MAFS522 – Quantitative and Statistical Risk Analysis

Topic One – Mixture models for modeling default correction

Bernuolli mixture model

The loss of a portfolio from a loss statistics $\boldsymbol{L} = (L_1, \cdots, L_m)$ with Bernuolli variables $L_i \sim B(1; P_i)$, where $B(m; p)$ denotes the binomial distribution with m independent trials and stationary probability of success p. The loss probabilities are random variables

$$\boldsymbol{P} = (P_1, \cdots, P_m) \sim \boldsymbol{F}$$

for some distribution function $\boldsymbol{F}$ with support in $[0, 1]^m$.

Conditional independence

Conditional on a realization $\widehat{\boldsymbol{P}} = (\widehat{P}_1, \cdots, \widehat{P}_m)$ of $\boldsymbol{P}$, the Bernuolli variables $L_1, \cdots, L_m$ are independent.

$$L_i|_{P_i=\widehat{P}_i} \sim B(1; \widetilde{P}_i), \quad (L_i|_{\boldsymbol{P}=\widehat{\boldsymbol{P}}})_{i=1,2,\cdots,m} \text{ are independent.}$$

The (unconditional) joint distribution of the Li's is

$$\mathbb{P}(L_1 = \ell_1, \cdots, L_m = \ell_m) = \int_{[0,1]^m} \prod_{i=1}^{m} \widehat{P}_i^{\ell_i}(1-\widehat{P}_i)^{1-\ell_i} d\boldsymbol{F}(\widehat{P}_1, \cdots, \widehat{P}_m), \quad \ell_i \in \{0, 1\}.$$

The first and second moments of the single losses L_i are given by

$$E[L_i] = E[P_i].$$

$$\begin{aligned}\text{var}(L_1) &= \text{var}(E[L_i|\boldsymbol{P}]) + E[\text{var}(L_1|\boldsymbol{P})] \\ &= \text{var}(P_i) + E[E[L_i^2|\boldsymbol{P}] - E[L_i|\boldsymbol{P}]^2] \\ &= \text{var}(P_i) + E[P_i(1-P_i)] = E[P_i](1-E[P_i]).\end{aligned}$$

The covariance between single losses

$$\text{cov}(L_i, L_j) = E[L_i L_j] - E[L_i]E[L_j] = \text{cov}(P_i, P_j)$$

so that the default correlation in a Bernuolli mixture model is

$$\text{corr}(L_i, L_j) = \frac{\text{cov}(P_i, P_j)}{\sqrt{E[P_i](1-E[P_i])}\sqrt{E[P_j](1-E[P_j])}}.$$

The dependence between losses in the portfolio is fully captured by the covariance structure of the multivariate distribution $\boldsymbol{F}$ of $\boldsymbol{P}$.

Uniform default probability

Retail banking portfolios and portfolios of smaller banks are often quite homogeneous. Assuming $L_i \sim B(1; p)$ with a common random default probability $\boldsymbol{P} \sim F$, F is a distribution function with support in $[0, 1]$. As the mixture distribution is dependent on the single distribution $F(p)$, this leads to the one-factor *Bernuolli mixture model*. The joint distribution of the L_i's:

$$\mathbb{P}[L_1 = \ell_1, \cdots, L_m = \ell_m] = \int_0^1 p^k (1-p)^{m-k} \, dF(p)$$

where $k = \sum_{i=1}^{m} \ell_i$ and $\ell_i \in \{0, 1\}$. Write L as the random number of defaults.

The probability that exactly k defaults occur is

$$\mathbb{P}[L = k] = \binom{m}{k} \int_0^1 p^k (1-p)^{m-k} \, dF(p).$$

(mixture of binomial probabilities with the mixing distribution F)

The uniform default probability of borrowers (obligors) in the portfolio

$$\overline{p} = \mathbb{P}[L_i = 1] = E[L_i] = \int_0^1 p \, dF(p).$$

The uniform default correlation of two different counterparties is

$$\begin{aligned} \rho &= \text{corr}(L_i, L_j) = \frac{\mathbb{P}[L_{i=1}, L_{j=1}] - \overline{p}^2}{\overline{p}(1-\overline{p})} \\ &= \frac{\int_0^1 p^2 \, dF(p) - \overline{p}^2}{\overline{p}(1-\overline{p})} = \frac{\text{var}(p)}{\overline{p}(1-\overline{p})}. \end{aligned}$$

Note that with a higher var(p), we have higher corr(L_i, L_j).

