denote the call price function. We choose Δ_t and M_t dynamically such that replication is achieved, which observes

$$V_t = \Delta_t S_t + M_t = c(S_t, t).$$

Full replication is achieved when the terminal portfolio value V_T observes the terminal payoff of the call option, where

$$V_T = (S_T - X)^+.$$

Under Q, the stochastic dynamics of S_t is governed by:

$$\frac{\mathrm{d}S_t}{S_t} = r\ \mathrm{d}t + \sigma\ \mathrm{d}Z_t^Q.$$

We would like to determine the hedge ratio Δ_t under Q measure and the governing equation of $c(S_t, t)$.

By Ito's lemma, without any finance concept, we have

$$
\begin{aligned}
V_t - V_0 &= c(S_t, t) - c(S, 0) = \int_0^t \mathrm{d}c(S_u, u) \\
&= \int_0^t \left[\frac{\partial c}{\partial u}(S_u, u) + rS\frac{\partial c}{\partial S}(S_u, u) + \frac{\sigma^2}{2}S^2\frac{\partial^2 c}{\partial S^2}(S_u, u) \right] \mathrm{d}u \\
&\quad + \int_0^t \sigma S_u \frac{\partial c}{\partial S}(S_u, u)\ \mathrm{d}Z_u^Q. \qquad \text{(i)}
\end{aligned}
$$

On the other hand, we advocate dynamic replication and apply $\mathrm{d}\Pi_t = r\Pi_t\ \mathrm{d}t$. Since (Δ_t, M_t) is self-financing without addition or withdrawal of fund, we have

$$
\begin{aligned}
V_t - V_0 &= \int_0^t \Delta_u\ \mathrm{d}S_u + \int_0^t rM_u\ \mathrm{d}u \\
&= \int_0^t \Delta_u r S_u\ \mathrm{d}u + \int_0^t \Delta_u \sigma S_u\ \mathrm{d}Z_u^Q + \int_0^t r(V_u - \Delta_u S_u)\ \mathrm{d}u \\
&= \int_0^t rV_u\ \mathrm{d}u + \int_0^t \sigma S_u \Delta_u\ \mathrm{d}Z_u^Q. \qquad \text{(ii)}
\end{aligned}
$$

By the definition of stochastic integral, Δ_u is fixed at the beginning instant of the differential time interval $(u, u + \mathrm{d}u)$.

The option price function $c(S_t, t)$ in equation (i) is replaced by gain from stock position and interest dollar collected in equation (ii). Comparing equations (i) and (ii), and observing $V_u = c(S_u, u)$, the equivalence in the two equations is achieved if we observe (changing the dummy time variable from u to t and setting $S_t = S$)

$$\Delta_t = \frac{\partial c}{\partial S}(S, t)$$

$$\text{and} \qquad \frac{\partial c}{\partial t}(S, t) + rS\frac{\partial c}{\partial S}(S, t) + \frac{\sigma^2}{2}S^2\frac{\partial^2 c}{\partial S^2}(S, t) - rc(S, t) = 0.$$

Under the risk neutral measure Q, where all discounted price processes are Q-martingales, we derive the required hedging strategy: $\Delta_t = \frac{\partial c}{\partial S}(S_t, t)$ and the governing equation for $c(S_t, t)$ simultaneously.

Exchange rate process under domestic risk neutral measure

Consider a foreign currency option whose payoff function depends on the exchange rate F, which is defined to be the domestic currency price of one unit of foreign currency.

Let M_d and M_f denote the value of the money market account in the domestic market and foreign market, respectively. The processes of $M_d(t), M_f(t)$ and $F(t)$ are governed by

$$dM_d(t) = rM_d(t)\,dt, \quad dM_f(t) = r_f M_f(t)\,dt, \quad \frac{dF(t)}{F(t)} = \mu\,dt + \sigma\,dZ,$$

where r and r_f denote the riskless domestic and foreign interest rates, respectively.

We may treat the domestic money market account and the foreign money market account in domestic dollars (whose value is given by FM_f) as traded securities in the domestic currency world.

With reference to the domestic equivalent martingale measure Q_d, the domestic money market account M_d is used as the numeraire.

The relative price process $X(t) = F(t)M_f(t)/M_d(t)$ is governed by

$$\frac{dX(t)}{X(t)} = (r_f - r + \mu)\, dt + \sigma\, dZ.$$

Here, $X(t)$ is the domestic currency price of one unit of foreign currency discounted in the domestic currency world. Therefore, $X(t)$ would be a martingale under Q_d.

With the choice of $\gamma = (r_f - r + \mu)/\sigma$, we define

$$dZ_d = dZ + \gamma\, dt,$$

where Z_d is a Brownian motion under Q_d.

Under the domestic equivalent martingale measure Q_d, the process of X now becomes

$$\frac{dX(t)}{X(t)} = \sigma\, dZ_d$$

so that X is Q_d-martingale.

Recall $F(t) = X(t)M_d(t)/M_f(t)$. The exchange rate process F under the Q_d-measure is given by

$$\frac{dF(t)}{F(t)} = (r - r_f)\, dt + \sigma\, dZ_d.$$

The risk neutral drift rate of F under Q_d is found to be $r - r_f$.

