MAFS523 – Advanced Credit Risk Models

Homework

1. Suppose that the corporate zero-coupon bonds issued by ABC Co. Ltd. trade at the yields shown in the following table.

Maturity (t)	Risk-free yield	Corporate bond yield
0.5	3.57%	3.77%
1.0	3.70%	3.92%
1.5	3.81%	4.04%
2.0	3.95%	4.20%

   (a) Assume the recovery rate of the above bonds is 40%, calculate the survival probability, $P(\tau > T)$, for $T = 0.5, 1.0, 1.5$ and 2.0. Here, τ is the default time of ABC Co. Ltd.

   (b) By considering a CDS on a 2-year maturity zero-coupon bond, which is issued by ABC Co. Ltd., with spread s and recovery rate 40%, we assume the CDS premium is paid half-yearly and the default payment is paid at the end of payment period. Assume the default and risk-free interest rate are independence under risk neutral measure, calculate s.

2. We denote by $\bar{s}(T_k)$ the spread (premium) of the CDS at time 0 with maturity T_k for $k = 1, \cdots, N$. We assume CDS premium is paid at T_i if no default until T_i, for $i = 1, \cdots, k$, and the default payment is paid at T_i if default occur in $(T_{i-1}, T_i]$. We also assume that the default and risk-free interest rate are independence under risk neutral measure. For $1 \leq k \leq N$, show that the CDS implied survival probability, $P(\tau > T_k)$, is given by

$$P(\tau > T_k) = \frac{(1-\pi)\bar{B}(0,T_k)P(\tau > T_{k-1})}{(\bar{s}(T_k) - \bar{s}(T_{k-1}))\sum_{n=1}^{k-1}\Delta_{n-1}\bar{B}(0,T_n) + \bar{B}(0,T_k)(\bar{s}(T_k)\Delta_{k-1} + 1 - \pi)}$$

   (**Note:** We employ the convention that $\sum_{n=1}^{0} \cdot = 0$.)

   where

   $\overline{B}(t,T)$: price at t of defaultable zero-coupon bond paying \$1 at T

   π: constant recovery rate

   $\Delta_{i-1} = T_i - T_{i-1}$

3. A defaultable 5-year par floater with a floating-rate coupon of LIBOR + 300 basis points. If there is a 5-year CDS on this bond with CDS spread 200 basis points, what is your favourable position in the CDS and bond? Explain your answer. (Position in the bond mean you buy or short the bond while the position in the CDS mean you act as protection buyer or seller.)

4. You have the option to enter into a five-year credit default swap at the end of one year for a swap spread of 100 basis points. The principal is \$100 million. Payments are made on the swap semiannually. The forward swap spread for the period between year 1 and year 6 is 90 basis points, the volatility of the forward swap spread is 15%, LIBOR is flat at 5% (continuously compounded). The risk-neutral probability of a default by the reference entity during the first year is 0.015. What is the value of the option? Assume that the option ceases to exist if there is a default during the first year.

5. Suppose that the LIBOR curve is flat at 5.8% with continuous compounding and a three-year bond with notional \$100 and a coupon of 6.5% (paid semiannually) sells for 94.5. What is the asset swap spread that would be calculated in this situation?

6. Suppose that the risk-free zero curve is flat at 6% per annum with continuous compounding and that defaults can occur at times 1 year, 2 years, 3 years, and 4 years in a four-year plain vanilla credit default swap with semiannual payments. Suppose that the recovery rate is 20% and the probabilities of default at times 1 year, 2 years, 3 years and 4 years are $0.01, 0.015, 0.02$, and 0.025, respectively. The reference obligation is a bond paying a coupon semiannually of 8% per year. Defaults always take place immediately before coupon-payment dates on this bond. What is the credit default swap spread?

7. Consider a defaultable coupon bond with coupon dates $t_1, t_2, \cdots, t_n$, where t_n is the maturity date T. The coupon payment is c and the par is unity. The i^{th} coupon payment is only made if the bond has not defaulted at time $t_i, i = 1, 2, \cdots, n$. If the bond defaults prior to maturity, a recovery rate δ of the par is paid at the default time T_d. Assuming T_d to be independent of $r(t)$ under the risk neutral measure Q. Show that the value of this defaultable coupon bond is given by

$$\begin{aligned} D(r,t;T) = &\sum_{i=1}^{n} cB(r,t;t_i)P[T_d > t_i] + B(r,t;T)P[T_d > T] \\ &+ \int_t^T \delta B(r,t;u)q(u)\, du, \end{aligned}$$

where $q(t)$ is the probability density of the default time.

8. Let $\Pi(t)$ be the discounted payoff of **B** in the CDS at time t for $0 \le t \le T_s$. Write down the expression of $\Pi(t)$.

