Topic 1C

Survival function

$\tau =$ continuous random variable that models the default time. Let $F(t)$ denote the distribution function of τ.

$$F(t) = P[\tau \leq t], \quad t \geq 0$$

Survival function $= S(t) = 1 - F(t) = P[\tau > t]$

$$\text{density function} \;=\; f(t) = F'(t) = -S'(t).$$

Conditional default and survival probabilities

$$\begin{aligned} {}_tq_x &= \text{conditional probability that the risky security will} \\ &\quad\ \text{default within the next } t \text{ years conditional on} \\ &\quad\ \text{its survival for } x \text{ year} \\ &= P[\tau - x \leq t | \tau > x], \quad t \geq 0 \\ {}_tp_x &= 1 - {}_tq_x, \quad t \geq 0. \end{aligned}$$

Note that $S(t) = {}_tp_0$

Hazard rate function (instantaneous) : $\lambda(x)$

Gives the instantaneous default probability for a security that has survived up to time x

$$P[x<\tau\le x+\Delta x|\tau>x]=\lambda(x)\Delta x=\frac{f(x)}{1-F(x)}\Delta x\approx\frac{F(x+\Delta x)-F(x)}{1-F(x)}$$

so that

$$\lambda(x) = -\frac{S'(x)}{S(x)},\ \text{giving } S(t)=e^{-\int_0^t\lambda(s)\,ds},\ \text{where } S(0)=1.$$

$$_tp_x = \text{probability of survival up to } x+t \text{ conditional on survival up to } x$$

$$= 1-e^{-\int_x^{x+t}\lambda(s)\,ds} = 1-e^{-\int_0^t\lambda(x+s)\,ds}.$$

Also, $F(t)=1-S(t)=1-e^{-\int_0^t\lambda(s)\,ds}$ and $f(t)=S(t)\lambda(t)$.

Recall that $S(t)=e^{-[y(t)-y^*(t)]t}$ where $y(t)$ = risky yield & $y^*(t)$ = riskfree yield

$$[y(t)-y^*(t)]t=\int_0^t\lambda(s)\,ds.$$

Also, $\lambda(t)=\bar{r}(t)-r(t)$, where $\bar{r}(t)$ = risky short rate
$r(t)$ = riskfree short rate.

Summary of formulas (deterministic hazard rate function)

- Survival probability

$$S(t) = P[\tau > t] = \exp\left(-\int_0^t \lambda(s) ds\right).$$

- Suppose we write $P[t < \tau \leq t + \Delta t] = f(t)\Delta t$, where $f(t)$ is the density function of the default time, we then have

$$f(t) = \lambda(t) \exp\left(-\int_0^t \lambda(s) ds\right) = \lambda(t) S(t).$$

- Probability of surviving until time t, given survival up to $s \leq t$,

$$P[\tau > t | \tau > s] = \frac{P[\tau > t]}{P[\tau > s]} = \exp\left(-\int_s^t \lambda(u) du\right).$$

- Default in $(t, t + \Delta t]$, conditional on no default up to time s,

$$P[t < \tau \leq t + \Delta t | \tau > s] = \lambda(t) \exp\left(-\int_s^t \lambda(u) du\right) \Delta t.$$

What happens when the hazard rate (intensity) λ_t is a random process?

It may be dependent on the history of a multidimensional Brownian motion up to time t, like a vector of macro-economic and firm specific variables.

$$p[\tau > t] = E\left[\exp\left(-\int_0^t \lambda_s \, ds\right)\right]$$

where the expectation is taken over all possible paths of the Brownian motion.

Let G_t denote the information set or filtration associated with the events on which the trajectory of λ_t depends up to time t.

The probability, based on information known at time t, of surviving until time T, given survival up to time t, is the following conditional expectation

$$p[\tau > T \mid \tau > t] = E\left[\exp\left(-\int_t^T \lambda_s \, ds\right) \Big| G_t\right].$$

If we vary t, for a fixed T, we obtain a random process Π_t^T, adapted to the filtration G_t.

Pricing of defaultable discount bond (zero recovery)

Money market account, $M_t = \exp\left(\int_0^t r_s\, ds\right)$.