Remarks

1. Since $\text{var}(p) \geq 0$, so $\text{corr}(L_i, L_j) \geq 0$. The non-negativity of correlation is obvious since L_i and L_j are dependent on the common mixture variable p. In other words, we cannot implement negative dependencies between the default risks of obligors under this model.

2. $\text{corr}(L_i, L_j) = 0$ if and only if $\text{var}(p) = 0$, implying no randomness at all regarding p. F is a Dirac measure concentrated in $\overline{p}$. The absolute portfolio loss L follows a binomial distribution with constant default probability $\overline{p}$.

3. corr$(L_i, L_j) = 1$ implies a "rigid" behavior of single losses in the portfolio. This corresponds to $p = 1$ with probability $\overline{p}$ and $p = 0$ with probability $1 - \overline{p}$, where the distribution F of p is a Bernoulli distribution. Financially speaking, when an event occurs with probability $\overline{p}$, all counterparties default, and the total portfolio is lost. Otherwise, with probability $1 - \overline{p}$, all obligors survive.

   Define $D_n = \sum_{i=1}^{n} L_i$, which is the total number of defaults. We then have

$$E[D_n] = n\overline{p}.$$

$$\begin{aligned}\text{var}(D_n) &= \sum_{i=1}^{n} \text{var}(L_i) + \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j \neq i}}^{n} \text{cov}(L_i, L_j) \\ &= n\overline{p}(1-\overline{p}) + n(n-1)(E[p^2] - E[p]^2)\end{aligned}$$

$$\text{var}\left(\frac{D_n}{n}\right) = \frac{\overline{p}(1-\overline{p})}{n} + \frac{n(n-1)}{n^2}\text{var}(\widetilde{p}) \longrightarrow \text{var}(\widetilde{p}) \text{ as } n \to \infty.$$

When considering the fractional loss for n large, the only remaining variance is that of the distribution of p.

- One can obtain any default correlation in $[0, 1]$; correlation of default events depends only on the first and second moments of F. However, the distribution of D_n can be quite different for different distribution F.

Remarks

1. For a given unconditional default probability $\overline{p}$, increasing correlation increases the probability of seeing large losses and of seeing small losses compared with a situation with no correlation.

2. It is the common dependence on the background variable p that induces the correlation in the default events. It requires assumptions of large fluctuations in p to obtain significant correlation.