3.5 European option pricing formulas and their greeks

Recall that the Black-Scholes equation for a European vanilla call option takes the form

$$\frac{\partial c}{\partial \tau} = \frac{\sigma^2}{2} S^2 \frac{\partial^2 c}{\partial S^2} + rS \frac{\partial c}{\partial S} - rc, \qquad 0 < S < \infty,\ \tau > 0, \quad \tau = T - t.$$

Initial condition that corresponds to $\tau = 0$ *(payoff at expiry)*

$$c(S, 0) = \max(S - X, 0), \quad X \text{ is the strike price.}$$

Using the transformation: $y = \ln S$ and $c(y, \tau) = e^{-r\tau} w(y, \tau)$, the Black-Scholes equation is transformed into

$$\frac{\partial w}{\partial \tau} = \frac{\sigma^2}{2} \frac{\partial^2 w}{\partial y^2} + \left(r - \frac{\sigma^2}{2} \right) \frac{\partial w}{\partial y}, \quad -\infty < y < \infty,\ \tau > 0.$$

The initial condition for the pricing model now becomes

$$w(y, 0) = \max(e^y - X, 0).$$

Green function approach

Note that $\frac{dS_t}{S_t} \neq d\ln S_t$ as in elementary calculus since S_t is an Ito process. Write $F(S_t) = \ln S_t$ as a function of S_t, by Ito's lemma:

$$dF_t = \frac{\partial F}{\partial t}dt + \frac{\partial F}{\partial S}dS + \frac{\sigma^2}{2}S^2\left(\frac{\partial^2 F}{\partial S^2}\right)dt.$$

since $\frac{\partial F}{\partial t} = 0$, $\frac{\partial F}{\partial S} = \frac{1}{S}$ and $\frac{\partial^2 F}{\partial S^2} = -\frac{1}{S^2}$, we obtain

$$dF_t = d\ln S_t = \left(r - \frac{\sigma^2}{2}\right)dt + \sigma\ dZ_t^Q.$$

The infinite domain Green function is known to be

$$\phi(y,\tau) = \frac{1}{\sigma\sqrt{2\pi\tau}}\ \exp\left(-\frac{[y + (r - \frac{\sigma^2}{2})\tau]^2}{2\sigma^2\tau}\right), \qquad -\infty < y < \infty.$$

Here, $\phi(y,\tau)$ satisfies the initial condition: $\lim_{\tau\to 0^+}\phi(y,\tau) = \delta(y)$, where $\delta(y)$ is the Dirac function representing a unit impulse at the origin.

The Green function can be identified as the density function of the Brownian motion starting at zero with drift rate $\mu = -\left(r - \frac{\sigma^2}{2}\right)$ and variance rate σ^2.

The drift rate of $\ln S_t$ is $r - \frac{\sigma^2}{2}$ forward in time. However, the drift rate is swapped in sign when we solve the option pricing problem backward in time.

The initial condition can be expressed via the Dirac function as

$$w(y, 0) = \int_{-\infty}^{\infty} w(\xi, 0)\delta(y - \xi)\, d\xi,$$

so that $w(y, 0)$ can be considered as the superposition of impulses with varying magnitude $w(\xi, 0)$ ranging from $\xi \to -\infty$ to $\xi \to \infty$.

Since the Black-Scholes equation is linear, the response in position y and at time to expiry τ due to an impulse of magnitude $w(\xi, 0)$ in position ξ at $\tau = 0$ is given by $w(\xi, 0)\phi(y - \xi, \tau)$. From the principle of superposition for a linear differential equation, the solution is obtained by summing up the responses due to these impulses. This gives

$$\begin{aligned} c(y, \tau) &= e^{-r\tau} w(y, \tau) \\ &= e^{-r\tau} \int_{-\infty}^{\infty} w(\xi, 0)\ \phi(y - \xi, \tau)\ d\xi \\ &= e^{-r\tau} \int_{\ln X}^{\infty} (e^{\xi} - X)\ \frac{1}{\sigma\ \sqrt{2\pi\tau}} \\ &\qquad \exp\left(- \frac{[y + (r - \frac{\sigma^2}{2})\tau - \xi]^2}{2\sigma^2 \tau} \right)\ d\xi. \end{aligned}$$

This integral representation agrees with the Feynman-Kac representation (risk neutral valuation formula).

It is relatively straightforward to show that

$$\begin{aligned}
&\int_{\ln X}^{\infty} \frac{1}{\sigma\sqrt{2\pi\tau}} \exp\left(-\frac{[y+(r-\frac{\sigma^2}{2})\tau-\xi]^2}{2\sigma^2\tau}\right) d\xi \\
= \quad & N\left(\frac{y+(r-\frac{\sigma^2}{2})\tau-\ln X}{\sigma\sqrt{\tau}}\right) = N\left(\frac{\ln\frac{S}{X}+(r-\frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}\right), \quad y=\ln S.
\end{aligned}$$