$\hat{I}_t$ = survival indicator process $= \mathbb{1}_{\{\tau > t\}}$

$\bar{B}(t,T)$ = time-t price of T-maturity defaultable discount bond (zero recovery)

$$= M_t\, E_t\left[\frac{\hat{I}_T}{M_T}\right],$$

where E_t denotes the expectation conditional on F_t; $F_t = G_t \cup I_t$, I_t is the default history (history of $\hat{I}_t$).

Consider $E_t[\hat{I}_T] = E[\hat{I}_T \mid G_t \cup I_t]$,

since $F_t = G_t \cup I_t \subset G_T \cup I_t$, by tower property,

$$E[\hat{I}_T \mid F_t] = E[\,E[\hat{I}_T \mid G_T \cup I_t] \mid F_t].$$

Recall $p(\tau > T \mid \tau > t) = \exp\left(-\int_t^T \lambda_u \, du\right)$

so that

$$E(\hat{I}_T \mid G_T \cup I_t] = \hat{I}_t \exp\left(-\int_t^T \lambda_u \, du\right).$$

Hence $E[\hat{I}_T \mid F_t] = \hat{I}_t \, E_t\left[\exp\left(-\int_t^T \lambda_u \, du\right)\right]$.

Now $\bar{B}(t,T) = M_t \, E\left[\frac{\hat{I}_T}{M_T} \mid F_t\right] = \hat{I}_t M_t E_t\left[\frac{\exp\left(-\int_t^T \lambda_u du\right)}{M_T}\right]$

$$= \hat{I}_t \, E_t\left[\exp\left(-\int_t^T (r_s + \lambda_s)\, ds\right)\right].$$

Credit-risky coupon paying instrument

An instrument that pays coupon Y_s on a continuous basis until time T.

$$\text{time-}t\text{ value} = M_t E_t\left[\int_t^T \frac{1}{M_s} Y_s \hat{I}_s \, ds\right].$$

Using the tower law again

$$E_t\left[\int_t^T \frac{1}{M_s} Y_s \hat{I}_s \, ds\right] = E_t\left[\int_t^T E\left[\frac{Y_s}{M_s} \hat{I}_s \,\middle|\, \mathcal{G}_T \cup \mathcal{I}_t\right] ds\right]$$

$$= \hat{I}_t \, E_t\left[\int_t^T \frac{Y_s}{M_s} \exp\left(-\int_t^s \lambda_u \, du\right) ds\right]$$

so that time-t value

$$= \hat{I}_t \, E_t\left[\int_t^T Y_s \exp\left(-\int_t^s (r_u + \lambda_u) \, du\right) ds\right].$$

An instrument that pays in the event of default

An instrument that pays Z_t at time t if default occurs.

If default does not occur over $[0,T]$, the payoff is zero. $\tilde{Z}_\tau$ denotes the compensation payment at the random default (stopping time τ.

$$\text{time-}t\text{ value} = M_t E_t\left[\frac{1}{M_\tau}\tilde{Z}_\tau\right]$$

$$= M_t E_t\left[E\left[\frac{\tilde{Z}_\tau}{M_\tau}\,\Big|\, G_T \cup I_t\right]\right]$$

$$= \hat{I}_t M_t E_t\left[\int_t^T \frac{Z_s}{M_s}\lambda_s \exp\left(-\int_t^s \lambda_u\,du\right) ds\right]$$

$$= \hat{I}_t E_t\left[\int_t^T Z_s \lambda_s \exp\left(-\int_t^s (\lambda_u + r_u)\,du\right) ds\right].$$

Note that $Z_s \lambda_s\,ds$ is analogous to $Y_s\,ds$ in the previous case of continuous coupon stream.

Counterparty risk

- If A defaults, no payment will be made to bondholders.
- The intensity process λ_t^A for A takes the form

$$\lambda_t^A = \hat{I}_t^B \alpha_t + (1 - \hat{I}_t^B)\beta_t,$$

where $\hat{I}_t^B$ is the survival indicator process for B.