Comparison with the case of independence of defaults

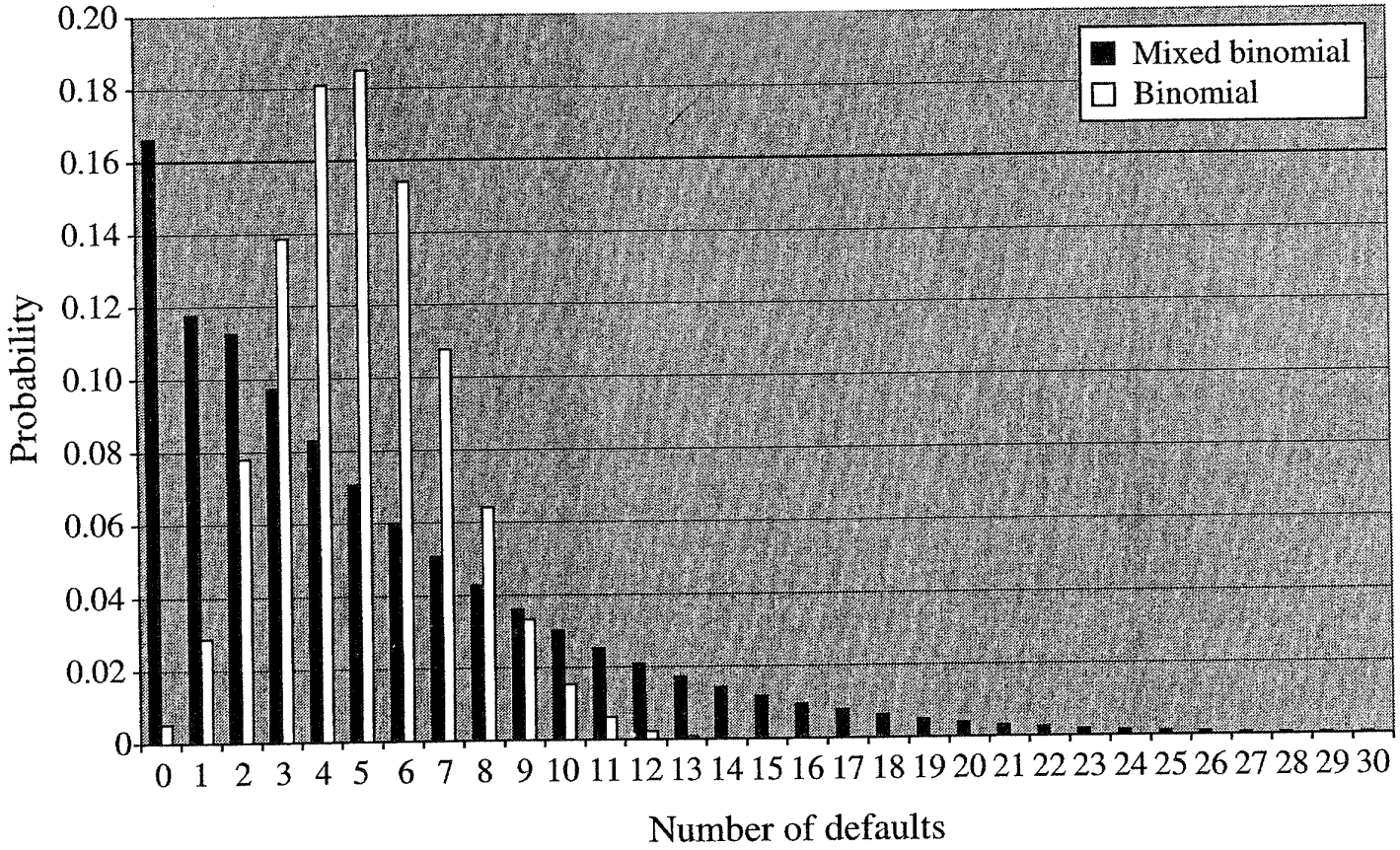
- The binomial distribution for independent defaults has a very **thin tail**, thus not representing the possibility of a large number of defaults realistically. Taking $N = 100$ obligors

Default Prob. (%)	1	2	3	4	5	6	7	8	9	10
99.0% VaR Level	5	7	9	11	13	14	16	17	19	20

  Maximum loss which is not exceeded with a given high probability (or confidence level).

$$VaR_\alpha(X) = \inf\{x \geq 0 | P[X \leq x] \geq \alpha\}.$$

  Take $p = 5\%$, the probability with 13 defaults or less is at least 99%, that is, 99% confidence level.



The distribution of the number of defaults among 50 issuers in the case of a pure binomial model with default probability 0.1 and in cases with beta distributions as mixture distributions over the default probability.

Keeping the default probability constant while having larger probabilities of many defaults requires the probability of very few defaults to increase as well.

Remark

For large portfolios, it is the distribution of p which determines the loss distribution. The more variability that there is in the mixture distribution, the more correlation of default events and more weight there in the tails of the loss distribution.

Choosing the mixing distribution using Merton's model

Consider n firms whose asset values V_t^i follow

$$dV_t^i = rV_t^i\,dt + \sigma V_t^i\,dB_t^i$$

with

$$B_t^i = \rho\widetilde{B}_t^0 + \sqrt{1-\rho^2}\,\widetilde{B}_t^i.$$

The GBM driving V_t^i can be decomposed into a common factor $\widetilde{B}_t^0$ and a firm-specific factor $\widetilde{B}_t^i$. Also, $\widetilde{B}^0, \widetilde{B}^1, \widetilde{B}^2, \cdots$ are independent standard Brownian motions. The firms are assumed to be identical in terms of drift rate and volatility.

Let D_i denote the default threshold of Firm i. Firm i defaults when

$$V_0^i \exp\left(\left(r - \frac{\sigma^2}{2}\right)T + \sigma B_T^i\right) < D^i$$

or

$$\ln V_0^i - \ln D^i + \left(r - \frac{\sigma^2}{2}\right)T + \sigma\left(\rho\widetilde{B}_T^0 + \sqrt{1-\rho^2}\,\widetilde{B}_T^i\right) < 0.$$

We write $\widetilde{B}_T^i = \epsilon_i\sqrt{T}$, where ϵ_i is a standard normal random variable.

