Modelling considerations in convertible bond pricing models

- equity risk, default risk, interest rate risk, call and put risks

Reference "The valuation of convertible bonds with credit risk," by E, Ayache, P.A. Forsyth and K.R. Vetzal (2003) Working paper of University of Waterloo.

Choices of the underlying state variables

- Firm value versus stock price

  Earliest works use the value of the issuing firm as the underlying state variable.

  The firm's debt and equity are claims contingent on the firm value, and options on its debt and equity are compound options on this variable.

Advantage:- dilution effect on equity upon conversion of the bond into shares can be modeled directly

Disadvantage:- value of the firm is not a traded asset, parameter estimation is difficult; any other liabilities that are more senior than the convertible must be simultaneously valued.

Recent models use the issuing firm's stock price

- stock is a traded asset, so parameter estimation is easy, use of risk neutral valuation principle is more convincing;
- no need to estimate the values of other more senior claims.

How about stochastic riskfree interest rate?

- Addition of stochastic interest rate as an additional state variable increases the dimensionality of the pricing model.
- Practitioners often regard a convertible bond primarily as an equity instrument, where the main factor is the stock price. The random nature of the riskfree interest rate is of second order importance.

Brennan and Schwartz (1980) conclude that "for a reasonable range of interest rates the errors from the non-stochastic interest rate model are likely to be slight."

Does negative correlation between the interest rate and stock price hasten or lessen the impact of interest rate fluctuation on convertible bond value? When interest rate increases, stock price tends to drop but the conversion option increases in value due to the drop in the bond floor value.

Other considerations in modeling

- Modeling of default risk
  - arrival of default event

    Structural approach and reduced form approach (hazard rate)
  - loss upon default

    What is the drop of the stock price upon default?
- Issuer's call provision
  - soft call requirement, trigger prices
  - call notice period
- Dilution upon holders' conversion – more shares are issued
- Holder's put right

Earlier attempt in modeling default risk/loss upon default

- Ad-hoc approach – use of risky discount rate

  However, different components of the convertible bond are subject to different default risk

  – cash-only part (bond part); equity part (conversion option)

- Stock price instantly jumps to zero upon default

  – Shares in firms filing for bankruptcy in the US had abnormal returns of about −65% during the three years prior to a bankruptcy announcement, and had abnormal returns of about −30% around the announcement.

  – characterized by a gradual erosion of the stock price prior to the event, followed by a significant decline upon announcement.

Pricing formulation — no credit risk case

Under a risk neutral measure Q,

$$\frac{dS_t}{S_t} = [r(t) - q(t)]\,dt + \sigma\, dZ_t.$$

Define

$$\mathcal{L}V = -V_t - \left[\frac{\sigma^2}{2} S^2 V_{SS} + (r-q) S V_S - rV\right].$$

A convertible has the following contractual features:

- A continuous (time-dependent) put provision with an exercise price B_p.
- A continuous (time-dependent) conversion provision — Conversion into k shares.
- A continuous (time-dependent) call provision; call price B_c, which is greater than B_p. Holders can convert the bond if called. We expect $B_c > B_p$ at all times.

Separate into two cases : $B_c > \kappa S$ or $B_c \le \kappa S$

- $B_c > \kappa S$ [linear complementarity formulation]

$$\begin{pmatrix} \mathcal{L}V = 0 \\ V - \max(B_p, \kappa S) > 0 \\ V - B_c < 0 \end{pmatrix} \vee \begin{pmatrix} \mathcal{L}V > 0 \\ V - \max(B_p, \kappa S) = 0 \\ V - B_c < 0 \end{pmatrix} \vee \begin{pmatrix} \mathcal{L}V < 0 \\ V - \max(B_p, \kappa S) > 0 \\ V - B_c = 0 \end{pmatrix}$$

$(C_1) \vee (C_2) \vee (C_3)$ is interpreted as at least one of the three conditions: C_1, C_2, C_3, holds at each point in the solution domain.

- $B_c \le \kappa S$

The convertible value is simply $V = \kappa S$ since the holder would choose to convert immediately [see p.18].