By performing the procedure of completing square with respect to ξ, we obtain

$$\begin{aligned}
&\int_{\ln X}^{\infty} e^{\xi} \frac{1}{\sigma\sqrt{2\pi\tau}} \exp\left(-\frac{[y+(r-\frac{\sigma^2}{2})\tau-\xi]^2}{2\sigma^2\tau}\right) d\xi \\
= \quad & \exp(y+r\tau)\int_{\ln X}^{\infty} \frac{1}{\sigma\sqrt{2\pi\tau}} \exp\left(-\frac{\left[y+\left(r+\frac{\sigma^2}{2}\right)\tau-\xi\right]^2}{2\sigma^2\tau}\right) d\xi \\
= \quad & e^{r\tau} S N\left(\frac{\ln\frac{S}{X}+(r+\frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}\right), \quad y=\ln S.
\end{aligned}$$

Hence, the price formula of the European call option is found to be

$$c(S,\tau) = SN(d_1) - Xe^{-r\tau}N(d_2),$$

where

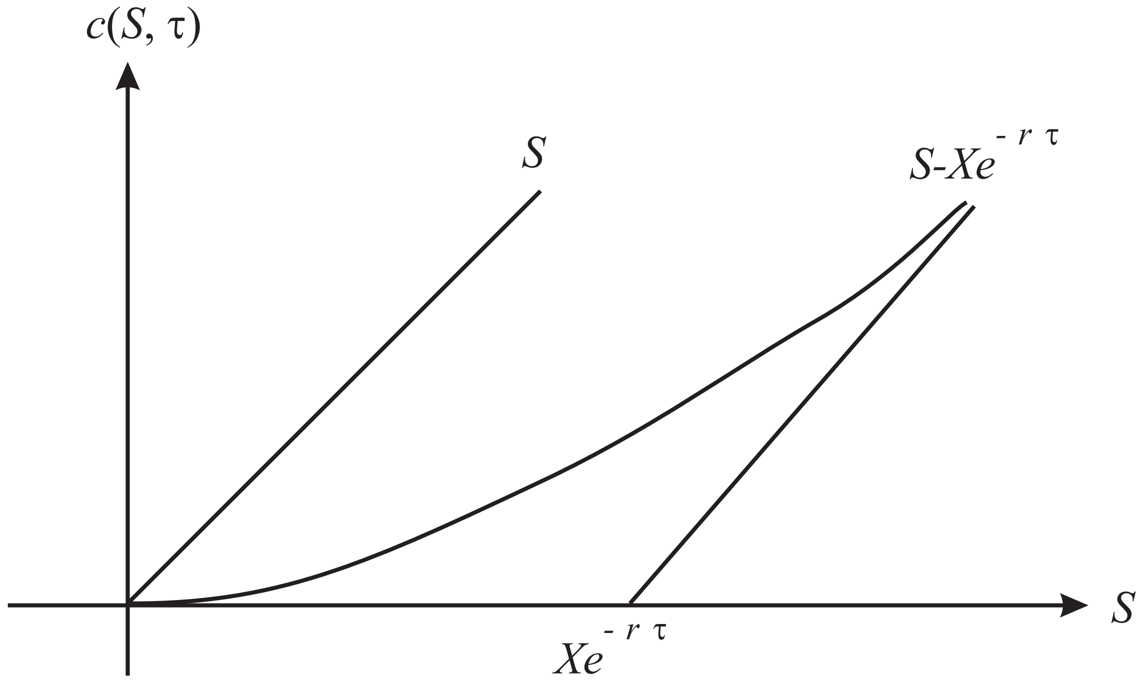
$$d_1 = \frac{\ln\frac{S}{X} + (r + \frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}, \quad d_2 = d_1 - \sigma\sqrt{\tau}.$$

- The initial condition is seen to be satisfied by observing that the limits of $N(d_1)$ and $N(d_2)$ tend to $N(\infty) = 1$ or $N(-\infty) = 0$, depending on $S > X$ or $S < X$. When $S = X$, $N(d_1) = N(d_2) = N(0) = \frac{1}{2}$, so $c(S,0) = SN(0) - XN(0) = 0$, as expected.

- The boundary conditions of $\lim_{S\to 0^+} c(S,\tau) = 0$ and $\lim_{S\to\infty} c(S,\tau) = S - Xe^{-r\tau}$ are satisfied by observing

$$\lim_{S\to\infty} N(d_1) = \lim_{S\to\infty} N(d_2) = 1; \quad \text{deep-in-the-money;}$$

$$\lim_{S\to 0^+} N(d_1) = \lim_{S\to 0^+} N(d_2) = 0; \quad \text{deep-out-of-the-money.}$$



The call value lies within the bounds

$$\max(S - Xe^{-r\tau}, 0) \leq c(S, \tau) \leq S, \quad S \geq 0, \tau \geq 0.$$

When $S \to \infty$, $c(S, \tau)$ tends to the forward value $S - Xe^{-r\tau}$. This is expected since the call is almost sure to expire in-the-money at expiry.