Then firm i defaults when

$$\frac{\ln V_0^i - \ln D^i + \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} + \rho\epsilon_0 + \sqrt{1-\rho^2}\epsilon_i < 0.$$

Conditional on a realization of the common factor, say, $\epsilon_0 = u$, firm i defaults when

$$\epsilon_i < -\frac{c_i + \rho u}{\sqrt{1-\rho^2}}$$

where

$$c_i = \frac{\ln \frac{V_0^i}{D_i} + \left(r - \frac{\sigma^2}{2}\right) T}{\sigma\sqrt{T}}.$$

Assume that V_0^i/D_i to be the same for all i so that $c_i = c$ for all i. For given $\epsilon_0 = u$, the probability of default is

$$p(u) = N\left(-\frac{c + \rho u}{\sqrt{1-\rho^2}}\right).$$

Given $\epsilon_0 = u$, defaults of the firms are independent. The mixing distribution is that of the common factor ϵ_0, and N transforms ϵ_0 into a distribution on $[0, 1]$.

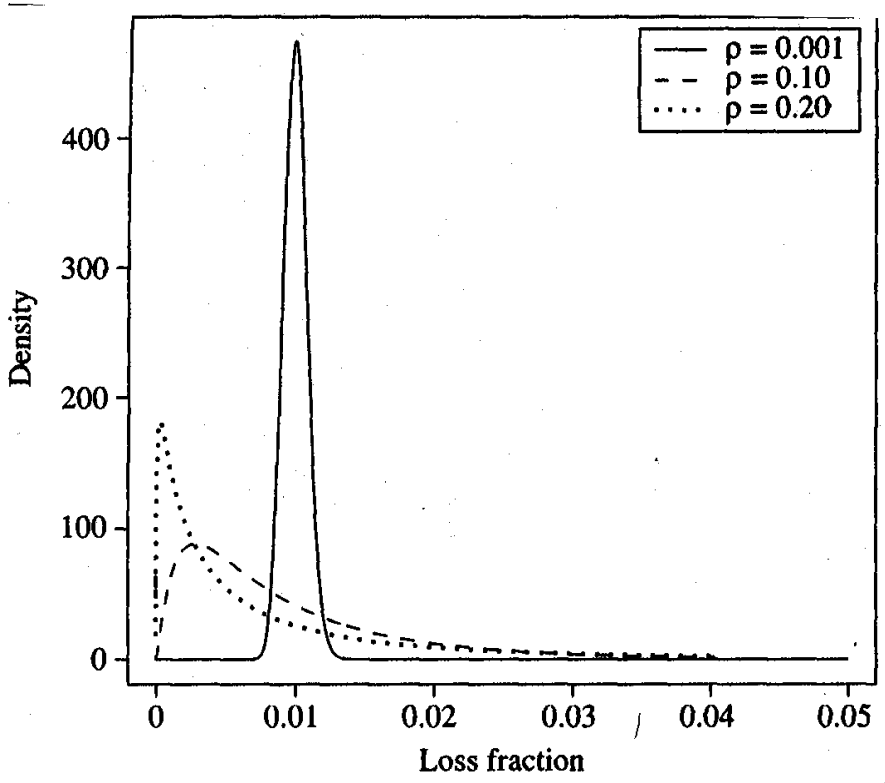
This distribution function $F(\theta)$ for the distribution of the mixing variable $\widetilde{p} = p(\epsilon_0)$ is

$$\begin{aligned} F(\theta) &= P[p(\epsilon_0) \le \theta] = P\left[N\left(-\frac{c+\rho\epsilon_0}{\sqrt{1-\rho^2}}\right) \le \theta\right] \\ &= P\left[-\epsilon_0 \le \frac{1}{\rho}\left(\sqrt{1-\rho^2}N^{-1}(\theta) + c\right)\right] \\ &= N\left(\frac{1}{\rho}\left(\sqrt{1-\rho^2}N^{-1}(\theta) - N^{-1}(\overline{p})\right)\right) \quad \text{where} \quad \overline{p} = N(-c). \end{aligned}$$

Here, $\overline{p}$ is the unconditional default probability corresponding to $\rho = 0$.