9. For $0 \le t \le T_s$, the $CDS(t; \bar{s}(T_s, T_{N+s}), T_s, T_{N+s})$ can be expressed as

$$CDS(t; \bar{s}(T_s, T_{N+s}), T_s, T_{N+s}) = E_Q\left[\Pi(t)|\mathcal{G}_t\right]$$

Show that

$$\begin{aligned} &CDS(t; \bar{s}(T_s, T_{N+s}), T_s, T_{N+s}) \\ &= \mathbf{1}_{\{\tau > t\}} \left\{ (1-R) \sum_{i=s+1}^{N+s} E_Q\left[e^{-\int_t^{T_i} r_u du - \int_t^{T_{i-1}} \lambda_u du} - e^{-\int_t^{T_i}(r_u+\lambda_u)du} | \mathcal{F}_t \right] - \right. \\ &\left. \bar{s}(T_s, T_{N+s}) \sum_{i=s+1}^{N+s} \Delta_{i-1} E_Q\left[e^{-\int_t^{T_i}(r_u+\lambda_u)du} | \mathcal{F}_t \right] \right\} \end{aligned}$$

where $\Delta_{i-1} = T_i - T_{i-1}$.

10. We now consider a CDS call option with maturity T_s and strike spread K which allows the option holder at T_s to have a right (not obligation) to use K as the CDS spread to enter into a CDS contract which starting at T_s and end at T_{N+s}. The periodic fee payment of the CDS is made at $T_{s+1}, T_{s+2}, \cdots, T_{N+s}$. The payoff of the option at T_s is given by

$$\Pi^{\text{Opt}}(T_s) = [CDS(T_s; K, T_s, T_{N+s}) - CDS(T_s; \bar{s}(T_s, T_{N+s}), T_s, T_{N+s})]^+.$$

Show that

$$\Pi^{\text{Opt}}(T_s) = \mathbf{1}_{\{\tau > T_s\}} \sum_{i=s+1}^{N+s} \Delta_{i-1} E_Q\left[e^{-\int_{T_s}^{T_i}(r_u+\lambda_u)du} | \mathcal{F}_{T_s} \right] (\bar{s}(T_s, T_{N+s}) - K)^+.$$

11. We would like to highlight the differences between entering a total return swap (TRS) and an outright purchase. We assume that **B** is the total return receiver, and **A** the total return payer. Using the example in the following table, we compare:

Portfolio I:

An outright purchase of the **C**-bond at $t = 0$ with a sale at $t = T_N$. The maturity of the **C**-bond may be longer than T_N. **B** finances this position with debt that is rolled over at LIBOR L_{i-1} for the period $(T_{i-1}, T_i]$, maturing at T_N.

Portfolio II:

A position as a total return receiver in a TRS with **A**. The TRS is unwound upon default of the underlying bond. Day count fractions are set to 1, i.e. $\delta_i = T_{i+1} - T_i = 1$.

The following table shows the payoff streams of the 2 portfolios to **B**.

	Portfolio I		Portfolio II		
Time	Defaultable bond	Funding	Periodic payment	Returns	Marking to market
$t = 0$	$-\bar{C}(0)$	$\bar{C}(0)$	0	0	0
$t = T_i$	$\bar{c}$	$-\bar{C}(0)L_{i-1}$	$-\bar{C}(0)(L_{i-1} + s^{TRS})$	$\bar{c}$	$\bar{C}(T_i) - \bar{C}(T_{i-1})$
$t = T_N$	$\bar{C}(T_N) + \bar{c}$	$-\bar{C}(0)(L_{N-1} + 1)$	$-\bar{C}(0)(L_{i-1} + s^{TRS})$	$\bar{c}$	$\bar{C}(T_N) - \bar{C}(T_{N-1})$
Default at $t = T_i$	Recovery	$-\bar{C}(0)L_{i-1}$	0	0	$-(\bar{C}(T_{i-1}) - \text{Recovery})$

Let $E_Q[\cdot]$ be the expectation operator under risk-neutral measure Q and $\beta(0,T)$ is the risk-free discount factor from T to 0.

We also denote $A(0)$ and $s(0)$ to be the value of an annuity paying \$1 per annum over $[0, T_N]$ and the fixed-for-floating (LIBOR) annual paying interest swap rate over $[0, T_N]$ respectively.

(a) Assuming there is no default in $[0, T_N]$, show that the value of the portfolio I to **B** at time 0 is given by

$$E_Q[(\bar{C}(T_N) - \bar{C}(0))\beta(0, T_N)] + \sum_{i=1}^{N} \bar{c}B(0, T_{i-1}) - \bar{C}(0)A(0)s(0),$$

where $B(0, T_i)$ is the time 0 price of default-free zero coupon bond with maturity T_i.

(b) Assuming there is no default in $[0, T_N]$, find the value of the portfolio II to **B** at time 0.

(c) Assuming there is no default in $[0, T_N]$, by neglecting the difference in the payoff at default, show that

$$s^{TRS} = \frac{\sum_{i=1}^{N} E_Q[(\beta(0,T_i) - \beta(0,T_N))(\bar{C}(T_i) - \bar{C}(T_{i-1}))]}{\sum_{i=1}^{N} B(0,T_i)\bar{C}(0)}.$$

12. Consider the Hull-White model for the credit default swap, show that an upper and a lower bound on the risky bond price maturing at t_j are given by

$$B_j \le B_j^0 - \sum_{i=1}^{j-1} q_i \beta_{ij}$$

and

$$B_j \ge B_j^0 - \sum_{i=1}^{j-1} q_i \beta_{ij} - \frac{\beta_{jj}}{t_j - t_{j-1}} \left[1 - \sum_{i=1}^{j-1} q_i (t_i - t_{i-1}) \right],$$

respectively.

Hint: Note that default probability densities must be non-negative and the cumulative probability of default must be less than one.