- time-t value of the risky bond issued by A

$$= \hat{I}_t^A E_t\left[\exp\left(-\int_t^T [r_s + \hat{I}_s^B \alpha_s + (1 - \hat{I}_s^B)\beta_s]\, ds\right)\right].$$

- Density function for the default time of B, given no default up to time t, conditional on economic information G_T is $\lambda_s^B \exp\left(-\int_t^s \lambda_u^B du\right)$.
- Final result for time-t value of the risky bond

$$\overline{B}(t,T) = \hat{I}_t^A E_t\left[\exp\left(-\int_t^T r_s\, ds\right) \int_t^T \underbrace{\lambda_s^B \exp\left(-\int_t^s \lambda_u^B du\right)}_{\text{density function of } \tau_B} \exp\left(-\int_t^s \alpha_x dx - \int_s^T \beta_x dx\right) ds\right].$$

Partial recovery on default

- If default occurs, the value drops to a fraction R_τ, τ = default time

$$\bar{B}(\tau, T) = R_\tau \bar{B}^*(\tau, T), \quad \bar{B}^*(\tau, T) = \text{value just before } \tau$$

R_t : recovery process ; random and adapted to the same economic information set that drives the intensity

- Value of the bond

= value of terminal payout if no default

+ amount recoverable in the event of a default

$$\tilde{Z}_\tau = R_\tau \bar{B}^*(\tau, T)$$

$$\bar{B}(t,T) = \hat{I}_t E_t\left[\exp\left(-\int_t^T (r_s+\lambda_s)\,ds\right) + \int_t^T R_s \lambda_s \bar{B}^*(s,T) \exp\left(-\int_t^s (r_u+\lambda_u)\,du\right) ds\right]$$

Recall $\bar{B}^*(t,T)$ is the value of the risky bond assuming no default has yet occurred, so we obtain the following recursive equation:

$$\bar{B}^*(t,T) = E_t\left[\exp\left(-\int_t^T (r_s+\lambda_s)\,ds\right) + \int_t^T R_s h_s \bar{B}^*(s,T)\exp\left(-\int_t^s (r_u+\lambda_u)\,du\right)ds\right].$$

- Define $\quad S_t^T = \bar{B}^*(t,T)\exp\left(-\int_0^t (r_s+\lambda_s)\,ds\right)$

and since $\bar{B}^*(T,T)=1$ (in the event of no default, payoff is unity)

$$S_T^T = \exp\left(-\int_0^T (r_s+\lambda_s)\,ds\right).$$

Observe

$$S_t^T = E_t\left[\exp\left(-\int_0^T (r_s+\lambda_s)\,ds\right) + \int_t^T \lambda_s R_s \bar{B}^*(s,T)\exp\left(-\int^s (r_u+\lambda_u)\,du\right)ds\right]$$

So that

$$S_t^T = E_t\left[S_T^T + \int_t^T \lambda_s R_s S_s^T\,ds\right].$$

$$= E_t\left[S_T^T + \int_0^T \lambda_s R_s S_s^T ds - \int_0^t \lambda_s R_s S_s^T ds\right].$$

The last term is known at time t, so

$$\underbrace{S_t^T + \int_0^t \lambda_s R_s S_s^T ds}_{\text{a martingale } m_t^T} = E_t\left[S_T^T + \int_0^T \lambda_s R_s S_s^T ds\right]$$

m_t^T and S_t^T are related by the following stochastic differential equation

$$dS_t^T + \lambda_s R_s S_t^T dt = d m_t^T$$

$$\Leftrightarrow d\left[S_t^T \exp \int_0^a \lambda_s R_s ds\right] = \exp\left(\int_0^a \lambda_s R_s ds\right) m_t^T.$$

We can deduce by integration that for each value of T, the process $\exp\left(\int_0^a \lambda_s R_s ds\right) m_t^T$ is a martingale. Using this martingale property: $\bar{B}(s,t) = \hat{I}_t \; E_t\left[\exp\left(-\int_t^T \left(r_s + \lambda_s (1-R_s)\right)\right] ds\right]$.

The hazard rate is adjusted by the loss fraction $1-R_s$ when we compute bond value.