In summary, the value of the convertible bond is given by $\mathcal{L}V = 0$ subject to the constraints:

$$V \ge \max(B_p, \kappa S), \quad V \le \max(B_c, \kappa S).$$

Either we are in the continuation region where $\mathcal{L}V = 0$, so neither the call constraint nor the put constraint are binding, or the put constraint is binding or the call constraint is binding.

- When it is optimal for the holder to put or convert, the rate of returns from the Δ-hedged portfolio is less than r so that $\mathcal{L}V > 0 \Leftrightarrow d\Pi < r\Pi dt$. In this case,

  $V = \max(B_p, kS)$ while $V < B_c$. [Recall $B_c > B_p$]

- When it is optimal for the issuer to call, $V = B_c$ [$B_c > kS$] and $V > \max(B_p, kS)$. Note that $\mathcal{L}V = -\left(\frac{\partial B_c}{\partial t} - rB_c\right) < 0$, on the assumption that the rate of growth of B_c is faster than r (a sensible contractual design of the bond contract).

Auxiliary conditions

We alter the operator $\mathcal{L}V$ at $S=0$ and $S\to\infty$.

- At $S=0$, $\mathcal{L}V$ becomes $-[V_t - r(t)V]$ [equity component disappears completely]
- As $S\to\infty$, the unconstrained solution is linear in S so that $\frac{\partial^2 V}{\partial S^2} = 0$. This is because $V = kS$ when $S\uparrow\infty$.

The terminal condition is given by

$$V(S,T) = \max(F, kS)$$

where F is the face value of the bond.

<u>Remark</u> The Japanese convertible bonds commonly contain the $\boxed{\text{reset}}$ provision where the conversion number k may be reset upward when the share price is NOT performing.

Calling policy

- The convertible bond indenture usually contains the hard call provision where the bond cannot be called for redemption or conversion by the bond issuer in the early life of the bond. This serves as a protection for the bondholders so that the privilege of awaiting growth of the equity component will not be called away too soon.
- Let $[T_c, T]$ denote the callable period, that is, the bond cannot be called during the earlier part of the bond life $[0, T_c]$.
- Upon calling, the bondholders can decide whether to redeem the bond for cash or convert into shares at the end of the notice period of t_n days.

Notice period requirement

- Let $\widehat{t}$ denote the date of call so that $\widehat{t}+t_n$ is the conversion decision date for the bondholders.

- The bondholders essentially replace the original bond at time $\widehat{t}$ by a new derivative that expires at the future time $\widehat{t}+t_n$ and with terminal payoff $\max(nS, K+\widehat{c})$, where $\widehat{c}$ is the accrued interest from the last coupon date to the time instant $\widehat{t}+t_n$, and K is the pre-specified call price of the convertible bond.

- We write $V_{new}(S,t;K,t_n)$ as the value of this new derivative. When there is no soft call requirement (a constraint that is related to stock price movement over a short period prior to calling), the convertible bond value should be capped by V_{new}. The convertible bond should be called once its value reaches $V_{new}(S,t;K,t_n)$.

$$
\begin{aligned}
V(S,t) &\leq V_{new}(S,t;K,t_n) && \text{within the callable period,} \\
V(S,\hat{t}) &= V_{new}(S,t;K,t_n) && \text{at the calling moment.}
\end{aligned}
$$

- When there is a soft call requirement, it is possible that $V(S,t)$ stays above $V_{new}(S,t;K,t_n)$.

Coupon payments

By no arbitrage argument, there is a drop in bond value of amount that equals the coupon payment c_i across a coupon payment date $t_i, i = 1, 2, \cdots, N$. We have

$$V(S, t_i^+) = V(S, t_i^-) - c_i, \quad i = 1, 2, \cdots, N.$$

Remark

The interaction of the optimal conversion and calling policies determines the potential early termination of the convertible bond. The [illegible] of these two features can be treated effectively via dynamic programming procedure in the numerical schemes.