Risk neutral transition density function

$$\begin{aligned} c(S,\tau) &= e^{-r\tau}E_Q[(S_T - X)\mathbf{1}_{\{S_T \geq X\}}] \\ &= e^{-r\tau}\int_0^\infty \max(S_T - X, 0)\psi(S_T, T; S, t)\ dS_T. \end{aligned}$$

Under the risk neutral measure Q, the asset price dynamics is

$$\frac{dS_t}{S_t} = r\ dt + \sigma\ dZ_t^Q \Leftrightarrow d\ln S_t = \left(r - \frac{\sigma^2}{2}\right)dt + \sigma\ dZ_t^Q,$$

giving

$$\ln\frac{S_t}{S_0} = \left(r - \frac{\sigma^2}{2}\right)t + \sigma Z_t^Q.$$

For the time period from t to T, where the width of time interval is $\tau = T - t$, we have

$$\ln\frac{S_T}{S} = \left(r - \frac{\sigma^2}{2}\right)\tau + \sigma Z^Q(\tau),\ \text{where } \tau = T - t,$$

so that $\ln\frac{S_T}{S}$ is normally distributed with mean $\left(r - \frac{\sigma^2}{2}\right)\tau$ and variance $\sigma^2\tau$, and S is time-t asset price.

In terms of $\ln S_T$ and $\ln S_t$, the transition density $\psi(\ln S_T, T; \ln S_t, t)$ can be expressed as

$$\psi(\ln S_T, T; \ln S_t, t)\ d\ln S_T = \psi(S_T, T; S_t, t)\ dS_T.$$

From the density function of a normal random variable, the transition density function is

$$\psi(S_T, T; S, t) = \frac{1}{S_T \sigma \sqrt{2\pi\tau}} \exp\left(- \frac{\left[\ln \frac{S_T}{S} - \left(r - \frac{\sigma^2}{2} \right) \tau \right]^2}{2\sigma^2 \tau} \right).$$

The scaling factor $\dfrac{1}{S_T}$ is appended since $d\ln S_T = \dfrac{1}{S_T}\ dS_T$. This should not be confused with the earlier claim that $\mathrm{d}\ln S_t \neq \frac{1}{S_t}\mathrm{d}S_t$. In the current context, S_T is visualized as a random variable rather than an Ito process, so the ususal scaling rule between density functions applies.

Letting $\xi = \ln S_T$ and $y = \ln S$, so $\psi(\xi, T; y, t)$ is equivalent to the fundamental solution of $\phi(y - \xi, \tau)$.

Recall

$$
\begin{aligned}
c(S,\tau) &= e^{-r\tau}\left\{E_Q\left[S_T \mathbf{1}_{\{S_T \geq X\}}\right] - XE_Q\left[\mathbf{1}_{\{S_T \geq X\}}\right]\right\} \\
&= SN(d_1) - Xe^{-r\tau}N(d_2),
\end{aligned}
$$

we can deduce that

$$
\begin{aligned}
N(d_2) &= E_Q[\mathbf{1}_{\{S_T \geq X\}}] = Q[S_T \geq X] \\
SN(d_1) &= e^{-r\tau}E_Q[S_T \mathbf{1}_{\{S_T \geq X\}}].
\end{aligned}
$$

- $N(d_2)$ is recognized as the probability under the risk neutral measure Q that the call expires in-the-money, so $Xe^{-r\tau}N(d_2)$ represents the present value of the risk neutral expectation of payment paid by the option holder at expiry.
- $SN(d_1)$ is the risk neutral discounted expectation of the terminal asset price conditional on the call being in-the-money at expiry.

Delta - derivative with respect to asset price

$$
\begin{aligned}
\triangle_c = \frac{\partial c}{\partial S} &= N(d_1) + S\,\frac{1}{\sqrt{2\pi}}\,e^{-\frac{d_1^2}{2}}\frac{\partial d_1}{\partial S} - Xe^{-r\tau}\,\frac{1}{\sqrt{2\pi}}\,e^{-\frac{d_2^2}{2}}\frac{\partial d_2}{\partial S} \\
&= N(d_1) + \frac{1}{\sigma\sqrt{2\pi\tau}}[e^{-\frac{d_1^2}{2}} - e^{-(r\tau + \ln\frac{S}{X})}e^{-\frac{d_2^2}{2}}] \\
&= N(d_1) > 0.
\end{aligned}
$$

Knowing that a European call can be replicated by Δ units of asset and riskless asset in the form of money market account, the factor $N(d_1)$ in front of S in the call price formula thus gives the hedge ratio Δ.

The value of $\triangle_c$ is bounded between 0 and 1 since $0 < N(d_1) < 1$. This is expected since the increase in call value cannot be more than the increase in stock price. The gain in holding the call is materialized only when the call expires in the money.

- $\triangle_c$ is an increasing function of S since $\dfrac{\partial}{\partial S}N(d_1)$ is always positive.
- The curve of $\triangle_c$ against S changes concavity at

