Topic 1B

Properties on implied survival probabilities, $P(t,T)$

1. $P(t,t) = 1$ and it is non-negative and decreasing in T. Also, $P(t,\infty) = 0$.

2. Normally $P(t,T)$ is continuous in its second argument, except that an important event *scheduled* at some time T_1 has direct influence on the survival of the obligor. For example, a coupon payment date if it is unclear if the coupon will be paid.

3. Viewed as a function of its first argument t, all survival probabilities for fixed maturity dates will tend to increase. This is obvious since
$$P(t_1,T) = P(t_2,T)\underbrace{P(t_1,t_2)}_{\text{less than one}}, \quad t_1 > t_2.$$

If we want to focus on the default risk over a given time interval in the future, we should consider conditional survival probabilities.

conditional survival probability over $[T_1, T_2]$ as seen from t, given that there was no default until time T_1

$$= P(t,T_1,T_2) = \frac{P(t,T_2)}{P(t,T_1)}, \quad \text{where } t \le T_1 < T_2;$$

$$= \frac{\bar{B}(t,T_2)}{B(t,T_2)} \frac{B(t,T_1)}{\bar{B}(t,T_1)}.$$

This is a direct consequence of $P[A|B] = \dfrac{P[A\cap B]}{P[B]}$; A = survival until T_2, B = survival until T_1.

Recall odds ratio of an event: (expected) number of events divided by the (expected) number of non-events. For example, "1 : 3 that the company will default in July" means : $p^{def}(t, \text{June } 30, \text{July } 31) : P(t, \text{June } 30, \text{July } 31) = 1:3$, where

$$p^{def}(t, T, T+\Delta T) = 1 - P(t, T, T+\Delta T).$$

Discrete implied hazard rate of default over $(T, T+\Delta T]$ as seen from time t

$$H(t,T,T+\Delta T)\Delta T = \frac{P_{def}(t,T,T+\Delta T)}{P(t,T,T+\Delta T)} = \frac{1-\frac{P(t,T+\Delta T)}{P(t,T)}}{\frac{P(t,T+\Delta T)}{P(t,T)}} = \frac{P(t,T)}{P(t,T+\Delta T)} - 1$$

so that

$$P(t,T) = P(t,T+\Delta T)[1 + H(t,T,T+\Delta T)\Delta T].$$

In the limit of $\Delta T \to 0$, the continuous hazard rate at time T as seen at time t is given by

$$h(t,T) = \lim_{\Delta T \to 0} H(t,T,T+\Delta T)$$

$$= \lim_{\Delta T \to 0} -\frac{1}{\Delta T} \frac{P(t,T+\Delta T) - P(t,T)}{P(t,T+\Delta T)}$$

$$= -\frac{1}{P(t,T)} \frac{\partial}{\partial T} P(t,T) = -\frac{\partial}{\partial T} \ln P(t,T),$$

provided that $\tau > t$ and the term structures of survival probabilities is differentiable with respect to T.

Forward spreads and implied hazard rate of default

For $t \leq T_1 < T_2$, the simply compounded forward rate over the period $(T_1, T_2]$ as seen from t is given by

$$F(t, T_1, T_2) = \frac{B(t, T_1)/B(t, T_2) - 1}{T_2 - T_1}.$$

This is the price of the forward contract with expiration date T_1 on a unit-par zero-coupon bond maturing on T_2. To prove, we consider the compounding of interest rates over successive time intervals:

$$\underbrace{\frac{1}{B(t, T_2)}}_{\text{compounding over } [t, T_2]} = \underbrace{\frac{1}{B(t, T_1)}}_{\text{compounding over } [t, T_1]} \underbrace{[1 + F(t, T_1, T_2)(T_2 - T_1)]}_{\text{simply compounding over } [T_1, T_2]}$$

Analogous to implied survival probabilities and implied hazard rate of default:

$$P(t, T) = P(t, T + \Delta T)[1 + H(t, T, T + \Delta T)\Delta T].$$

Defaultable simply compounded forward rate over $[T_1, T_2]$

$$\overline{F}(t, T_1, T_2) = \frac{\overline{B}(t, T_1)/\overline{B}(t, T_2) - 1}{T_2 - T_1}.$$

Instantaneous continuously compounded forward rates

$$\begin{aligned} f(t, T) &= \lim_{\Delta T \to 0} F(t, T, T + \Delta T) = -\frac{\partial}{\partial T} \ln B(t, T) \\ \overline{f}(t, T) &= \lim_{\Delta T \to 0} \overline{F}(t, T, T + \Delta T) = -\frac{\partial}{\partial T} \ln \overline{B}(t, T). \end{aligned}$$