Pricing of risky convertible bonds

Incorporation of default risk, call and conversion features

- stock price process follows the binomial random walk
- interest rates are deterministic

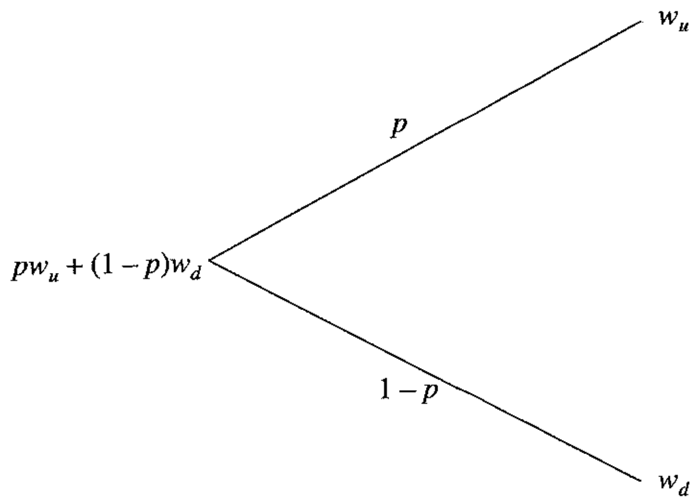
Two discount rates

1. If the convertible is certain to remain a bond, it is appropriate to use a discount rate corresponding to the creditworthiness of the issuer, namely, the risky rate.
2. Suppose the bond is certain to be converted, it is then appropriate to use the riskfree rate since the fair value of the equity part is priced under the risk neutral valuation principle.

The discount rate to be used when we roll back is given by

$$pw_u + (1-p)w_d.$$

Here, p is the probability to an upward node where the discount rate is w_u and $(1-p)$ is probability to a downward node with w_d. The appropriate discount rate is the weighted average of the discounted rates at the nodes in the next time step.



Remark Only the discount rate is taken care in this formulation. Does the potential default risk have any impact on the risk neutral drift rate?

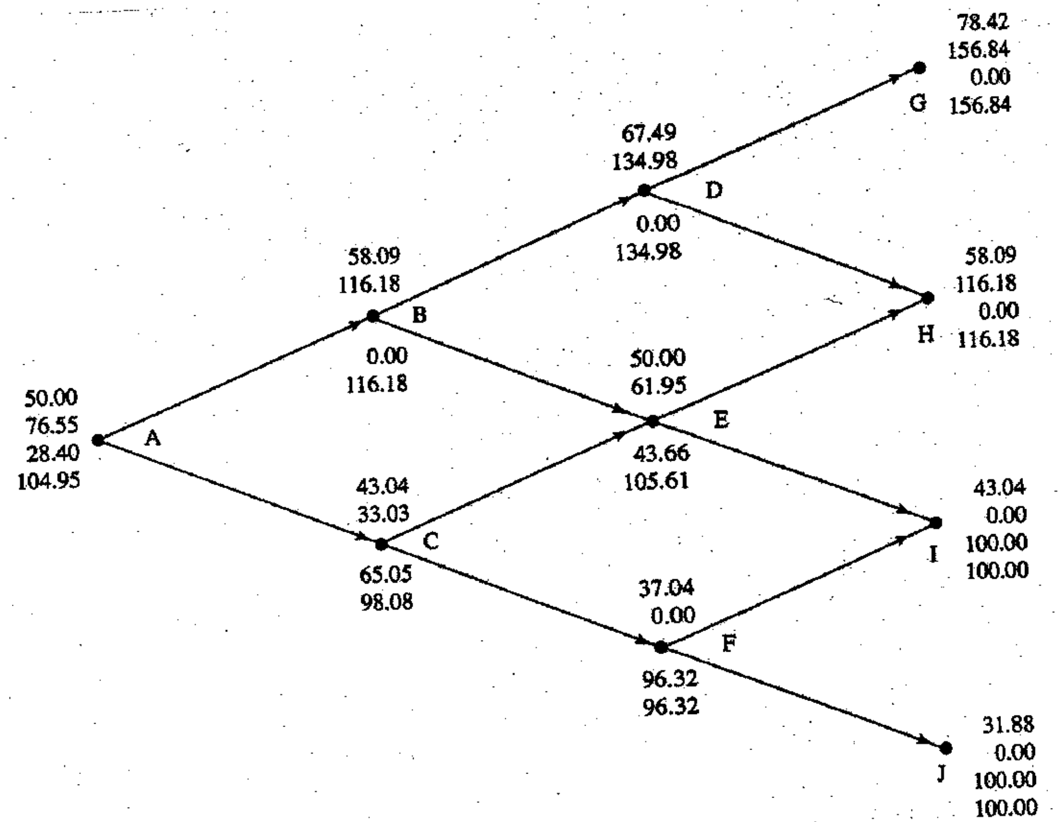
Example

A 9-month discount bond issued by XYZ company with a face value of \$100. Assume that it can be exchanged for 2 shares of company's stock at any time during the 9 months.

- It is callable for \$115 at any time.
- Initial stock price $= \$50, \sigma = 30\%$ per annum and no dividend; risk-free yield curve to be flat at 10% per annum.
- Yield curve corresponding to bonds issued by the company to be flat at 15%.
- Tree parameters are: $u = 1.1618, d = 0.8607, p = 0.5467$,

$$R = e^{0.1\Delta t} = 1.0253.$$

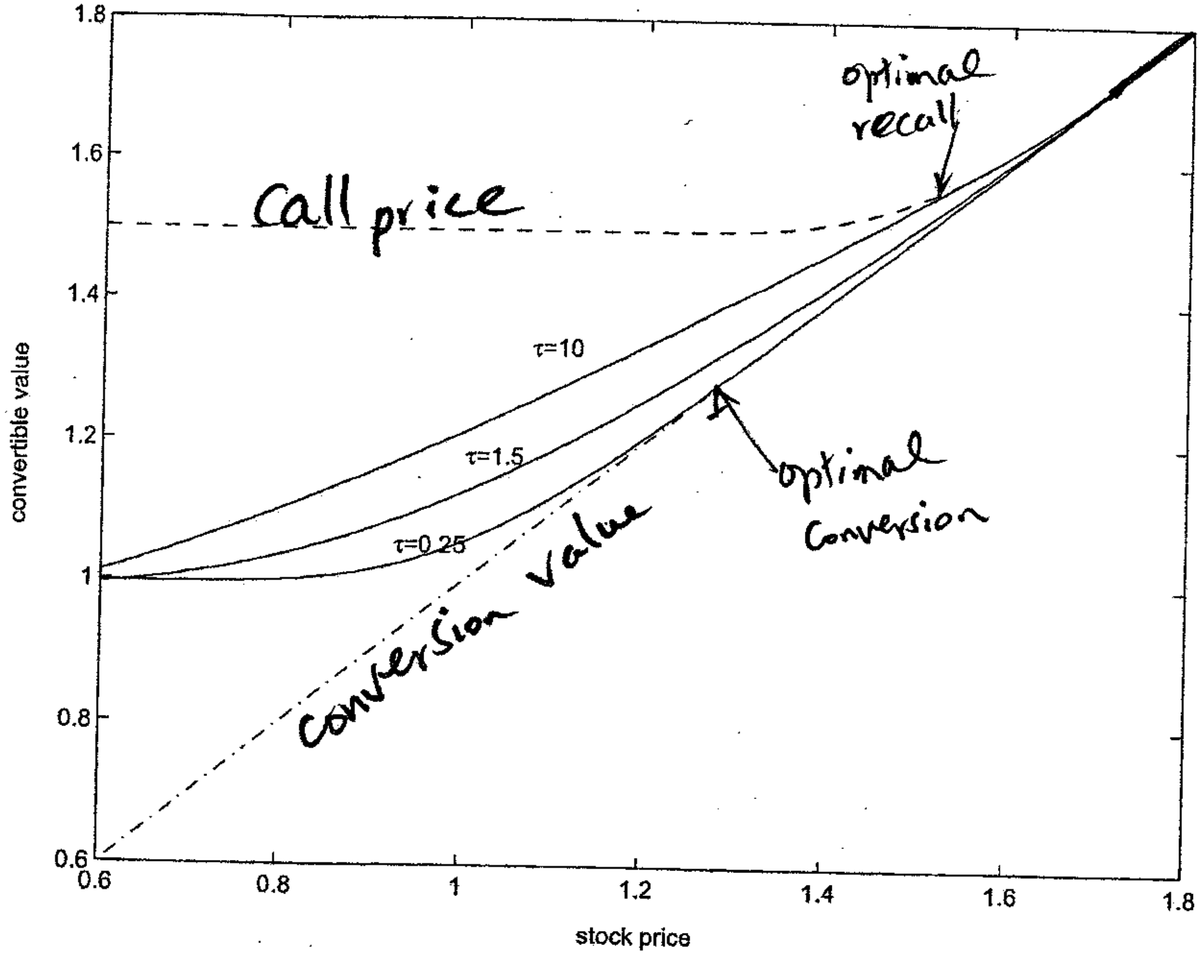
- At maturity, the convertible is worth $\max(100, 2S_T)$.



Tree for valuing convertible

Top figure: stock price; bottom figure: total bond value;

third figure: debt component; second figure: equity component



When $\tau = 0.25$ and $\tau = 10$, the bond price curves intersect the conversion value line (shown as dotted-dashed line) and the cap value curve $\widetilde{c}_n(S, \tau_n)$ (shown as dashed line), respectively. When $\tau = 1.5$, the price curve ends at the intersection point of the conversion value line and the cap value curve.

Interaction between conversion and calling

conv = value of stocks received if conversion takes place

call = call price

roll = value given by the rollback (neither converted nor recalled)

Six possible permutations on relative magnitude

(i) $conv < call$ (relevant case)

$$\begin{array}{ccccc} conv & < & call & < & roll \\ conv & < & roll & < & call \\ roll & < & conv & < & call \end{array}$$

(ii) $call < conv$ (irrelevant case)

$$\begin{array}{ccccc} call & < & conv & < & roll \\ call & < & roll & < & conv \\ roll & < & call & < & conv \end{array}$$

Premature conversion into shares is optimal only when the stock pays dividends at a yield higher than the coupon rate.

Dynamic programming procedure:

$$\max(\min(roll, call), conv).$$

At each node, the optimal strategy of the holder is exemplified by taking the maximum of $\min(roll, call)$ and $conv$.

- The maximum reflects the conversion right, which persists with or without recall by the issuer.
- We take the bond value before potential conversion to be $\min(roll, call)$ since the issuer would initiate calling when the roll value shoots beyond the call price.

Alternative dynamic programming procedure:

$$\text{min}(\text{max}(roll, conv), \text{max}(call, conv))$$

- $\text{max}(roll, conv)$ represents the optimal strategy of the holder.
- Upon recall, the holder chooses to accept the call price or convert into shares. This can be represented by $\text{max}(call, conv)$.

The issuer chooses to recall or to abstain from recalling in order to minimize the option value.

These two procedures cna be shown to be mathematically equivalent if we apply the distributive rule of sequencing the order on the "max" and "min" operations.

What would happen when $call < conv$?

- This occurs when the stock price level is sufficiently high, that is, the conversion option is sufficiently deep-in-the-money. Since the convertible bond value is always equal or above $conv$, so the issuer initiates calling immediately.

- Upon calling, the holder chooses to convert into stocks since $conv > call$.

- This represents a straightforward case since convertible value $= conv$ for sure, and there is no need to perform any calculations.

Provided that we rule out the scenario where $call < conv$, the third alternative is given by

$$\min(call, \max(conv, roll)).$$

Under the scenario $conv < call$, the holder has the optionality to convert but the convertible bond value is always capped by $call$.

The 3 alternative dynamic programming procedures give the same set of outcomes

	outcome
$conv < call < roll$	$call$
$conv < roll < call$	$roll$
$roll < conv < call$	$conv$

- We are interested to find the price function within the range of stock price such that

$$\text{call price} > \text{conversion value.}$$

- The convertible bond value lies between the "call price" (upper obstacle function) and conversion value (lower obstacle function).

- If call price $<$ conversion value, then the convertible bond value is simply given by the conversion value (straightforward case).

- The call price has "stock price" dependence if we include the consideration of the notice period requirement.

- Actually, the convertible bond will be either optimally converted by the holder or recalled by the issuer before the stock price can increase beyond the level where call price $<$ conversion value.

- In the binomial calculations, it suffices to limit the calculations to the range of stock price such that

$$\text{call price} > \text{conversion price.}$$

- Implicitly, we are interested to compute the price function whose value is capped by "call price". Hence, the dynamic programming procedure

$$\min(call, \max(conv, roll))$$

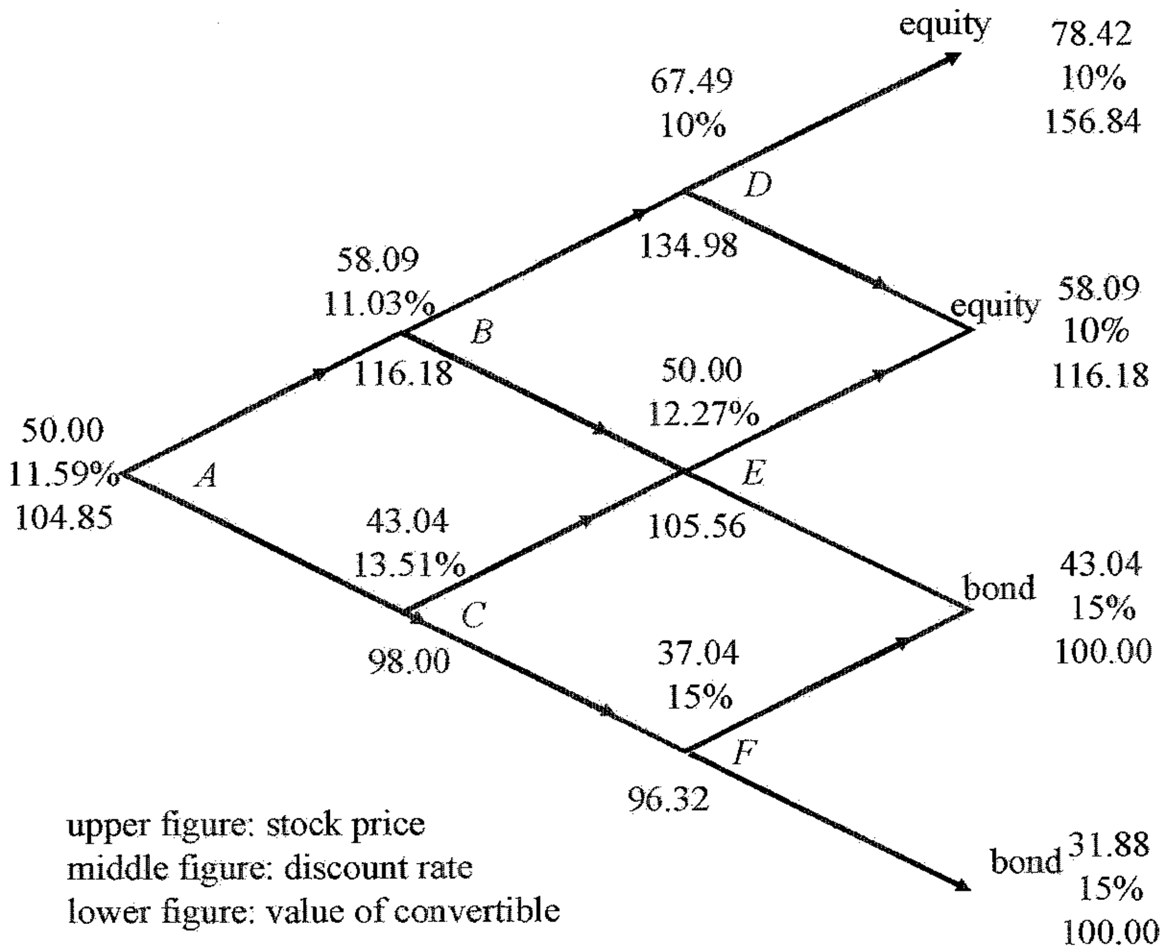
makes good sense.

Delayed call phenomenon

Convertible bonds are recalled by issuers only when the stock price is sufficiently well above the "theoretical" critical recall price. In this case, the holders are almost sure to make the "so called" forced conversion into shares. This is consistent with one of the corporate finance considerations – delayed equity financing.

- How to incorporate this behavior into the pricing model?
- What other factors that affect the determination of the optimal "recall price"?

Binomial tree for pricing a risky convertible bond



At node D

Roll back gives the convertible bond value

$$(0.5467 \times 156.84 + 0.4533 \times 116.18)e^{-0.1\times 0.25} = 134.98.$$

The bondholder is indifferent to conversion or hold. Upon call, the holder will choose to convert, so the issuer is also indifferent as to whether the bond is called. The correct discount rate at node D is 10% since the convertible is considered 100% equity at this node.

At node F

The correct discount rate is 15% since the convertible is certain not to be converted if node F is reached.

At node E

The discount rate is given by the weighted average:

$$0.5467 \times 10\% + 0.4533 \times 15\% = 12.27\%.$$

The roll back value of convertible at E

$$(0.5467 \times 116.18 + 0.4533 \times 100)e^{-0.1227 \times 0.25} = 105.56.$$

Since roll back value is higher than the conversion value (which is $2 \times 50 = 100$) and lower than the call price of 115, so the bond should be neither converted nor called. The discount rate is then kept at 12.27%.

At node B

The discount rate is given by the weighted average of those at nodes D and E:

$$0.5467 \times 10\% + 0.4533 \times 12.27\% = 11.03\%$$

and roll back value of convertible is

$$(0.5467 \times 134.99 + 0.4533 \times 105.56)e^{-0.1103 \times 0.25} = 118.34.$$

It is optimal to call the bond at node B so that it causes immediate conversion and lead to \$116.18.

Note that continuation is assumed when the roll back value is calculated, we use the weighted averaged discount rate in the preliminary calculation. However, the discount rate at node B should be taken to be 10% subsequently since optimal conversion into shares would take place at this node.

At node A

The discount rate is

$$0.5467 \times 10\% + 0.4533 \times 13.51\% = 11.59\%.$$

The convertible value at node A is given by the roll back value

$$(0.5467 \times 116.18 + 0.4533 \times 98.00)e^{-0.1159 \times 0.25} = 104.85,$$

since neither call or conversion would occur.

Value of the embedded option features

If the bond has no conversion option, its value is

$$e^{-0.75 \times 0.15} = 89.36.$$

The value of the conversion option minus the issuer's call right $= 104.85 - 89.36 = 15.49$.

Pricing formulation of a risky bond - motivation

- Simple coupon bearing bond having a non-zero default risk.
- Let the probability of default in t to $t+dt$, conditional on on-default in $[0,t]$ be $\lambda(S,t)\,dt$, where $\lambda(S,t)$ is a deterministic hazard rate.
- Let $B(S,t)$ denote the price of a risky convertible bond.

  Construct the standard hedging portfolio : $\Pi = B - \beta S$, with $\beta = \frac{\partial B}{\partial S}$.

  In the absence of default, the usual argument gives

$$d\Pi = \left[B_t + \frac{\sigma^2 S^2}{2} B_{SS}\right] dt + o(dt),$$

  where $o(dt)$ denotes terms that go to zero faster than dt.

  The probability of default in $t \to t+dt$ is $\lambda\,dt$.

Assumptions

- The value of the bond immediately after default is RX, where $0 \le R \le 1$ is the recovery factor. Here, X may be (i) face value, (ii) pre-default value.
- If we assume that S jumps to zero in the case of default, then

$$d\Pi = (1 - \lambda dt)\left[B_t + \frac{\sigma^2 S^2}{2} B_{SS}\right] dt - \lambda dt(B - RX - \beta S) + o(dt)$$
$$= \left[B_t + \frac{\sigma^2 S^2}{2} B_{SS}\right] dt - \lambda dt(B - RX - \beta S) + o(dt),$$

where $\beta = \frac{\partial B}{\partial S}$ as usual. Assuming default risk is fully diversifiable, $\boxed{E[d\Pi] = r(t)\Pi}$

so that

$$B_t + [r(t) + \lambda] S B_S + \frac{\sigma^2 S^2}{2} B_{SS} - [r(t) + \lambda] B + \lambda RX = 0. \qquad (A)$$

Note that λ appears in the drift term as well as in the discounting term. There is a source term λRX.

Hedge Model – Convertible bonds with credit risk

- For illustration on modeling stock price downward jump upon default, we assume that there are no put or call features and conversion is only allowed at maturity or upon default.

$$S^+ = S^-(1-\eta), \qquad 0 \le \eta \le 1, \quad \eta \text{ is the loss ratio.}$$

- Upon default, the holder receives RX, $0 \le R \le 1$, R is the recovery factor; or shares worth $\kappa S(1-\eta)$.

- The change in value of the hedging portfolio during $t \to t+dt$ is

$$d\Pi = (1-\lambda dt)\left[V_t + \frac{\sigma^2 S^2}{2} V_{SS}\right] dt - \lambda dt[V - \beta S\eta] + \lambda\, dt \max(\kappa S(1-\eta), RX) + o(dt)$$

$$= \left[V_t + \frac{\sigma^2 S^2}{2} V_{SS}\right] dt - \lambda dt[V - V_S S\eta] + \lambda\, dt \max(\kappa S(1-\eta), RX) + o(dt),$$

where $V = V(S,t)$ = value of convertible.

Using $E[d\Pi] = r(t)\,\Pi\,dt$, and taking the expectation of $d\Pi$ above, we obtain

$$r[V - SV_S]\,dt = \left[V_t + \frac{\sigma^2 S^2}{2} V_{SS}\right] dt - \lambda[V - V_S S\eta]\,dt + \lambda[\max(\kappa S(1-\eta), RX)]\,dt + o(dt).$$

this implies

$$V_t + [r(t) + \lambda\eta] SV_S + \frac{\sigma^2 S^2}{2} V_{SS} - [r(t) + \lambda]V + \lambda \max(\kappa S(1-\eta), RX) = 0. \qquad (B)$$

- $r(t) + \lambda\eta$ appears in the drift term and $r(t) + \lambda$ appears in the discounting term.
  - $\lambda\eta$ appears together as the adjusted amount of the risk neutral drift rate of S. This can be considered as the negative dividend yield.
  - If the residual value upon default is taken to be V [taking $X = V$ and $\eta = 1$ in $\max(\kappa S(1-\eta), RX)$], then the adjusted risky rate is $r(t) + (1-R)\lambda$.

- Defining $\mathcal{M}V \equiv -V_t - \left\{\frac{\sigma^2}{2}S^2 V_{SS} + [r(t)+\lambda\eta - q]SV_S - [r(t)+\lambda]V\right\}$,

then Eq (B) becomes

$$\mathcal{M}V - \max(\kappa S(1-\eta), RX) = 0.$$

The new linear complementarity formulation becomes

- $B_c > kS$

$$\begin{pmatrix} \mathcal{M}V - \lambda\max(\kappa S(1-\eta), RX) = 0 \\ (V - \max(B_p, \kappa S)) \geq 0 \\ (V - B_c) \leq 0 \end{pmatrix}$$
$$\vee \begin{pmatrix} \mathcal{M}V - \lambda\max(\kappa S(1-\eta), RX) \geq 0 \\ (V - \max(B_p, \kappa S)) = 0 \\ (V - B_c) \leq 0 \end{pmatrix}$$
$$\vee \begin{pmatrix} \mathcal{M}V - \lambda\max(\kappa S(1-\eta), RX) \leq 0 \\ (V - \max(B_p, \kappa S) \geq 0 \\ (V - B_c) = 0 \end{pmatrix} \qquad (\mathrm{I})$$

- $B_c \leq kS$

$$V = \kappa S. \qquad (\mathrm{II})$$

In summary, the value of the convertible is given by

$$\mathcal{M}V - \lambda \max(\kappa S(1-\eta), RX) = 0,$$

Subject to the constraints:

$$V \geq \max(B_p, \kappa S)$$
$$V \leq \max(B_c, \kappa S).$$

If X represents the bond component of the convertible (whose value should be affected by put/call provisions), then we need to solve another equation for the bond component B, which must be coupled to the total value V.

PDE formulation of the "two discount rates" approach

Equity component E is discounted at riskfree rate r;

bond component B is discounted at a risky rate $r+s$, s = spread.

$$\text{Convetible value} = V = B + E$$

Bond component satisfies

$$B_t + rS\,B_S + \frac{\sigma^2 S^2}{2} B_{SS} - (r+s)B = 0.$$

We may recover eq.(A) if we assume $X = B$, S does not jump upon default and treat $s = \lambda(1-R)$ [see P. 33].

The convetible value satisfies

$$V_t + \frac{\sigma^2}{2} S^2 V_{SS} + (r-q)SV_S - r(V-B) - (r+s)B = 0$$

Subject to the constraints: $V \geq \max(B_p, kS)$ & $V \leq \max(B_c, kS)$.

Summary

The convertible pricing equation is developed by the steps:

- usual hedging portfolio is constructed
- Poisson default process is specified
- