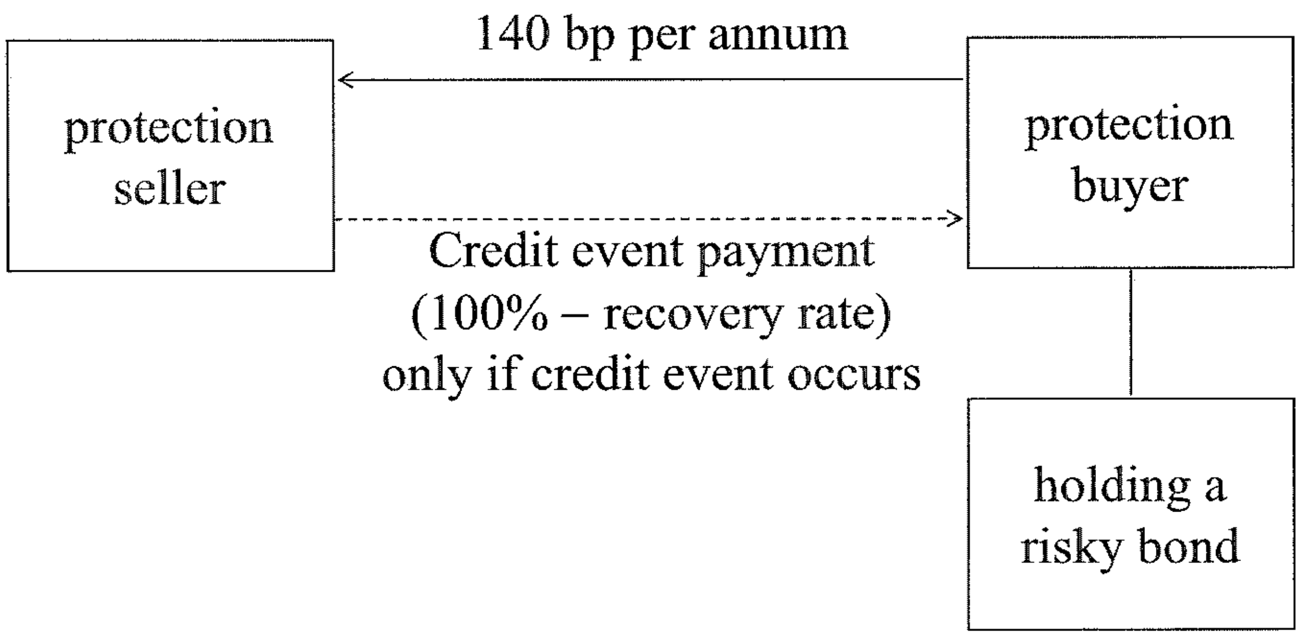
Topic 2B

Credit default swaps

The protection seller receives fixed periodic payments from the protection buyer in return for making a single contingent payment covering losses on a reference asset following a default.



Protection seller

- earns premium income with no funding cost
- gains customized, synthetic access to the risky bond

Protection buyer

- hedges the default risk on the reference asset

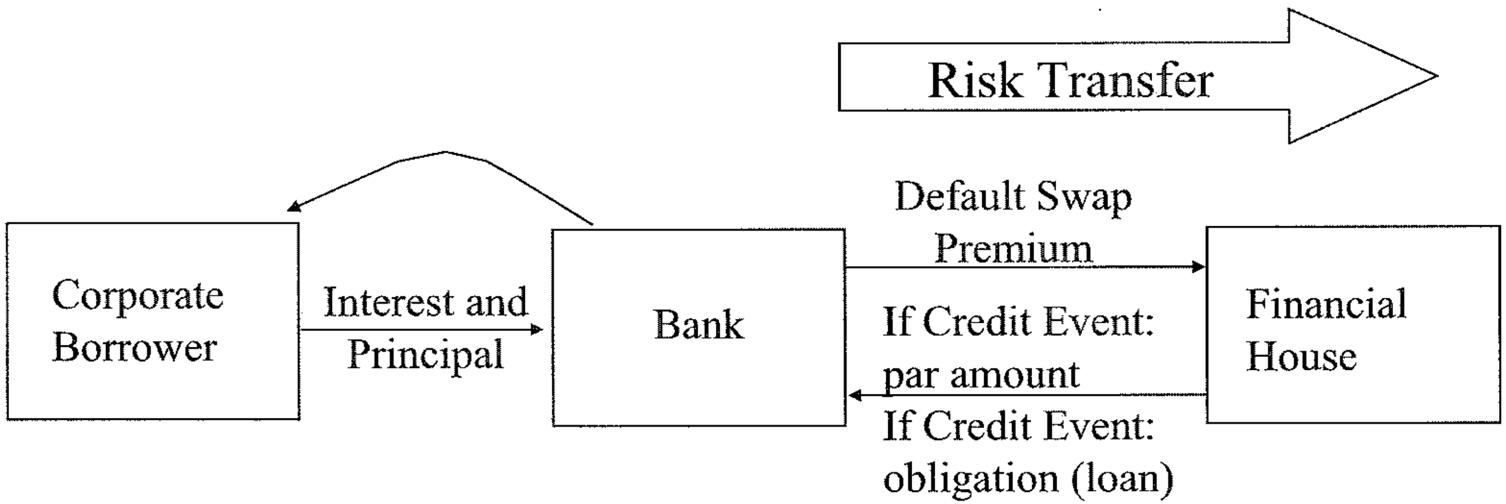
1. Very often, the bond tenor is longer than the swap tenor. In this way, the protection seller does not have exposure to the full period of the bond.

2. Basket default swap – gain additional yield by selling default protection on several assets.

A bank lends 10mm to a corporate client at L + 65bps. The bank also buys 10mm default protection on the corporate loan for 50bps.

Objective achieved

- maintain relationship
- reduce credit risk on a new loan



Settlement of compensation payment

1. Physical settlement:

   The defaultable bond is put to the Protection Seller in return for the par value of the bond.

2. Cash compensation:

   An independent third party determines the loss upon default at the end of the settlement period (say, 3 months after the occurrence of the credit event).

   $$\text{Compensation amount} = (1 - \text{recovery rate}) \times \text{bond par}.$$

Selling protection

To receive credit exposure for a fee or in exchange for credit exposure to better diversify the credit portfolio.

Buying protection

To reduce either individual credit exposures or credit concentrations in portfolios. Synthetically to take a short position in an asset which are not desired to sell outright, perhaps for relationship or tax reasons.

Credit default swap

Fixed leg Payment of $\delta_{n-1}\overline{s}$ at T_n if no default until T_n.

The value of the fixed leg is

$$\overline{s}\sum_{n=1}^{N}\delta_{n-1}\overline{B}(0,T_n).$$

Floating leg Payment of $1-\pi$ at T_n if default in $(T_{n-1},T_n]$ occurs. The value of the floating leg is

$$\begin{aligned}&(1-\pi)\sum_{n=1}^{N}e(0,T_{n-1},T_n)\\ =\ &(1-\pi)\sum_{n=1}^{N}\delta_{n-1}H(0,T_{n-1},T_n)\overline{B}(0,T_n).\end{aligned}$$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad = P_{def}(0,T_{n-1},T_n)\,/\,P(0,T_{n-1},T_n)$

$*$ $e(0,T_{n-1},T_n)$ can be expressed as

$$B(0,T_n)\underbrace{P(0,T_{n-1})}_{\text{survival prob}}\ \underbrace{P_{def}(0,T_{n-1},T_n)\,\delta_{n-1}}_{\text{conditional default prob}}.$$

The market CDS spread is chosen such that the fixed leg and floating leg of the CDS have the same value. Hence

$$\overline{s} = (1-\pi)\frac{\sum_{n=1}^{N} \delta_{n-1} H(0, T_{n-1}, T_n)\overline{B}(0, T_n)}{\sum_{n=1}^{N} \delta_{n-1}\overline{B}(0, T_n)}.$$

Define the weights

$$w_n = \frac{\delta_{n-1}\overline{B}(0, T_n)}{\sum_{k=1}^{N} \delta_{k-1}\overline{B}(0, T_k)}, \quad n = 1, 2, \cdots, N, \quad \text{and} \quad \sum_{n=1}^{N} w_n = 1,$$

then the fair swap premium rate is given by

$$\overline{s} = (1-\pi)\sum_{n=1}^{N} w_n H(0, T_{n-1}, T_n).$$

1. $\overline{s}$ depends only on the defaultable and default free discount rates, which are given by the market bond prices. CDS is an example of a cash product — pricing derived from yield curves.