$$S_c = X \exp\left(-\left(r + \frac{3\sigma^2}{2}\right)\tau\right)$$

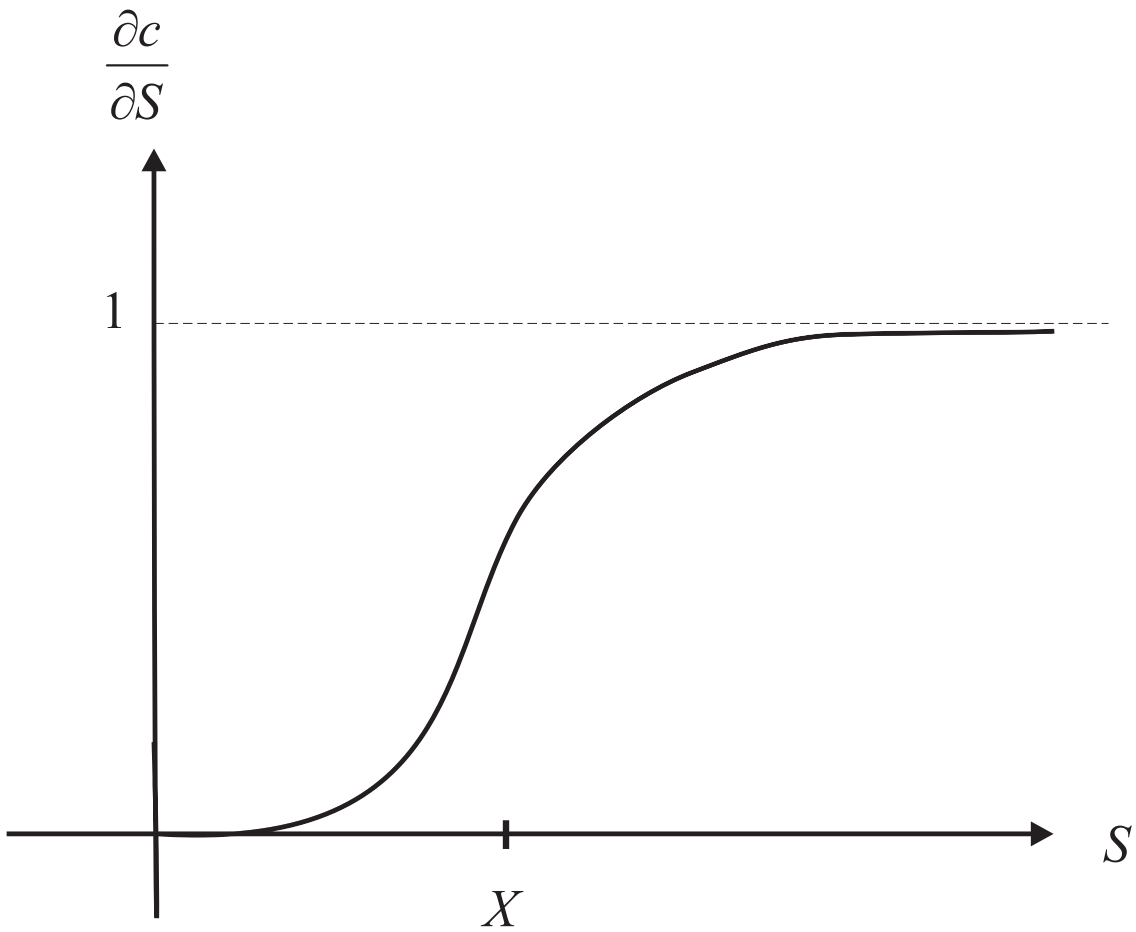
so that the curve is concave upward for $0 \leq S < S_c$ and concave downward for $S_c < S < \infty$.

Asymptotic limits of delta at $\tau \to \infty$ and $\tau \to 0^+$

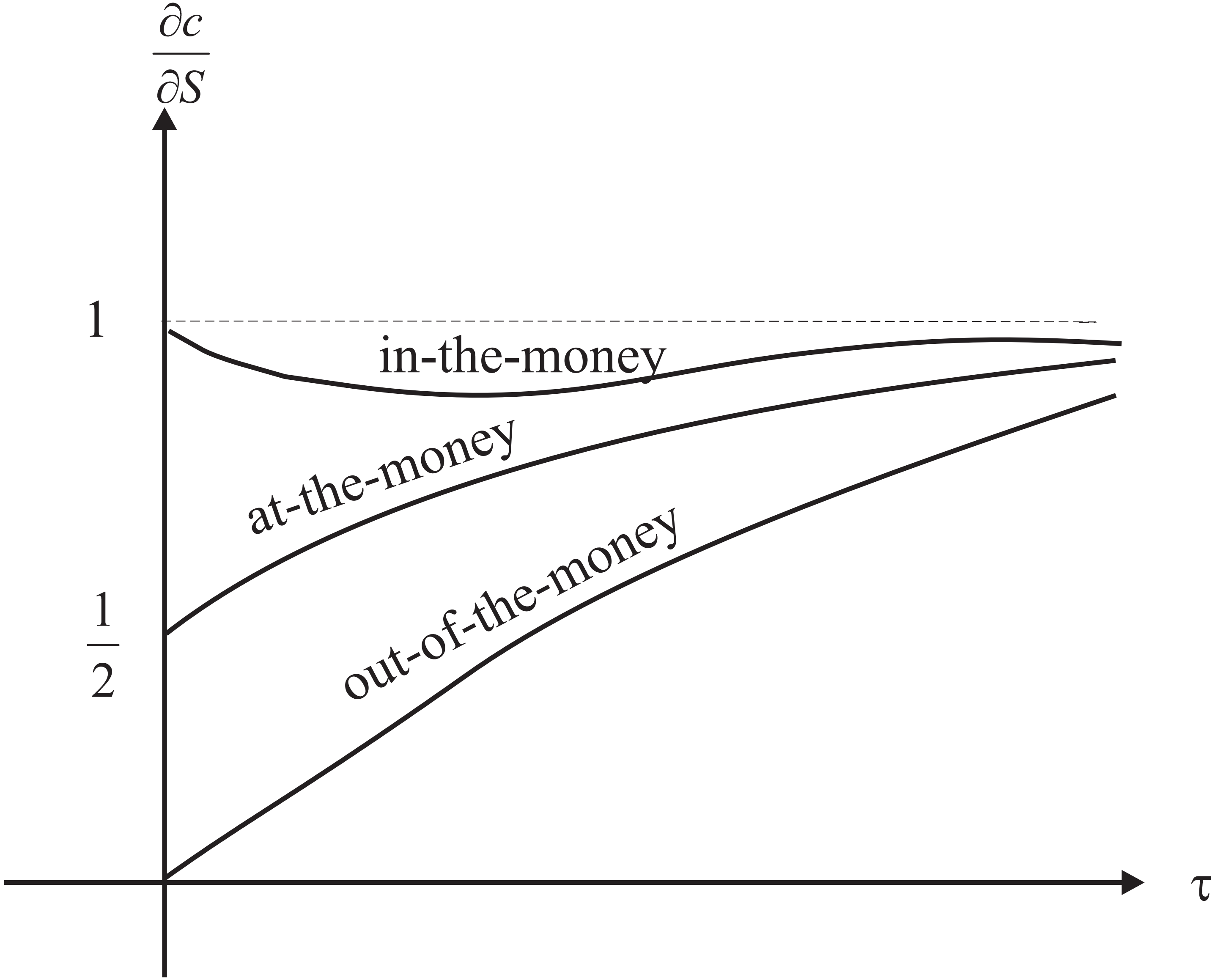
$$\lim_{\tau\to\infty} \frac{\partial c}{\partial S} = 1 \quad \text{for all values of } S,$$