Appendix

Interpretation of $E[X|\mathcal{F}]$
It is quite often that we would like to consider all conditional expectations of the form $E[X|B]$ where the event B runs through the algebra $\mathcal{F}$. Let B_j, $j = 1, 2, \cdots, n$, be the atoms of the algebra $\mathcal{F}$. We define the quantity $E[X|\mathcal{F}]$ by

$$E[X|\mathcal{F}] = \sum_{j=1}^{n} E[X|B_j]\mathbf{1}_{B_j}. \tag{2.2.4}$$

We see that $E[X|\mathcal{F}]$ is actually a random variable that is measurable with respect to the algebra $\mathcal{F}$. In the above numerical example, we have $\mathcal{F}_1 = \{\phi, \{\omega_1, \omega_2\}, \{\omega_3, \omega_4\}, \Omega\}$, and the atoms of $\mathcal{F}_1$ are $B_1 = \{\omega_1, \omega_2\}$ and $B_2 = \{\omega_3, \omega_4\}$. Since we have

$$E[S(2)|S(1) = 3] = 2.8 \quad \text{and} \quad E[S(2)|S(1) = 5] = 4.6,$$

so that

$$E[S(2)|\mathcal{F}_1] = 2.8\mathbf{1}_{B_1} + 4.6\mathbf{1}_{B_2}.$$

$$
\begin{aligned}
E[S(2)|S(1)=3] &= \frac{S(2;\omega_1)P(\omega_1)+S(2;\omega_2)P(\omega_2)}{P(\omega_1)+P(\omega_2)} \\
&= (4\times 0.2 + 2\times 0.3)/0.5 = 2.8; \\
E[S(2)|S(1)=5] &= \frac{S(2;\omega_3)P(\omega_3)+S(2;\omega_4)P(\omega_4)}{P(\omega_3)+P(\omega_4)} \\
&= (4\times 0.35 + 6\times 0.15)/0.5 = 4.6.
\end{aligned}
$$

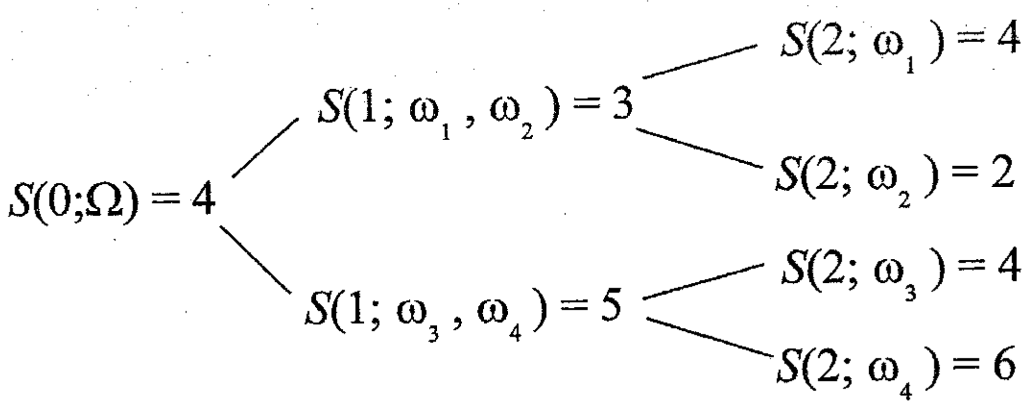

Fig. 2.5. The tree representation of an asset price process in a two-period securities model.

Tower Property
Since $E[X|\mathcal{F}]$ is a random variable, we may compute its expectation. We find that

$$E[E[X|\mathcal{F}]] = \sum_{B\in\mathcal{F}} E[X|B]P(B) = \sum_{B\in\mathcal{F}}\sum_{\omega\in B} X(\omega)(P[\omega]/P(B))P(B)$$
$$= \sum_{B\in\mathcal{F}}\sum_{\omega\in B} X(\omega)P(\omega) = E[X]. \qquad (2.2.5)$$

The above result can be generalized as follows. If $\mathcal{F}_1 \subset \mathcal{F}_2$, then

$$E[E[X|\mathcal{F}_2]|\mathcal{F}_1] = E[X|\mathcal{F}_1]. \qquad (2.2.6)$$

If we condition first on the information up to $\mathcal{F}_2$ and later on the information $\mathcal{F}_1$ at an earlier time, then it is the same as conditioning originally on $\mathcal{F}_1$. This is called the *tower property* of conditional expectations.