2. It is similar to the calculation of fixed rate in the interest rate swap

$$s = \sum_{n=1}^{N} w'_n F(0, T_{n-1}, T_n)$$

where $$w'_n = \frac{\delta_{n-1} B(0, T_n)}{\sum_{k=1}^{N} \delta_{k-1} B(0, T_k)}, \qquad n = 1, 2, \cdots, N.$$

$\delta_{n-1} F(0, T_{n-1}, T_n)\, B(0, T_n) =$ time-0 value of riskfree floating rate payment at T_n (reset at T_{n-1})

$\delta_{n-1}\, H(0, T_{n-1}, T_n)\, \overline{B}(0, T_n) = \delta_{n-1}\, P(0, T_{n-1})\, P_{def}(0, T_{n-1}, T_n)\, B(0, T_n)$

$=$ time-0 value of compensation payment at T_n if default occurs in $(T_{n-1}, T_n]$.

Marked-to-market value

original CDS spread $= \overline{s}'$; new CDS spread $= \overline{s}$

Let $\Pi = \text{CDS}_{old} - \text{CDS}_{new}$, and observe that $\text{CDS}_{new} = 0$, then

$$\text{marked-to-market value} = \text{CDS}_{old} = \Pi = (\overline{s} - \overline{s}') \sum_{n=k}^{N} \overline{B}(0, T_n)\delta_{n-1},$$

where $\{T_k, \ldots, T_n\}$ is the remaining tenor.

Why? If an offsetting trade is entered at the current CDS rate $\overline{s}$, the fee difference $(\overline{s} - \overline{s}')$ will be received over the life of the CDS. Should a default occurs, the protection payments will cancel out, and the fee difference payment will be cancelled, too. The fee difference stream is defaultable and must be discounted with $\overline{B}(0, T_n)$.

- CDS's are useful instruments to gain exposure against spread movements, not just against default arrival risk.

Hedge strategy using fixed-coupon bonds

Portfolio 1

- One defaultable coupon bond $\overline{C}$; coupon $\overline{c}$, maturity t_N.
- One CDS on this bond, with CDS spread $\overline{s}$

Portfolio 2

- One default-free coupon bond C: with the same payment dates as the defaultable coupon bond and coupon size $\overline{c} - \overline{s}$.

Remark When does this default free bond become a par bond? This requires $\bar{c} - \bar{s}$ to be set equal to the market swap rate. Implicitly, the deduction of coupon size by $\bar{s}$ is applied over the whole life of the bond.

Comparison of cash flows of the two portfolios

1. In survival, the cash flows of both portfolio are identical.

	Portfolio 1	*Portfolio 2*
$t = 0$	$-\overline{C}(0)$	$-C(0)$
$t = t_i$	$\overline{c} - \overline{s}$	$\overline{c} - \overline{s}$
$t = t_N$	$1 + \overline{c} - \overline{s}$	$1 + \overline{c} - \overline{s}$

2. At default, portfolio 1's value = par = 1 (full compensation by the CDS); that of portfolio 2 is $C(\tau)$, τ is the time of default.

   The price difference at default $= 1 - C(\tau)$. This difference is very small when the default-free bond is a par bond.

Remark

The issuer can choose $\overline{c}$ to make the bond be a par bond such that the initial value of the bond is at par.

This is an approximate replication.

Recall that the value of the CDS at time 0 is zero. Neglecting the difference in the values of the two portfolios at default, the no-arbitrage principle dictates

$$\overline{C}(0) = C(0) = B(0, t_N) + \overline{c}A(0) - \overline{s}A(0).$$

Here, $(\overline{c} - \overline{s})A(0)$ is the sum of present value of the coupon payments at the fixed coupon rate $\overline{c} - \overline{s}$. The equilibrium CDS rate $\overline{s}$ can be solved:

$$\overline{s} = \frac{B(0, t_N) + \overline{c}A(0) - \overline{C}(0)}{A(0)}.$$

$B(0, t_N) + \overline{c}A(0)$ is the time-0 price of a default free coupon bond paying coupon at the rate of $\overline{c}$.

Query: $\overline{s}$ is paid during the survival period of the risky bond, unlike s^A in asset swap which continues even when the bond defaults.

Cash-and carry arbitrage with par floater

A par floater $\overline{C}'$ is a defaultable bond with a floating-rate coupon of $\overline{c}_i = L_{i-1} + s^{par}$, where the par spread s^{par} is adjusted such that at issuance the par floater is valued at par.

Portfolio 1

- One defaultable par floater $\overline{C}'$ with spread s^{par} over LIBOR.
- One CDS on this bond: CDS spread is $\overline{s}$.

The portfolio is unwound after default.

Portfolio 2

- One default-free floating-coupon bond C': with the same payment dates as the defaultable par floater and coupon at LIBOR, $c_i = L_{i-1}$.

The bond is sold after default.

Time	*Portfolio 1*	*Portfolio 2*
$t = 0$	-1	-1
$t = t_i$	$L_{i-1} + s^{par} - \overline{s}$	L_{i-1}
$t = t_N$	$1 + L_{N-1} + s^{par} - \overline{s}$	$1 + L_{N-1}$
τ (default)	1	$C'(\tau) = 1 + L_i(\tau - t_i)$

The hedge error in the payoff at default is caused by accrued interest $L_i(\tau - t_i)$, accumulated from the last coupon payment date t_i to the default time τ. If we neglect the *small* hedge error at default, then

$$s^{par} = \overline{s}.$$

Remarks

- The non-defaultable bond becomes a par bond (with initial value equals the par value) when it pays the floating rate equals LIBOR. The extra coupon s^{par} paid by the defaultable par floater represents the credit spread demanded by the investor due to the potential credit risk. The above result shows that the credit spread s^{par} is just equal to the CDS spread $\overline{s}$.

- The above analysis neglects the counterparty risk of the Protection Seller of the CDS. Due to potential counterparty risk, the actual CDS spread will be lower.

Valuation of Credit Default Swap

- Suppose that the probability of a reference entity defaulting during a year conditional on no earlier default is 2%. Here, constant hazard rate is assumed.

- Table 1 shows survival probabilities and unconditional default probabilities (i.e., default probabilities as seen at time zero) for each of the 5 years. The probability of a default during the first year is 0.02 and the probability that the reference entity will survive until the end of the first year is 0.98.

- The probability of a default during the second year is $0.02 \times 0.98 = 0.0196$ and the probability of survival until the end of the second year is $0.98 \times 0.98 = 0.9604$.

- The probability of default during the third year is $0.02 \times 0.9604 = 0.0192$, and so on.

Table 1 Unconditional default probabilities and survival probabilities

Time (years)	*Default probability*	*Survival probability*
1	0.0200	0.9800
2	0.0196	0.9604
3	0.0192	0.9412
4	0.0188	0.9224
5	0.0184	0.9039

- In the calculation of the expected compensation payment upon default as observed at time 0, the unconditional default probabilities are required.

- We will assume the defaults always happen halfway through a year and that payments on the credit default swap are made once a year, at the end of each year. We also assume that the risk-free (LIBOR) interest rate is 5% per annum with continuous compounding and the recovery rate is 40%.

- Table 2 shows the calculation of the expected present value of the payments made on the CDS assuming that payments are made at the rate of s per year and the notional principal is \$1.