The implied hazard rate of default at time $T > t$ as seen from time t is the spread between the forward rates:

$$h(t, T) = \overline{f}(t, T) - f(t, T)$$

obtained using

$$\begin{aligned} \overline{f}(t, T) - f(t, T) &= -\frac{\partial}{\partial T} \ln \frac{\overline{B}(t, T)}{B(t, T)} \\ &= -\frac{\partial}{\partial T} \ln P(t, T) = h(t, T). \end{aligned}$$

Implied hazard rate of default

Recall

$$\begin{aligned} P(t,T_1,T_2) &= \frac{\overline{B}(t,T_2)\,B(t,T_1)}{B(t,T_2)\,\overline{B}(t,T_1)} \\ &= \frac{1+F(t,T_1,T_2)(T_2-T_1)}{1+\overline{F}(t,T_1,T_2)(T_2-T_1)} = 1 - P_{def}(t,T_1,T_2), \end{aligned}$$

and upon expanding, we obtain

$$P_{def}(t,T_1,T_2)\underbrace{[1+\overline{F}(t,T_1,T_2)(T_2-T_1)]}_{\overline{B}(t,T_1)/\overline{B}(t,T_2)} = \underbrace{[\overline{F}(t,T_1,T_2)-F(t,T_1,T_2)]}_{\text{spread between risky and riskfree discrete forward rates}}(T_2-T_1).$$

If we take $T_1 = T$ and $T_2 = T + \Delta T$, then

$$\frac{P_{def}(t,T,T+\Delta T)}{P(t,T,T+\Delta T)} = H(t,T,T+\Delta T)\,\Delta T = \overline{F}(t,T,T+\Delta T) - F(t,T,T+\Delta T).$$

Instantaneous short rates and hazard rate

The *local default probability* at time t over the next small time step Δt

$$\frac{1}{\Delta t} Q[\tau \leq t + \Delta t | \mathcal{F}_t \wedge \{\tau > t\}] \approx \overline{r}(t) - r(t) = \lambda(t)$$

where $r(t) = f(t,t)$ is the riskfree short rate and $\overline{r}(t) = \overline{f}(t,t)$ is the defaultable short rate.

Recovery value

View an asset with positive recovery as an asset with an additional positive payoff at *default*. The recovery value is the *expected* value of the recovery shortly after the occurrence of a default.

* This definition ignores the difficulties that are involved in the real world determination of recovery in credit derivatives, like time delays, dealer polls or delivery options.

Payment upon default

Define $e(t,T,T+\Delta T)$ to be the value at time $t < T$ of a deterministic payoff of \$1 paid at $T+\Delta T$ if and only if a default happens in $[T,T+\Delta T]$.

$$e(t,T,T+\Delta T) = E_Q\left[\beta(t,T+\Delta T)[I(T)-I(T+\Delta T)]|\mathcal{F}_t\right].$$

Note that

$$I(T)-I(T+\Delta T) = \begin{cases} 1 & \text{if default occurs in } [T,T+\Delta T] \\ 0 & \text{otherwise} \end{cases},$$

$$\begin{aligned} E_Q[\beta(t,T+\Delta T)I(T)] &= E_Q[\beta(t,T+\Delta T)]E_Q[I(T)] \quad \text{(independence assumption)} \\ &= B(t,T+\Delta T)P(t,T), \end{aligned}$$

$$E_Q[\beta(t,T+\Delta T)I(T+\Delta T)] = \overline{B}(t,T+\Delta T),$$

and

$$B(t,T+\Delta T) = \overline{B}(t,T+\Delta T)/P(t,T+\Delta T).$$

It is seen that

$$\begin{aligned} e(t,T,T+\Delta T) &= B(t,T+\Delta T)P(t,T) - \overline{B}(t,T+\Delta T) \\ &= \overline{B}(t,T+\Delta T)\left[\frac{P(t,T)}{P(t,T+\Delta T)} - 1\right] \\ &= \Delta T \overline{B}(t,T+\Delta T) H(t,T,T+\Delta T). \end{aligned}$$

$$= \Delta T\, \overline{B}(t, T+\Delta T)\, \frac{P_{def}(t,T,T+\Delta T)}{P(t,T,T+\Delta T)}$$