while

$$\lim_{\tau\to 0^+} \frac{\partial c}{\partial S} = \begin{cases} 1 & \text{if } S > X \\ \frac{1}{2} & \text{if } S = X \\ 0 & \text{if } S < X \end{cases}.$$



Variation of the delta of the European call value with respect to the asset price S. The curve changes concavity at $S = Xe^{-\left(r+\frac{3\sigma^2}{2}\right)\tau}$.



Delta of the European call value with respect to time to expiry τ

- The delta value always tends to one from below when the time to expiry tends to infinity since $d_1 \to \infty$ as $\tau \to \infty$. This is because the stock price is expected to grow above the strike price with sufficiently long time to expiry.

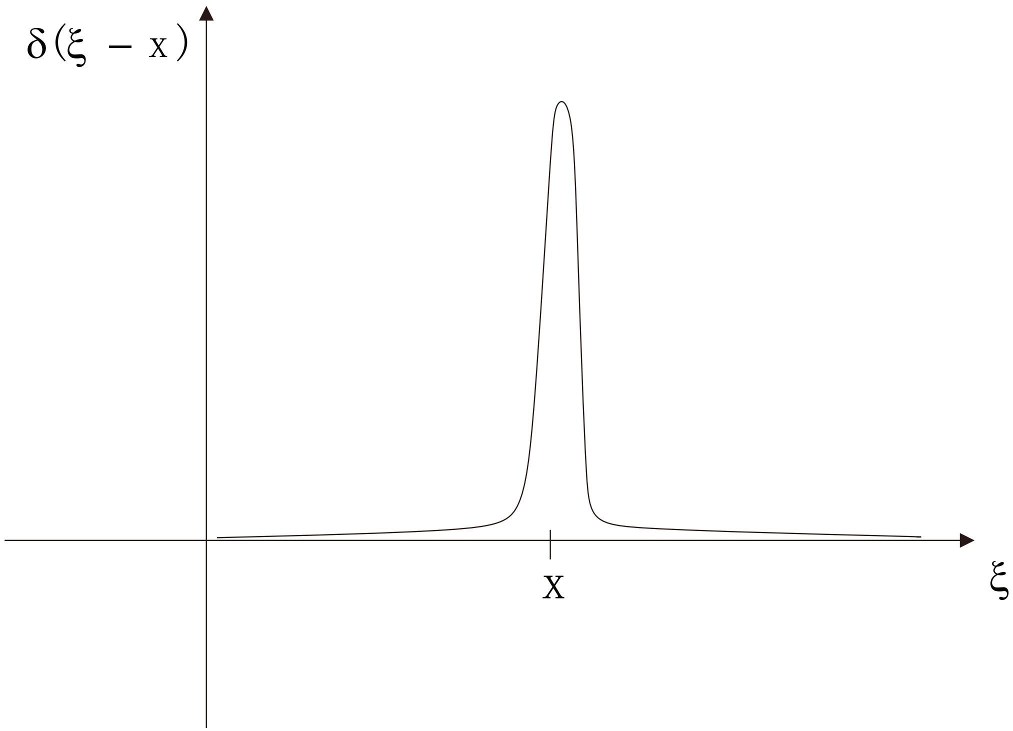
- The delta value tends to different asymptotic limits as time comes close to expiry, depending on the moneyness of the option. This would make hedging very challenging when the option is around-the-money at time near expiry. Since the hedger needs to hold $\Delta = 1$ unit of stock or $\Delta = 0$ in the next moment. Lucky enough, the potential loss is not too high since the stock price is close to the strike price.