For example, there is a 0.9412 probability that the third payment of s is made. The expected payment is therefore $0.9412s$ and its present value is $0.9412se^{-0.05\times 3} = 0.8101s$. The total present value of the expected payments is $4.0704s$.

Table 2 Calculation of the present value of expected payments. Payment $= s$ per annum.

Time (years)	*Probability of survival*	*Expected payment*	*Discount factor*	*PV of expected payment*
1	0.9800	$0.9800s$	0.9512	$0.9322s$
2	0.9604	$0.9604s$	0.9048	$0.8690s$
3	0.9412	$0.9412s$	0.8607	$0.8101s$
4	0.9224	$0.9224s$	0.8187	$0.7552s$
5	0.9039	$0.9039s$	0.7788	$0.7040s$
Total				$4.0704s$

Table 3 Calculation of the present value of expected payoff. Notional principal = \$1.

Time (years)	*Expected payoff (\$)*	*Recovery rate*	*Expected payoff (\$)*	*Discount factor*	*PV of expected payoff (\$)*
0.5	0.0200	0.4	0.0120	0.9753	0.0117
1.5	0.0196	0.4	0.0118	0.9277	0.0109
2.5	0.0192	0.4	0.0115	0.8825	0.0102
3.5	0.0188	0.4	0.0113	0.8395	0.0095
4.5	0.0184	0.4	0.0111	0.7985	0.0088
Total					0.0511

For example, there is a 0.0192 probability of a payoff halfway through the third year. Given that the recovery rate is 40% the expected payoff at this time is $0.0192 \times 0.6 \times 1 = 0.0115$. The present value of the expected payoff is $0.0115e^{-0.05\times2.5} = 0.0102$.

The total present value of the expected payoffs is \$0.0511.

Table 4 Calculation of the present value of accrual payment.

Time (years)	*Probability of default*	*Expected accrual payment*	*Discount factor*	*PV of expected accrual payment*
0.5	0.0200	$0.0100s$	0.9753	$0.0097s$
1.5	0.0196	$0.009s$	0.9277	$0.0091s$
2.5	0.0192	$0.0096s$	0.8825	$0.0085s$
3.5	0.0188	$0.0094s$	0.8395	$0.0079s$
4.5	0.0184	$0.0092s$	0.7985	$0.0074s$
Total				$0.0426s$

As a final step we evaluate in Table 4 the accrual payment made in the event of a default.

- There is a 0.0192 probability that there will be a final accrual payment halfway through the third year.
- The accrual payment is $0.5s$.
- The expected accrual payment at this time is therefore $0.0192 \times 0.5s = 0.0096s$.
- Its present value is $0.0096se^{-0.05\times 2.5} = 0.0085s$.
- The total present value of the expected accrual payments is $0.0426s$.

From Tables 2 and 4, the present value of the expected payment is

$$4.0704s + 0.0426s = 4.1130s.$$

From Table 3, the present value of the expected payoff is 0.0511. Equating the two, we obtain the CDS spread for a new CDS as

$$4.1130s = 0.0511$$

or $s = 0.0124$. The mid-market spread should be 0.0124 times the principal or 124 basis points per year.

In practice, we are likely to find that calculations are more extensive than those in Tables 2 to 4 because

(a) payments are often made more frequently than once a year

(b) we might want to assume that defaults can happen more frequently than once a year.

Marking-to-market a CDS

- At the time it is negotiated, a CDS, like most like swaps, is worth close to zero. Later it may have a positive or negative value.

- Suppose, for example the credit default swap in our example had been negotiated some time ago for a spread of 150 basis points, the present value of the payments by the buyer would be $4.1130 \times 0.0150 = 0.0617$ and the present value of the payoff would be 0.0511.

- The value of swap to the seller would therefore be $0.0617 - 0.0511$, or 0.0166 times the principal.

- Similarly the mark-to-market value of the swap to the buyer of protection would be -0.0106 times the principal.

Valuing credit default swap I: No counterparty default risk

by John Hull and Alan White, *Journal of Derivatives* (Fall 2000) p.29-40.

- Estimation of the risk neutral probability that the reference bond will default at different times in the future. The market prices of bonds issued by the same obligor and Treasury bonds are used to provide the market information of the expected default loss of the reference entity.

– Choose a set of N bonds issued by the obligor with maturity dates $t_j, j = 1, 2, \cdots, N$, where $t_{j-1} < t_j$ and $t = 0$. The life of CDS is $[0, t_N]$.

$B_j =$ market price of Treasury bond with maturity date t_j

$\overline{B}_j =$ market price of defaultable bond with maturity date t_j

$B_j - \overline{B}_j$ gives the market estimation of the present value of expected default loss of the j^{th} defaultable bond over the period $[0, t_j], j = 1, 2, \cdots, N$.

Assume the risk neutral default probability density function $q(t)$ to be piecewise constant over $[0, t_N], q(t) = q_i, t \in (t_{i-1}, t_i], i = 1, 2, \cdots, N$. Define β_{ij} be the present value of the expected default loss of the j^{th} risky bond defaulting within $(t_{i-1}, t_i], i \leq j$. We deduce that

$$\sum_{i=1}^{j} q_i \beta_{ij} = B_j - \overline{B}_j, \quad j = 1, 2, \cdots, N. \qquad (i)$$

Try to determine q_i by estimating β_{ij}.

$F_j(t)$: Forward price of the j^{th} bond for a forward contract maturing at time t $(t \leq t_j)$ assuming the bond is default-free. Note that $F_j(t)$ is the forward value of B_j, not $\bar{B}_j$, that is observable at time t

$v(t)$: Present value of \$1 received at time t with certainty

$C_j(t)$: Claim made by holders of the j^{th} defaultable bond if there is a default at time $t\,(t < t_j)$ (depends on the recovery mechanism)

$R_j(t)$: Recovery rate for holders of the j^{th} defaultable bond in the event of a default at time t $(t < t_j)$

β_{ij}: Present value of the loss from a default on the j^{th} defaultable bond at time t_i

q_i: The probability of default at time t_i

- The no-arbitrage price at time t of the no-default value of the j^{th} bond is $F_j(t)$. If there is a default at time t, the bondholder makes a recovery at rate $\widehat{R}$ on a claim of $C_j(t)$. It follows that

$$\beta_{ij} = v(t_i)[F_j(t_i) - \widehat{R}C_j(t_i)].$$

- There is a probability q_i of the loss β_{ij} being incurred. The total present value of the losses on the j^{th} bond is therefore given by

$$B_j - \bar{B}_j = \sum_{i=1}^{j} q_i \beta_{ij}.$$

- The first probability, q_1, is $(B_1 - \bar{B}_1)/\beta_{11}$. The remaining probabilities are given by

$$q_j = \frac{B_j - \bar{B}_j - \sum_{i=1}^{j-1} q_i \alpha_{ij}}{\beta_{jj}}.$$

Assumptions

1. We estimate the expected recovery rate from historical data.
2. We assume that all risky bonds have the same seniority and the expected recovery rate is time independent. Let $\widehat{R}$ be this expected recovery rate, which is independent of j and t.
3. Let $C_j(t)$ denote the claim amount on the j^{th} bond defaulting at time t, then

$$C_j(t) = L[1 + A(t)],$$

$L =$ face value, $A(t) =$ accrued interest at time t as percentage of its face value.

4. The protection buyer has to pay at default the accrued payment covering the period between the default time and the last payment date.

5. The default event, Treasury interest rates and recovery rates are mutually independent (under the risk neutral measure).

6. From the riskless Treasury interest rate, we can compute the discount factor $v(t)$, which is the present value of \$1 received at time t with certainty.