$$= \Delta T\, \overline{B}(t, T+\Delta T)\, P(t,T)\, P_{def}(t,T,T+\Delta T)\, \frac{B(t,T,T+\Delta T)}{\overline{B}(t,T,T+\Delta T)}$$

$$= P(t,T)\left[P_{def}(t,T,T+\Delta T)\,\Delta T\right] B(t,T+\Delta T).$$

The above result indicates that

$$\begin{aligned} e(t,T,T+\Delta T) &= E_Q\left[\beta(t,T+\Delta T)\left[I(T) - I(T+\Delta T)\right] \middle| \mathcal{F}_t\right] \\ &= \underbrace{E_Q\left[\beta(t, T+\Delta T) \middle| \mathcal{F}_t\right]}_{B(t,\,T+\Delta T)} \underbrace{E_Q\left[I(T) - I(T+\Delta T) \middle| \mathcal{F}_t\right]}_{P(t,T)\,P_{def}(t,\,T,\,T+\Delta T)\Delta T}. \end{aligned}$$

On taking the limit $\Delta T \to 0$, we obtain

$$\text{rate of default compensation} \quad = \quad e(t,T) = \lim_{\Delta T \to 0} \frac{e(t,T,T+\Delta T)}{\Delta T}$$
$$= \quad \overline{B}(t,T)h(t,T) = B(t,T)P(t,T)h(t,T).$$

The value of a security that pays $\pi(s)$ if a default occurs at time s for all $t < s < T$ is given by

$$\int_t^T \pi(s)e(t,s)\,ds = \int_t^T \pi(s)\overline{B}(t,s)h(t,s)\,ds.$$

This result holds for deterministic recovery rates.

Stochastic recovery

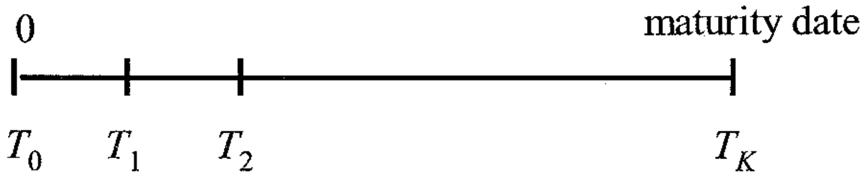
Let π' denote the random recovery amount received at $T+\Delta T$ if a default occurs in $(T, T+\Delta T]$. The value of this payoff is

$$\pi^e(t,T,T+\Delta T)\, e(t,T,T+\Delta T),$$

where $\pi^e(t,T,T+\Delta T)$ is the conditional expectation of π' conditional on a default occurring in $(T, T+\Delta T]$.

Building blocks for credit derivatives pricing

Tenor structure



$$\delta_k = T_{k+1} - T_k, 0 \leq k \leq K-1$$

Coupon and repayment dates for bonds, fixing dates for rates, payment and settlement dates for credit derivatives all fall on $T_k, 0 \leq k \leq K$.

- At every date T_k, a default can occur, or the obligor can continue until the next date T_{k+1}. After default, the default-free world goes on but the defaultable assets only earn their recovery payoff and cease to exist after default.

Fundamental quantities of the model

- Term structure of default-free interest rates $F(0,T)$
- Term structure of implied hazard rates $H(0,T)$
- Expected recovery rate π (rate of recovery as percentage of par)

From $B(0,T_i) = \dfrac{B(0,T_{i-1})}{1+\delta_{i-1}F(0,T_{i-1},T_i)}, i = 1,2,\cdots,k$, and $B(0,T_0) = B(0,0) = 1$, we obtain

$$B(0,T_k) = \prod_{i=1}^{k} \frac{1}{1+\delta_{i-1}F(0,T_{i-1},T_i)}.$$

Similarly, from $P(0,T_i) = \dfrac{P(0,T_{i-1})}{1+\delta_{i-1}H(0,T_{i-1},T_i)}$, we deduce that

$$
\begin{aligned}
\overline{B}(0,T_k) &= B(0,T_k)P(0,T_k) = B(0,T_k)\prod_{i=1}^{k}\frac{1}{1+\delta_{i-1}H(0,T_{i-1},T_i)}. \\
e(0,T_k,T_{k+1}) &= \delta_k H(0,T_k,T_{k+1})\overline{B}(0,T_{k+1}) \\
&= \text{value of \$1 at } T_{k+1} \text{ if a default} \\
&\quad\ \text{has occurred in } (T_k,T_{k+1}].
\end{aligned}
$$

- Forward hazard rates have a simple intuitive interpretation both in the economic sense (via forward zerobond spreads) as well as in the probability sense (via local conditional default probabilities).