Appendix – Dirac function

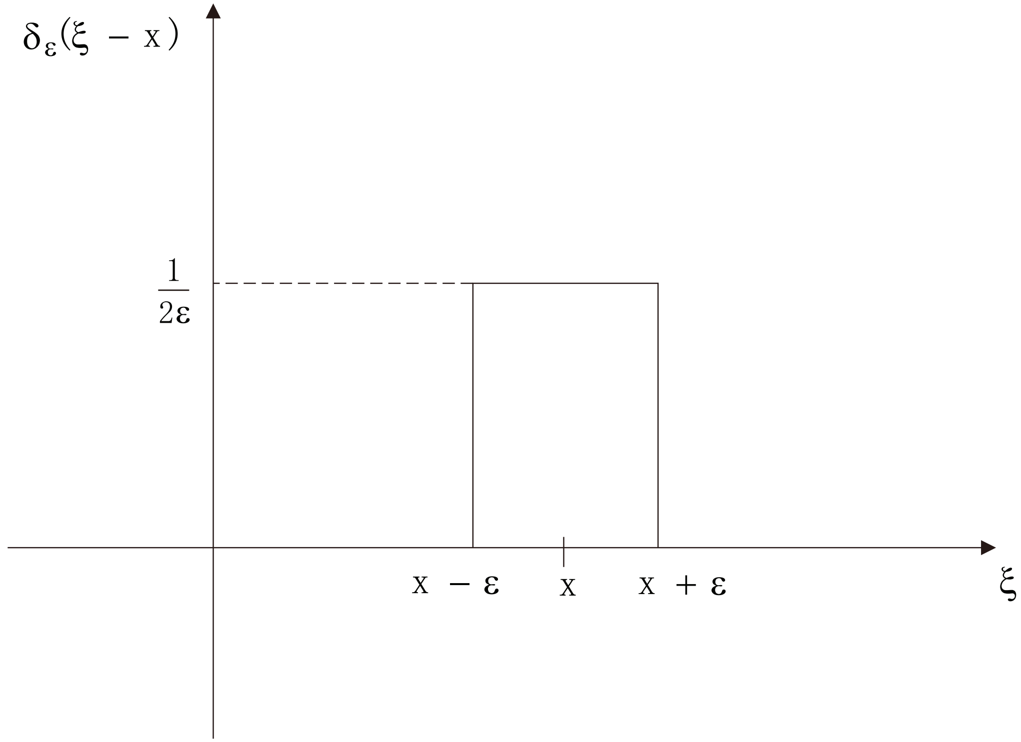
The Dirac function is a generalized function that resembles the impulse effect in physics where the force is applied over an infinitesimal time interval but with infinite magnitude. The (finite) size of the impulse is given by the integration of the infinite magnitude of force over an infinitesimal time interval.

The defining properties of the Dirac function $\delta(\xi - x)$ are

$$\int_{-\infty}^{\infty} \delta(\xi - x)\ d\xi = 1 \quad \text{and} \quad \delta(\xi - x) = \begin{cases} 0 & \text{if } \xi \neq x \\ \infty & \text{if } \xi = x \end{cases} .$$



δ(ξ − x)
x
ξ



δε(ξ − x)
1/2ε
x − ε
x
x + ε
ξ

Suppose we approximate $\delta(\xi - x)$ by

$$\delta_\epsilon(\xi - x) = \begin{cases} \frac{1}{2\epsilon} & \xi \in (x-\epsilon, x+\epsilon) \\ 0 & \text{otherwise} \end{cases},$$

then

$$\int_{-\infty}^{\infty} f(\xi)\delta_\epsilon(\xi - x)\ d\xi = \frac{1}{2\epsilon}\int_{x-\epsilon}^{x+\epsilon} f(\xi)\ d\xi$$

can be visualized as the average value of $f(\xi)$ over the interval $(x-\epsilon, x+\epsilon)$. In the limit $\epsilon \to 0^+$, we obtain $\delta_\epsilon(\xi - x) \to \delta(\xi - x)$ and

$$\int_{-\infty}^{\infty} f(\xi)\delta(\xi - x)\ d\xi = f(x).$$

Solution of an initial value problem using the Green function approach

Consider the initial value problem

$$LV(y,\tau) = 0, \quad V(y,0) = V_0(y)$$

where L is a linear differential operator and $V_0(y)$ is the initial condition.