Let $F_j(t)$ be the forward price of the j^{th} default-free bond for a forward contract maturing at time t. Assuming deterministic interest rate, then the price at time t of the no-default value of the j^{th} bond is $F_j(t)$. We then have

$$\beta_{ij} = \int_{t_{i-1}}^{t_i} v(t) \left[F_j(t) - \widehat{R}C_j(t)\right] dt \quad i \leq j, j = 1, 2, \cdots, N. \qquad (ii)$$

From Eq. (i), we can deduce

$$q_j = \frac{B_j - \overline{B}_j - \sum_{i=1}^{j-1} q_i \beta_{ij}}{\beta_{jj}}, \qquad j = 1, 2, \cdots, N. \tag{iii}$$

The risk neutral expected payoff paid by the protection seller upon default at time t is $L\{1 - \widehat{R}[1 + A(t)]\}$. Therefore, the present value of the expected payoff is

$$\int_0^T L\{1 - \widehat{R}[1 + A(t)]\} q(t) v(t)\, dt.$$

$u(t) =$ present value of payments at the rate of \$1 per year on payment dates between time zero and t

$e(t) =$ present value of an accrual payment at time t of the time interval $t - t^*$, where t^* is the last payment date.

How to compute the annuity premium rate w paid by the protection buyer?

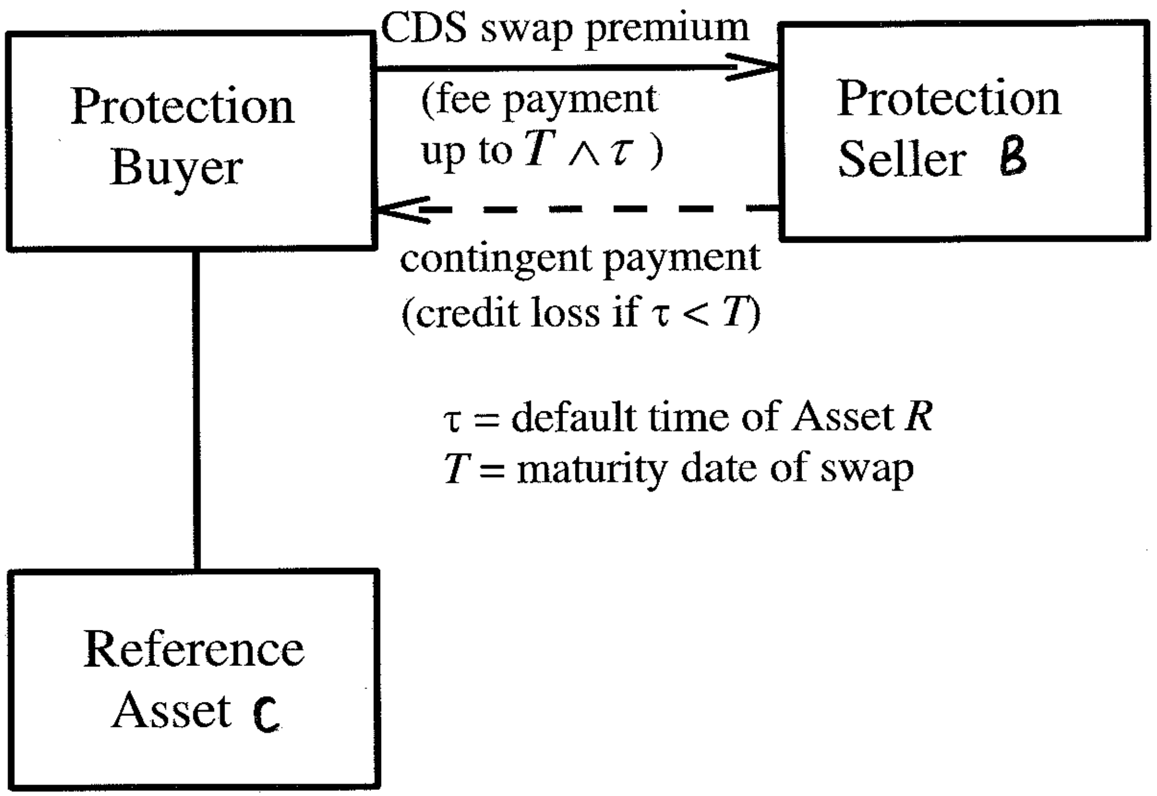
If there is no default prior to CDS maturity, the present value of payments is $wu(T)$.

$$\begin{aligned}&\text{Expected value of payments}\\ =\ & w\int_0^T Lq(t)[u(t)+e(t)]\,dt + wLu(T)\left[1-\int_0^T q(t)\,dt\right].\end{aligned}$$

Lastly, w is determined such that the present value of payments equals the present value of expected default loss. Hence

$$w = \frac{\int_0^T \{1-\widehat{R}[1+A(t)]\}q(t)v(t)\,dt}{\int_0^T q(t)[u(t)+e(t)]\,dt + u(T)\left[1-\int_0^T q(t)\,dt\right]}.$$

Counterparty risk of credit default swap



3 parties: Protection Seller (Counterparty), Protection Buyer, Reference Obligor

- Protection Buyer pays periodic swap premium (insurance fee) to Protection Seller (counterparty) to acquire protection on a risky reference asset (compensation upon default).

- Before the 1997 crisis in Korea, Korean financial institutions are willing to offer protection on Korean bonds. The financial melt down caused failure of compensation payment on defaulting Korean bonds by the Korean Protection Sellers.

- Given that the counterparty (Protection Seller) may default on the contingent compensation payment, what is the impact of the counterparty risk on the swap premium?

How does the inter-dependent default risk structure between the Protection Seller and the Reference Obligor affect the swap rate?

1. *Replacement cost* (Protection Seller defaults earlier)

    - If the Protection Seller C defaults prior to the Reference Entity, then the Protection Buyer renews the CDS with a new counterparty.
    - Supposing that the default risks of the Protection Seller C and Reference Entity R are positively correlated, then there will be an increase in the swap rate of the new CDS.

2. *Settlement risk* (Reference Entity defaults earlier)

    - The Protection Seller defaults during the settlement period after the default of Reference Entity.

Two-firm contagion model

The inter-dependent default risk structure between firm B and firm C is characterized by the interacting default intensities:

$$\lambda_t^B = b_0 + b_2 \mathbb{1}_{\{\tau^C \le t\}}$$

$$\lambda_t^C = c_0 + c_2 \mathbb{1}_{\{\tau^B \le t\}}.$$

- We take the notional to be \$1 and assume zero recovery upon default.
- $\{T_1, \ldots, T_n\}$ is the swap tenor with $0 = T_0 < T_1 < \cdots < T_n = T$

  δ = length of the settlement period

Since it takes no cost to enter a CDS, the swap rate $S_2(T)$:

$$\sum_{i=1}^{n} E\left[e^{-rT_i}\, S_2(T)\, 1_{\{\tau^B \wedge \tau^C > T_i\}}\right] + S_2(T) A_2(T)$$
$$= E\left[e^{-r(\tau^C+\delta)} 1_{\{\tau^C \le T\}} 1_{\{\tau^B > \tau^C + \delta\}}\right],$$

where $\tau^C + \delta$ is the settlement date.

- The first term gives the present value of the sum of periodic swap payment (terminated when either B or C defaults or at maturity).
- $S_2(T) A_2(T)$ is the present value of the acrued swap premium for the fraction of period between τ^C and the last payment date.

$$S_2(T)A_2(T) = S_2(T)\sum_{i=1}^{n} E\left[e^{-r\tau^C}\left(\frac{\tau^C - T_{i-1}}{\Delta T}\right)1_{\{T_{i-1}<\tau^C<T_i\}}1_{\{\tau^B>\tau^C\}}\right]$$

The accrued premium is paid at τ^c and $\frac{\tau^c - T_{i-1}}{\Delta T}$ represents the fraction of the time interval between successive payment dates.

- The buyer may face potential replacement cost when $\tau_B < \min(\tau_C, T)$. However, since $S_2(T)$ represents the fair swap rate charged by the seller party B, the replacement cost should not be included.

The mathematical procedure requires the determination of the joint density of default times (τ^B, τ^C).

Default contagion with interacting intensities

The default status of the assets in the portfolio is given by the default indicator process

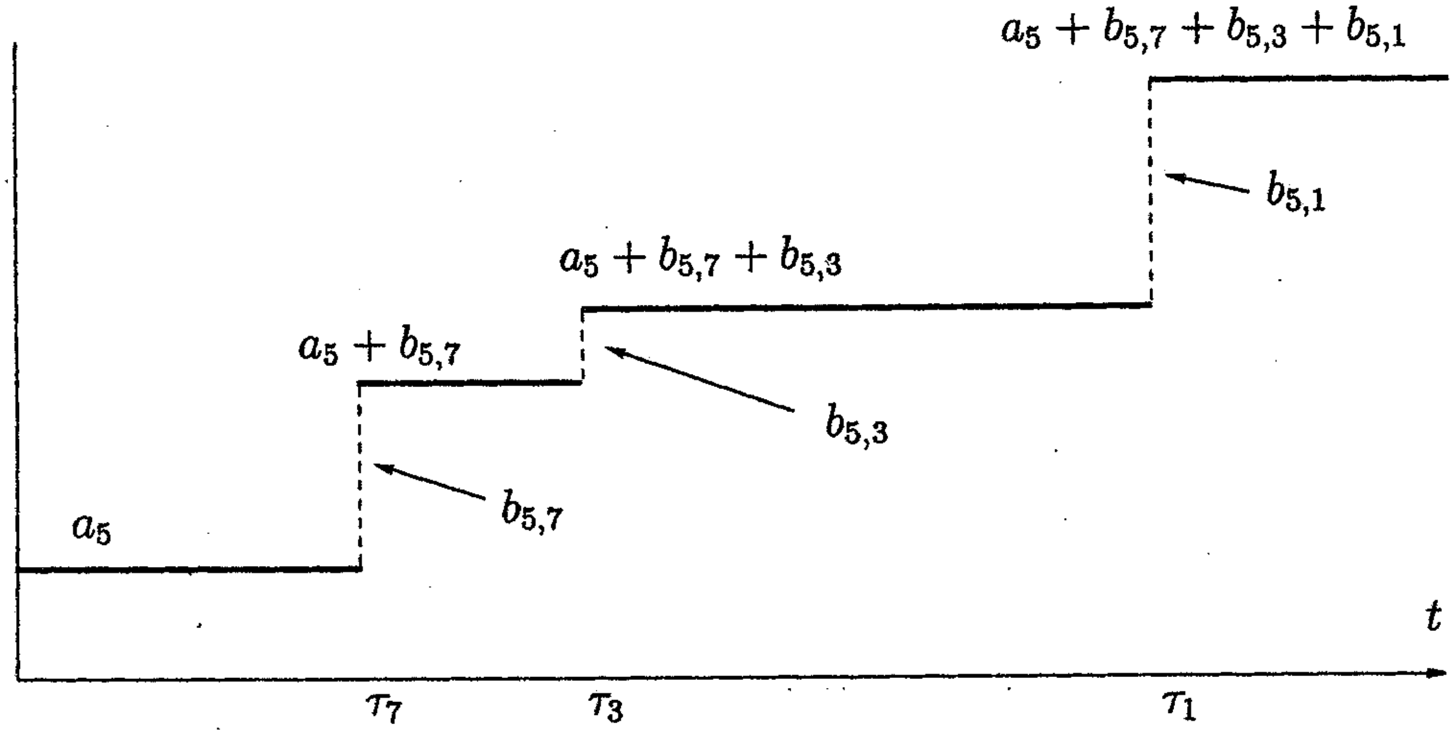
$$\boldsymbol{H}_t = (H_t^1 \quad H_t^2 \cdots H_t^N) \in \{0,1\}^N = \boldsymbol{E},$$

where $H_t^i = \mathbf{1}_{\{\tau_i \leq t\}}$ and $\boldsymbol{E}$ is the state space of the default status.

Characterize the default intensity of firm i by

$$\lambda_{t,i}(\boldsymbol{H}_t) = a_i + \sum_{j \neq i} b_{i,j} \mathbf{1}_{\{\tau_j \leq t\}}, \quad t \leq \tau_i, \tag{A}$$

and $\lambda_{t,i} = 0$ for $t > \tau_i$. Also, $a_i \geq 0$ and $b_{i,j}$ are constants such that $\lambda_{t,i}$ is non-negative. The jumps are independent of the order in which the defaults have occurred.



The default intensity for Firm 5 when the first default time $T_1 = \tau_7$, the second default time $T_2 = \tau_3$ and the third default time $T_3 = \tau_1$. The successive defaults of Firm 7, Firm 3 and Firm 1 put Firm 5 at a higher risk.

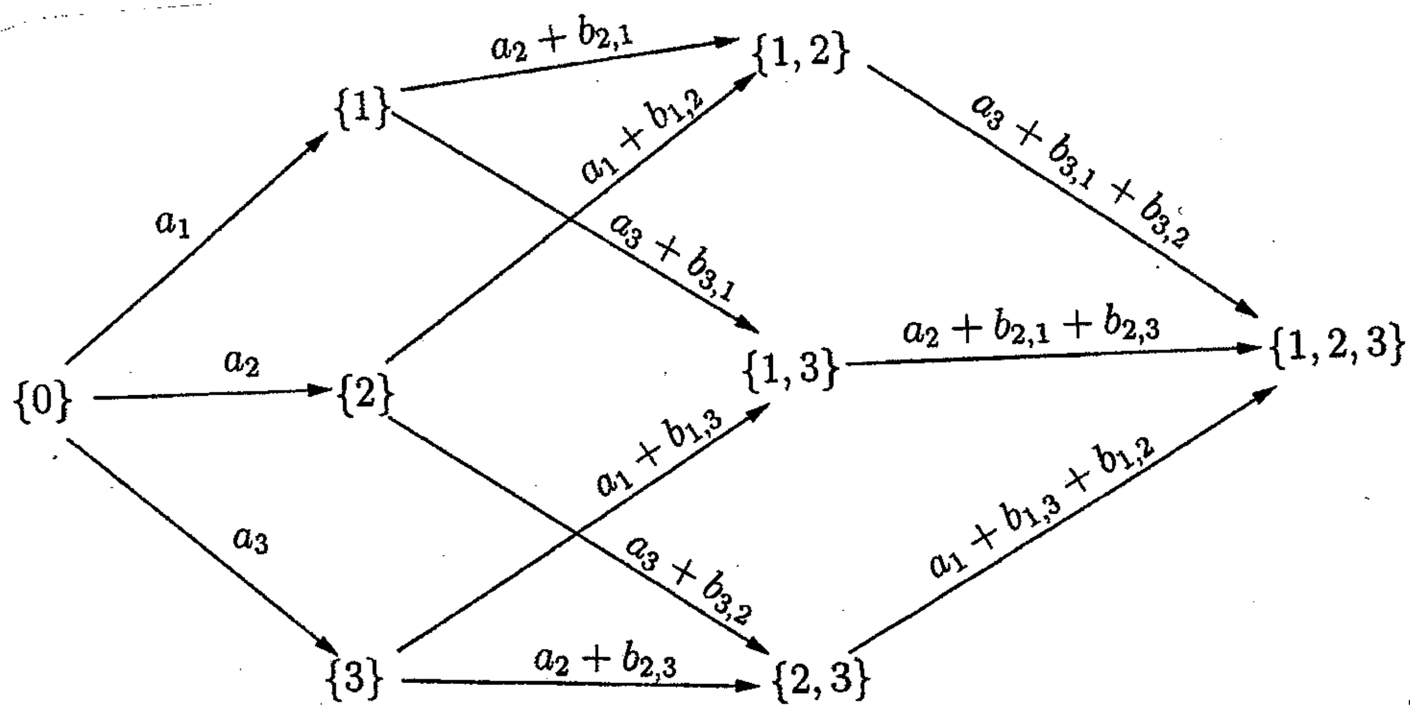


Illustration of the construction for $m = 3$. Arrows indicate possible transitions, and the transition intensities are given on top of the arrows.