Continuous limits

Taking the limit $\delta_i \to 0$, for all $i = 0, 1, \cdots, k$

$$\begin{aligned} B(0,T_k) &= \exp\left(-\int_0^{T_k} f(0,s)\,ds\right) \\ \overline{B}(0,T_k) &= \exp\left(-\int_0^{T_k} [h(0,s)+f(0,s)]\,ds\right) \\ e(0,T_k) &= h(0,T_k)\overline{B}(0,T_k). \end{aligned}$$

Alternatively, the above relations can be obtained by integrating

$$\begin{aligned} f(0,T) &= -\frac{\partial}{\partial T}\ln B(0,T) \quad \text{with} \quad B(0,0)=1 \\ \overline{f}(0,T) &= h(0,T)+f(0,T) = -\frac{\partial}{\partial T}\ln \overline{B}(0,T) \quad \text{with} \quad \overline{B}(0,0)=1. \end{aligned}$$

Defaultable fixed coupon bond

$$\begin{aligned}\overline{c}(0) = & \sum_{n=1}^{K} \overline{c}_n \overline{B}(0, T_n) && \text{(coupon)} && \overline{c}_n = \overline{c}\delta_{n-1} \\ & + \overline{B}(0, T_K) && \text{(principal)} \\ & + \pi \sum_{k=1}^{K} e(0, T_{k-1}, T_k) && \text{(recovery)}\end{aligned}$$

The recovery payment can be written as

$$\pi \sum_{k=1}^{K} e(0, T_{k-1}, T_k) = \sum_{k=1}^{K} \pi\delta_{k-1} H(0, T_{k-1}, T_k)\overline{B}(0, T_k).$$

The recovery payments can be considered as an additional coupon payment stream of $\pi\delta_{k-1}H(0, T_{k-1}, T_k)$.

Defaultable floater

Recall that $L(T_{n-1}, T_n)$ is the reference LIBOR rate applied over $[T_{n-1}, T_n]$ at T_{n-1} so that $1 + L(T_{n-1}, T_n)\delta_{n-1}$ is the compounding factor over $[T_{n-1}, T_n]$. Application of no-arbitrage argument gives

$$B(T_{n-1}, T_n) = \frac{1}{1 + L(T_{n-1}, T_n)\delta_{n-1}}.$$

- The coupon payment at T_n equals LIBOR plus a spread

$$\delta_{n-1}\left[L(T_{n-1}, T_n) + s^{par}\right] = \left[\frac{1}{B(T_{n-1}, T_n)} - 1\right] + s^{par}\delta_{n-1}.$$

- Consider the payment of $\dfrac{1}{B(T_{n-1}, T_n)}$ at T_n, its value at T_{n-1} is $\dfrac{\overline{B}(T_{n-1}, T_n)}{B(T_{n-1}, T_n)} = P(T_{n-1}, T_n)$. Why? We use the defaultable discount factor $\overline{B}(T_{n-1}, T_n)$ since the coupon payment may be defaultable over $[T_{n-1}, T_n]$.

- Seen at $t = 0$, the value becomes

$$
\begin{aligned}
& E_Q\left[\beta(0,T_{n-1})\, I(T_{n-1})\, P(T_{n-1},T_n)\right] \\
= \;& B(0,T_{n-1})\, E_Q\left[I(T_{n-1})\, P(T_{n-1},T_n)\right] \qquad \text{(independence assumption)} \\
= \;& B(0,T_{n-1})\, P(0,T_n).
\end{aligned}
$$

Combining with the fixed part of the coupon payment and observing the relation

$$
\begin{aligned}
[B(0,T_{n-1}) - B(0,T_n)]P(0,T_n) &= \left[\frac{B(0,T_{n-1})}{B(0,T_n)} - 1\right]\overline{B}(0,T_n) \\
&= \delta_{n-1}F(0,T_{n-1},T_n)\overline{B}(0,T_n),
\end{aligned}
$$

the model price of the defaultable floating rate bond is

$$
\begin{aligned}
\overline{c}(0) &= \sum_{n=1}^{K} \delta_{n-1}F(0,T_{n-1},T_n)\overline{B}(0,T_n) + s^{par}\sum_{n=1}^{K}\delta_{n-1}\overline{B}(0,T_n) \\
&\quad + \overline{B}(0,T_K) + \pi\sum_{k=1}^{K} e(0,T_{k-1},T_k).
\end{aligned}
$$