Note that $F(\theta)$ has the appealing feature that it has dependence on ρ and $\overline{p}$. The probability that no more than a fraction θ default is

$$P\left[\frac{D_n}{n} \le \theta\right] = \int_0^1 \sum_{k=0}^{n\theta} {}_nC_k p(u)^k [1-p(u)]^{n-k} f(u)\, du.$$



The figure shows the loss distribution in an infinitely diversified loan portfolio consisting of loans of equal size and with one common factor of default risk. The unconditional default probability is fixed at 1% but the correlation in asset values varies from nearly 0 to 0.2.

Contagion models

Contagion means that once a firm defaults, it may bring down other firms with it. Define Y_{ij} to be an "infection" variable. Both X_i and Y_{ij} are Bernuolli variables.

X_i is the default indicator of firm i due to its firm specific causes（自身原因）

$$Y_{ij} = \begin{cases} 1 & \text{if default of firm } i \text{ bring down firm } j \\ 0 & \text{if default of firm } i \text{ does not bring down firm } j \end{cases}.$$

$$\mathbb{P}[X_i] = p \quad \text{and} \quad \mathbb{P}[Y_{ij}] = q.$$

The default indicator of firm i is

$$Z_i = X_i + (1 - X_i)\left[1 - \prod_{j \neq i}(1 - X_j Y_{ji})\right].$$

Note that Z_i equals one either when there is a direct default of firm i or if there is no direct default and $\prod_{j \neq i}(1 - X_j Y_{ji}) = 0$. The latter case occurs when at least one of the factor $X_j Y_{ji}$ is 1, which happens when firm j defaults and infects firm i.

Define $D_n = Z_1 + \cdots + Z_n$, Davis and Lo (2001) find that

$$\begin{aligned} E[D_n] &= n[1 - (1 - p)(1 - pq)^{n-1}] \\ \text{var}(D_n) &= n(n-1)\beta_n^{pq} - (E[D_n])^2 \end{aligned}$$

where

$$\begin{aligned}\beta_n^{pq} &= p^2 + 2p(1-p)[1-(1-q)(1-pq)^{n-2}] \\ &\quad +(1-p)^2[1-2(1-pq)^{n-2}+(1-2pq+pq^2)^{n-2}].\end{aligned}$$

$$\mathrm{cov}(Z_i, Z_j) = \beta_n^{pq} - \mathrm{var}(D_n/n)^2.$$

Remarks

1. Zero contagion gives a pure binomial model.

2. Increasing the contagion brings more mass to high and low default numbers.

3. To preserve the mean, we must compensate for an increase in the infection parameter by decreasing the probability of direct default.

Estimating default through variation in frequencies

- Clustering of defaults should lead to larger fluctuations in default frequencies as years with many defaults are followed by years of few defaults compared with the overall default frequency.

- Consider a period of T years, each year we observe a population of n firms and record for year t the number of defaults D_t.

  Let ρ denote the correlation coefficient of default events contributing to D_i with a given year.

- Assume that the default events from year to year are independent, then

$$\text{var}\left(\frac{D_t}{n}\right) = \frac{\overline{p}(1-\overline{p})}{n} + \frac{n(n-1)}{n^2}(E[p]^2 - \overline{p}^2) \sim E[p]^2 \text{ as } n \to \infty.$$

- The overall default frequency is estimated as

$$\widehat{\overline{p}} = \frac{1}{nT} \sum_{t=1}^{T} D_t.$$

- The empirical variance of the default frequencies is obtained as

$$\widehat{\text{var}\left(\frac{D_t}{n}\right)} = \frac{1}{T} \sum_{t=1}^{T} \left(\frac{D_t}{n} - \widehat{\overline{p}}\right)^2,$$

  which serves as an estimator for $E[p]^2 - E[p]^2$.

- A moment-based estimator for the correlation is

$$\widehat{\rho} = \frac{\widehat{\text{var}}(D_t/n)}{\widehat{\overline{p}}(1 - \widehat{\overline{p}})}.$$

Moody's Binomial Expansion Method

- For a Binomial distribution with independent obligors, the tail with fewer (but larger) obligors is "fatter" than the tail with many (but smaller) independent obligors. Actually, variance of fractional loss decreases as $1/n$ as n increases.

- Moody's are aware that pure Binomial distribution with independent defaults is unrealistic.