$\{0\}$ – no default; $\{i\}$ – default of Firm i;

$\{i, j\}$ – default of Firms i and j;

$\{1, 2, 3\}$ – default of all 3 firms.

Three-Firm Model

The inter-dependent default intensities of the 3 firms are defined as

$$\begin{aligned}
\lambda_t^A &= a_1 + b_{12}\mathbf{1}_{\{\tau_B \le t\}} + b_{13}\mathbf{1}_{\{\tau_C \le t\}} + b_{14}\mathbf{1}_{\{\tau_B \le t, \tau_C \le t\}} \\
\lambda_t^B &= a_2 + b_{21}\mathbf{1}_{\{\tau_A \le t\}} + b_{23}\mathbf{1}_{\{\tau_C \le t\}} + b_{24}\mathbf{1}_{\{\tau_A \le t, \tau_C \le t\}} \\
\lambda_t^C &= a_3 + b_{31}\mathbf{1}_{\{\tau_A \le t\}} + b_{32}\mathbf{1}_{\{\tau_B \le t\}} + b_{34}\mathbf{1}_{\{\tau_A \le t, \tau_B \le t\}}.
\end{aligned}$$

We assume an extra jump in default intensity if the other two firms have defaulted, allowing the interaction between the default events on the intensity of surviving firms.

The state space S of $\boldsymbol{H} = (H_t^A, H_t^B, H_t^C)$ is given by

$S = \{(0,0,0), (1,0,0), (0,1,0), (0,0,1), (1,1,0), (1,0,1), (0,1,1), (1,1,1)\}$.

State 1	$(0,0,0)$	State 2	$(1,0,0)$	State 3	$(0,1,0)$	State 4	$(0,0,1)$
State 5	$(1,1,0)$	State 6	$(1,0,1)$	State 7	$(0,1,1)$	State 8	$(1,1,1)$

The infinitesimal generator Λ of the process $\boldsymbol{H}$ is given by

$$\Lambda = \begin{bmatrix} -(a_1+a_2+b_3) & a_1 & a_2 & a_3 & 0 & 0 & 0 & 0 \\ 0 & \begin{matrix}-(a_2+b_{21}\\+a_3+b_{31})\end{matrix} & 0 & 0 & a_2+b_{21} & a_3+b_{31} & 0 & 0 \\ 0 & 0 & \begin{matrix}-(a_0+b_{12}\\+a_3+b_{32})\end{matrix} & 0 & a_1+b_{12} & 0 & a_3+b_{32} & 0 \\ 0 & 0 & 0 & \begin{matrix}-(a_1+b_{13}\\+a_2+b_{21})\end{matrix} & a_0+b_{13} & a_2+b_{23} & 0 & \\ 0 & 0 & 0 & 0 & \begin{matrix}-(a_3+b_{31}\\+b_{32}+b_{34})\end{matrix} & 0 & 0 & \begin{matrix}(a_3+b_{31}\\+b_{32}+b_{33})\end{matrix} \\ 0 & 0 & 0 & 0 & 0 & \begin{matrix}-(a_2+b_{21}\\+b_{23}+b_{24})\end{matrix} & 0 & \begin{matrix}-(a_2+b_{21}\\+b_{23}+a_{24}\end{matrix} \\ 0 & 0 & 0 & 0 & 0 & 0 & \begin{matrix}-(a_1+b_{12}\\+b_{13}+b_{14})\end{matrix} & \begin{matrix}-(a_1+b_{12}\\+b_{13}+b_{14})\end{matrix} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

For example, consider the transition rate from State $2:\ (1 \quad 0 \quad 0)$ to State $5:\ (1 \quad 1 \quad 0)$ and State $6:\ (1 \quad 0 \quad 1)$:

$$\begin{aligned} \Lambda_{25} &= a_2 + b_{21} \\ \Lambda_{26} &= a_3 + b_{31} \end{aligned}$$

$\Lambda_{21} = \Lambda_{23} = \Lambda_{24} = \Lambda_{27} = \Lambda_{28} = 0$ and $\Lambda_{22} = 1 - \Lambda_{25} - \Lambda_{26}$.

Define the conditional transition density matrix on $\psi_s = \psi'$ as

$$P(t, s|\psi') = (p_{ij}(t, s|\psi'))_{|S|\times|S|} = (p(t, s, \boldsymbol{y}_i, \boldsymbol{y}_j|\psi'))_{|S|\times|S|}.$$

Kolmogorov backward equation

$$\frac{dP(t, s|\psi')}{dt} = -\Lambda_{[\psi]}(t)P(t, s|\psi'), \quad P(s, s|\psi') = I.$$

The $(i, j)^{\text{th}}$ entry $p_{ij}(t, s|\psi')$ satisfies the following system of ODE:

$$\begin{cases} \dfrac{dp_{ij}(t, s, \boldsymbol{y}_i, \boldsymbol{y}_j|\psi')}{dt} = -\displaystyle\sum_{k=1}^{|S|} \Lambda_{ik}(t|\psi) p_{kj}(t, s, \boldsymbol{y}_k, \boldsymbol{y}_j|\psi') \\ p_{ij}(s, s, \boldsymbol{y}_i, \boldsymbol{y}_j|\psi') = \mathbf{1}_{\{\boldsymbol{y}_j\}}(\boldsymbol{y}_i) \end{cases}. \qquad (2a)$$

Since default state is absorbing, Λ is upper triangular.

Using the results in eqs (1a,b), eq. (2a) can be expressed as

$$\frac{dp_{ij}(t,s,\boldsymbol{y}_i,\boldsymbol{y}_j|\psi')}{dt} + \sum_{k=1}^{N}[1-\boldsymbol{y}_i(k)]\lambda_k(\psi,\boldsymbol{y}_i)[p_{ij}(t,s,\tilde{\boldsymbol{y}}_i^k,\boldsymbol{y}_j|\psi') - p_{ij}(t,s,\boldsymbol{y}_i,\boldsymbol{y}_j|\psi')] = 0$$

with auxiliary condition:

$$p_{ij}(s,s,\boldsymbol{y}_i,\boldsymbol{y}_j|\psi') = \mathbf{1}_{\{\boldsymbol{y}_j\}}(\boldsymbol{y}_i). \qquad (2b)$$

Marginal distribution of default times

$$F_i(t_i) = P_r[\tau_i \le t_i] = \int \sum_{\boldsymbol{y}_j(i)=1} p_{ij}(0,t_i|\psi)\, d\mu_{\boldsymbol{\psi}}(\omega),$$

where we sum over all states j with default of the i^{th} obligor $[\boldsymbol{y}_j(i) = 1]$ and subsequently integrate over the distribution $\mu_{\boldsymbol{\psi}}(\omega)$. Here, $\mu_{\boldsymbol{\psi}}(\omega)$ is the probability measure which gives the law of $\boldsymbol{\Psi}$.

Assume that the pre-default intensities of A and B are

$$\lambda_t^{\mathrm{A}} = a_1 + a_2 1_{\{\tau^{\mathrm{B}} \leqslant t\}},$$
$$\lambda_t^{\mathrm{B}} = b_1 + b_2 1_{\{\tau^{\mathrm{A}} \leqslant t\}}.$$

Consider a four-state Markov chain in continuous time whose state space is $\{(\mathrm{N},\mathrm{N}), (\mathrm{D},\mathrm{N}), (\mathrm{N},\mathrm{D}), (\mathrm{D},\mathrm{D})\}$, where "N" signifies nondefault and "D" is default and the first coordinate refers to issuer A and the second to B. We can then reformulate the analysis by looking at the generator:

$$\Lambda = \begin{pmatrix} -(a_1+b_1) & a_1 & b_1 & 0 \\ 0 & -(b_1+b_2) & 0 & b_1+b_2 \\ 0 & 0 & -(a_1+a_2) & a_1+a_2 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Because of the simple upper triangular structure of the generator it is easy to compute its matrix exponential. Clearly, the eigenvalues of Λ are just its diagonal elements.