The first step is to solve for the Green function (fundamental solution) $\phi(y,\tau)$, where

$$L\phi(y,\tau) = 0, \quad \phi(y,0) = \delta(y).$$

Recall that the initial value function can be expressed as

$$V_0(y) = \int_{-\infty}^{\infty} V_0(\xi)\delta(y-\xi)\ d\xi,$$

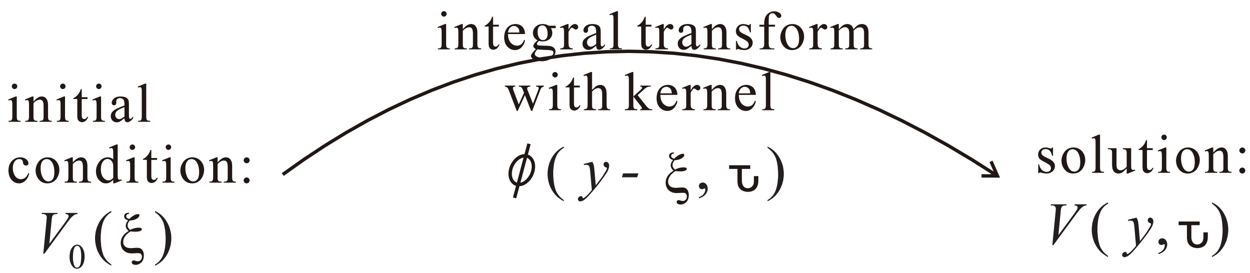
which is visualized as an **infinite sum of impulses**. By the superposition principle of linear differential equation, we obtain

$$V(y,\tau) = \int_{-\infty}^{\infty} V_0(\xi)\phi(y-\xi,\tau)\ d\xi.$$

Interpretation of the fundamental solution

The fundamental solution $\phi(y-\xi,\tau)$ can be visualized as the kernel in the integral transform that gives the solution $V(y,\tau)$ given the initial condition $V_0(\xi)$, where

$$V(y,\tau) = \int_{-\infty}^{\infty} V_0(\xi)\phi(y-\xi,\tau)\ \mathrm{d}\xi.$$



Remark

The integral transform formulation coincides with the expectation integral formulation derived from the Feynman-Kac representation Theorem. We may visualize the Green function as the transition density function.

Summary of risk neutral pricing

Option pricing equation before the Black-Scholes-Merton risk neutral pricing framework, which is derived based on calculus rule and complete absence of financial economics concepts of hedging and no-arbitrage:

$$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} + \rho S \frac{\partial V}{\partial S} - \rho_V V = 0 \qquad \text{(A)}$$

where the dynamics of S_t and V_t under P measure are given by

$$\frac{dS_t}{S_t} = \rho\, dt + \sigma\, dZ_t^P \quad \text{and} \quad \frac{dV_t}{V_t} = \rho_V\, dt + \sigma_V\, dZ_t^P.$$

These results are purely from Ito's lemma, no finance concepts are involved.

By the Feynman-Kac formula, V_t admits the following expectation representation:

$$V_t = e^{-\rho_V (T-t)} E_t^P[V_T], \text{ where } P \text{ is the physical measure.}$$

One has to estimate ρ and ρ_V.

If we take the very **strong and restrictive** assumption that all investors are risk neutral so that $\rho = \rho_V = r$, then we obtain the Black-Scholes equation directly. However, the option pricing theory derived from this strong assumption has very little value and would not warrant the Nobel prize award.

What would happen when the riskless hedging procedure is adopted?

1. Hedging concept plus $d\Pi = r\Pi dt$ (no-arbitrage argument) lead to

   $\frac{\rho_V - r}{\sigma_V} = \frac{\rho - r}{\sigma}$, same market price of risk for both securities.

   Together with $\sigma_V V = \frac{\partial V}{\partial S}\sigma S$, where $\frac{\partial V}{\partial S}$ is the hedge ratio. We then obtain

   $$\rho_V V = (\rho - r)\frac{\sigma_V V}{\sigma} + rV = (\rho - r)\frac{\partial V}{\partial S}S + rV. \qquad \text{(B)}$$

   Putting eq.(B) into eq.(A), we obtian

   $$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2}S^2\frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0. \qquad \text{(C)}$$

The no-arbitrage price is given by

$$V_t = e^{-r(T-t)} E_t^Q [V_T],$$

where Q is the martingale measure. Note that r appears since a hedged portfolio should earn the riskfree interest rate. Under Q, the dynamics of S_t is governed by

$$\frac{dS_t}{S_t} = r\,dt + \sigma\,dZ_t^Q \quad \text{or} \quad \frac{dS_t^*}{S_t^*} = \sigma\,dZ_t^Q.$$

Apparently, we can set $\rho = \rho_V = r$. Under this case, the investor is said to be risk neutral since she demands zero excess expected rate of return above the risk free rate on risky instruments.

It is not necessary to restrict investors on derivatives to be risk neutral in order to derive the Black-Scholes equation. We simply use the convenience of risk neutrality under the framework of hedging of options.

2. When the underlying asset is non-tradeable, the governing differential equation becomes

$$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} + (\rho - \lambda_V \sigma) \frac{\partial V}{\partial S} - rV = 0,$$

where

$$\lambda_V = \frac{\rho_V - r}{\sigma_V}.$$

- Under hedgeability of the two derivatives on S, the rate of return on V is set to be r, but not ρ_V. This gives rises to the discount term $-rV$ in the governing pde.

- However, the drift rate is modified to $\rho - \lambda_V \sigma$.

- When S becomes tradeable, market price of risk of tradeable Q comes in and $\lambda_V = \lambda$ so that we have

$$\lambda_V = \frac{\rho_V - r}{\sigma_V} = \frac{\rho - r}{\sigma} = \lambda$$

so that $\rho - \lambda_V \sigma = \rho - \lambda \sigma = r$. This recovers the standard Black-Scholes-Merton equation.