- Make tails of distribution fatter by assuming fewer obligors (the *diversity score*). Adjustment are made for:

  – industry concentration

The idea is to approximate the loss on a portfolio of n positively correlated loans with the loss on a smaller number of independent loans with larger face value.

Moody's Diversity Scores

Number of Firms in the Same Industry	Diversity Score
1	1.0
2	1.5
3	2.0
4	2.3
5	2.6
6	3.0
7	3.2
8	3.5
9	3.7
10	4.0
10 or more	Evaluated case by case

Consider a portfolio of 60 bonds

Portfolio distribution

No of issuers in sector	1	2	3	4	5
No of incidences	2	7	3	4	2
Diversity	2	10.5	9	9.2	5.2

- Based on the idea that issuers in the same industry sector are related, while issuers in different industry sectors can be treated as independent.

- For the example, there are 2 cases in which issuers are the only representatives of hteir industry sector, 7 cases in which pairs of issuers are in the same sector, etc. The total diversity score $= 2 + 10.5 + 9 + 9.2 + 5.2 = 36$.

- The original portfolio of 60 bonds is treated to be equivalent to a portfolio of 36 independent bonds with the same default probability but with notional value 60/36 times the original notional.

Moody's binomial expansion technique (BET)

The two parameters in a binomial experiment are n and p.

- Diversity score, weighted average rating factor and binomial expansion technique.
- Generate the loss distribution.

To build a hypothetical pool of uncorrelated and homogeneous assets that mimic the default behaviors of the original pool of correlated and inhomogeneous assets.

Moody's diversity score

The diversity score of a given pool of participations is the number n of bonds in a idealized comparison portfolio that meets the following criteria:

- Comparison portfolio and collateral pool have the same face value.
- Bonds in the comparison portfolio have equal face values.
- Comparison bonds are equally likely to default, and their defaults are independent.
- Comparison bonds are of the same average default probability as the participations of the collateral pool.
- Comparison portfolio has, according to some measure of risk, the same total risk as does the collateral pool.

Binomial approximation using diversity scores

Seek reduction of problem of multiple defaults to binomial distributions.

If n loans each with equal face value are independent, have the same default probability, then the distribution of the loss is a binomial distribution with n as the number of trials.

Let F_i be the face value of each bond, p_i be the probability of default within the relevant time horizon and ρ_{ij} between the linear correlation coefficient of default events.

Assuming zero recovery, the loss variable L_i associated with bond i with face value F_i is given by

$$L_i = F_i \mathbf{1}_{\{D_i\}},$$

where D_i is the default event of bond i.

With n bonds, the total principal is $\sum_{i=1}^{n} F_i$ and the mean and variance of the loss of principal $\widehat{P}$ is

$$\begin{aligned}
E[\widehat{P}] &= E[L_1 + \cdots + L_n] = \sum_{i=1}^{n} F_i E[\mathbf{1}_{\{D_i\}}] = \sum_{i=1}^{n} p_i F_i \\
\text{var}(\widehat{P}) &= \sum_{i=1}^{n}\sum_{j=1}^{n} E[L_i L_j] - E[L_i]E[L_j] \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} F_i F_j (E[\mathbf{1}_{\{D_i\}}\mathbf{1}_{\{D_j\}}] - E[\mathbf{1}_{\{D_j\}}]E[\mathbf{1}_{\{D_j\}}]) \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} F_i F_j \rho_{ij} \sqrt{p_i(1-p_i)p_j(1-p_j)}.
\end{aligned}$$

We construct an approximating portfolio consisting D independent loans, each with the same face value F and the same default probability p.

$$\sum_{i=1}^{n} F_i = DF$$
$$\sum_{i=1}^{n} p_i F_i = DFp$$
$$\mathrm{var}(\widehat{P}) = F^2 Dp(1-p).$$

Solving the equations

$$\begin{aligned} p &= \frac{\sum_{i=1}^{n} p_i F_i}{\sum_{i=1}^{n} F_i} \\ D &= \frac{\sum_{i=1}^{n} p_i F_i \sum_{i=1}^{n} (1-p_i) F_i}{\sum_{i=1}^{n} \sum_{j=1}^{n} F_i F_j \rho_{ij} \sqrt{\rho_i (1-p_i) \rho_j (1-p_j)}} \\ F &= \sum_{i=1}^{n} F_i \Big/ D. \end{aligned}$$

Here, D is called the *diversity score*.