Letting

$$D = \begin{pmatrix} -(a_1+b_1) & 0 & 0 & 0 \\ 0 & -(b_1+b_2) & 0 & 0 \\ 0 & 0 & -(a_1+a_2) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and relying on the assumption of distinct parameter values,

$$B = \begin{pmatrix} 1 & \dfrac{a_1}{a_1-b_2} & \dfrac{b_1}{b_1-a_2} & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

we have

$$\Lambda = BDB^{-1}.$$

In fact, B^{-1} is also easy to compute as

$$B^{-1} = \begin{pmatrix} 1 & \frac{-a_1}{b_2 - a_2} & \frac{b_1}{a_2 - b_1} & -1 + \frac{b_1}{b_1 - a_2} + \frac{a_1}{a_1 - b_2} \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1, \end{pmatrix}$$

and hence we can compute all relevant transition probabilities:

$$P(t) = B \begin{pmatrix} \exp(-(a_1 + b_1)t) & 0 & 0 & 0 \\ 0 & \exp(-(b_1 + b_2)t) & 0 & 0 \\ 0 & 0 & \exp(-(a_1 + a_2)t) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} B^{-1}.$$

In particular, this gives us the marginal probability of A being in default before t:

$$Q(\tau^{\mathrm{A}} \leqslant t) = P_{12}(t) + P_{14}(t),$$

where we have added the case where B has not defaulted and where both have defaulted.

Simple computation gives us

$$Q(\tau^{\mathrm{A}} \leqslant t) = \frac{a_2 - a_2 \exp(-(a_1 + b_1)t) + b_1[\exp(-(a_1 + a_2)t) - 1]}{a_2 - b_1}.$$

The important thing to note is that this expression does *not* depend on b_2. This is intuitively clear. b_2 only takes effect after the default of A and from that time it controls the waiting time for B to follow A in default. In other words, changing b_2 will only serve to move probability mass between $P_{12}(t)$ and $P_{14}(t)$ but will not alter the sum.

Had we tried to compute

$$E \exp\left(-\int_0^t \lambda_s^{\mathrm{A}}\,\mathrm{d}s\right) = E \exp\left(-\int_0^t (a_1 + a_2 1_{\{\tau^{\mathrm{B}} \leqslant s\}})\,\mathrm{d}s\right),$$

we would obtain an expression that does depend on b_2, since the distribution of τ^{B} obviously depends on b_2 and this enters into the expression.

$$f(t_1,t_2)=\begin{cases} c_0(b_0+b_2)e^{-(b_0+b_2)t_1-(c_0-b_2)t_2}, & t_2\le t_1, \\ b_0(c_0+c_2)e^{-(c_0+c_2)t_2-(b_0-c_2)t_1}, & t_2>t_1. \end{cases}$$

The marginal density of the default times τ^B and τ^C can be obtained by integrating the joint density $f(t_1,t_2)$. This gives

$$\frac{P[\tau^B\in dt_1]}{dt_1}=\frac{(b_0+b_2)c_0}{c_0-b_2}\left[e^{-(b_0+b_2)t_1}-e^{-(b_0+c_0)t_1}\right]+b_0e^{-(b_0+c_0)t_1}$$

and

$$\frac{P[\tau^C\in dt_2]}{dt_2}=\frac{(c_0+c_2)b_0}{b_0-c_2}\left[e^{-(c_0+c_2)t_2}-e^{-(b_0+c_0)t_2}\right]+c_0e^{-(b_0+c_0)t_2}.$$

Consequently, the marginal survival probabilities are given by

$$P[\tau^B>t_1]=\frac{c_0e^{-(b_0+b_2)t_1}-b_2e^{-(b_0+c_0)t_1}}{c_0-b_2},$$

and

$$P[\tau^C>t_2]=\frac{b_0e^{-(c_0+c_2)t_2}-c_2e^{-(b_0+c_0)t_2}}{b_0-c_2}.$$

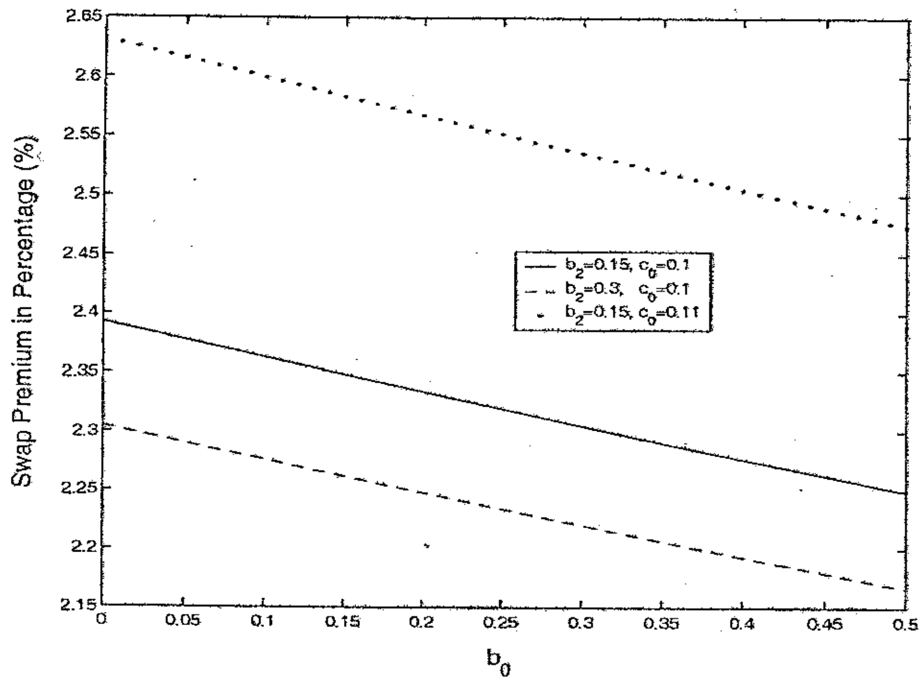
Swap Premium in the Two-firm Model

$$S_2(T) = \frac{c_0 e^{-(b_0+b_2+r)\delta}(1-e^{-\beta T})}{\beta}\left[\frac{e^{-\beta\Delta T}(1-e^{-\beta n\Delta T})}{1-e^{-\beta\Delta T}} + A_2(T)\right]^{-1},$$

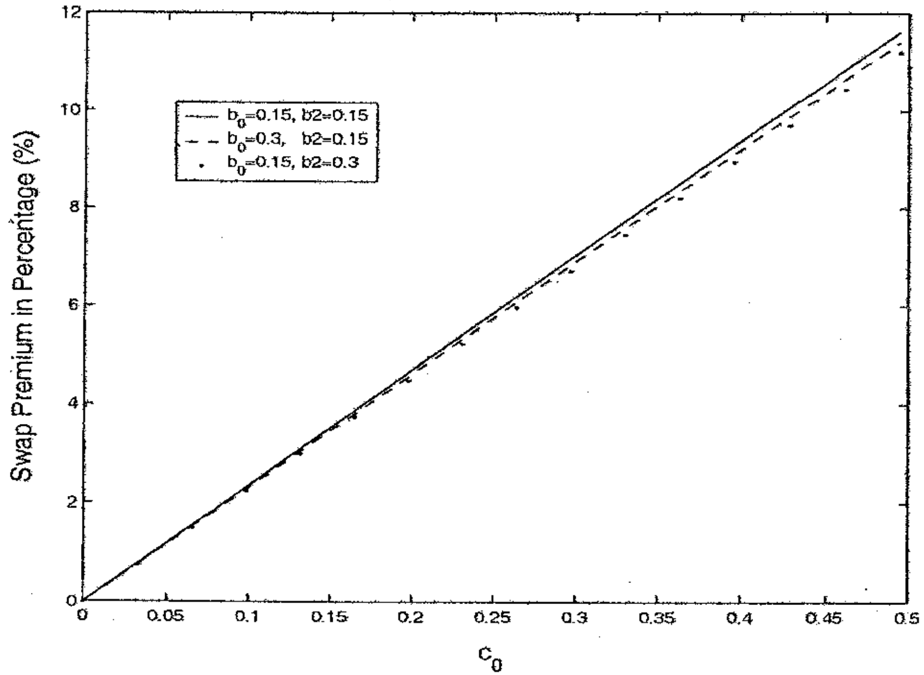
where $\beta = b_0 + c_0 + r$ and

$$\begin{aligned} A_2(T) &= \frac{c_0}{\Delta T}\left[\frac{1-e^{-(b_0+c_0+r)T}}{(b_0+c_0+r)^2} - \frac{Te^{-(b_0+c_0+r)T}}{b_0+c_0+r}\right] \\ &\quad - \frac{c_0}{b_0+c_0+r}\sum_{i=1}^{N} T_{i-1}\left[e^{-(b_0+c_0+r)T_{i-1}} - e^{-(b_0+c_0+r)T_i}\right]. \end{aligned}$$

- $S_2(T)$ is independent of C_2, though $E[e^{-rT_i} 1_{\{\tau^B \wedge \tau^C > T_i\}}]$ is involved. This is reasonable since an increase of λ_t^c would have impact only on the by C_2 due to B' default replacement cost.



The swap premium decreases with b0 as the protection buyer is willing to pay a lower premium when dealing with a more risky protection seller.



The swap premium is highly sensitive to the underlying default risk of C proxied by c_0. Impact of other parameters on $S_2(T)$ is minor compared to that of c_0.

- $S_2(T)$ is not sensitive to the length of the protection period

Settlement risk

If firm B is default free, then the swap premium is

$$\sum_{i=1}^{n} E\left[e^{-rT_i}\bar{S}_2(T)\, 1_{\{\tau^C > T_i\}}\right] + \bar{S}_2(T)\bar{A}_2(T) = E\left[e^{-r(\tau^C+\delta)} 1_{\{\tau^C \le T\}}\right],$$

where

$$\bar{A}_2(T) = \sum_{i=1}^{n} E\left[e^{-r\tau^C}\left(\frac{\tau^C - T_{i-1}}{\Delta T}\right) 1_{\{T_{i-1} < \tau^C < T_i\}}\right].$$

To examine the effect of settlement risk on the swap premium, we define the swap premium spread $V(T)$ to be the difference of the swap premium with and without settlement risk, that is,

$$V(T) = \bar{S}_2(T) - S_2(T).$$

Intuitively speaking, it is not clear that whether $V(T)$ is strictly positive. In a CDS the protection buyer inevitably faces a trade-off between a higher present value c compensation for its loss in the event of C's default, that is,

$$E\left[e^{-r(\tau^C+\delta)}1_{\{\tau^C\leq T\}}\right] \geq E\left[e^{-r(\tau^C+\delta)}1_{\{\tau^C\leq T\}}1_{\{\tau^B>\tau^C+\delta\}}\right]$$

and a higher present value of total swap payments due to an obligation to mak compensation to swap buyer upon the default of underlying asset, that is,

$$E\left[e^{-rT_i}1_{\{\tau^C>T_i\}}\right] + \bar{A}_2(T) \geq E\left[e^{-rT_i}1_{\{\tau^B\wedge\tau^C>T_i\}}\right] + A_2(T).$$

It is quite straightforward to derive $\overline{S}_2(T)$, which can be obtained by setting b_0 $b_2 = 0$ in $S_2(T)$.

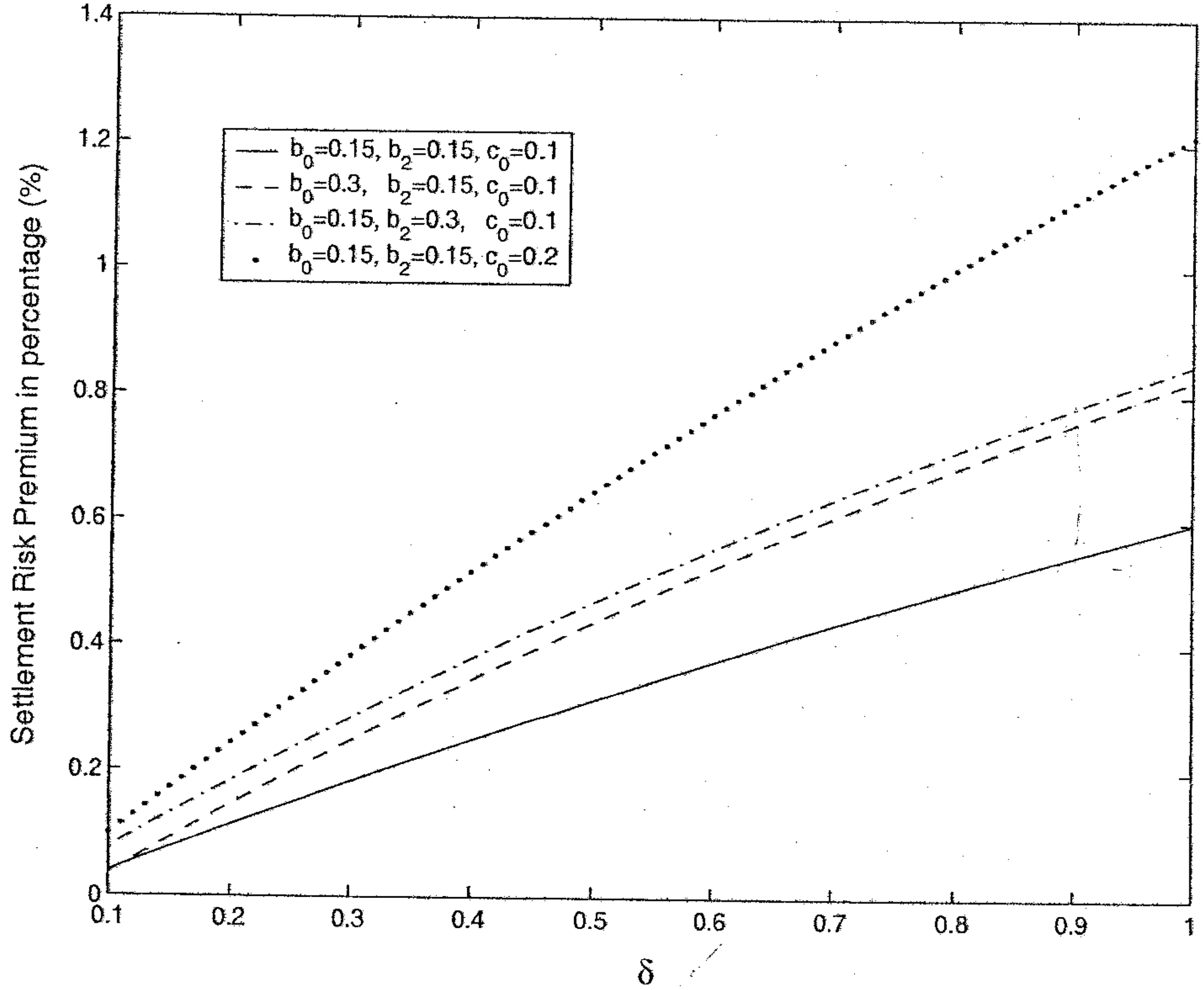


Figure 2 Change of settlement risk premium on δ. The base parameter values are: $r = 0.05$, $\Delta T =$ 0.25, $b_0 = 0.15$, $b_2 = 0.15$, $c_0 = 0.1$, $c_2 = 0.1$.

- The settlement risk premium increases as δ increases, and its sensitivity is very significant.
- The effect of b_0 and b_2 have relatively less influence on the settlement risk premium; b_2 is slightly more important than b_0.

<u>Key considerations</u>

1. Credit rating of the counterparty B (protection seller);
2. Correlated default risk between seller and reference asset;
3. Length of settlement period.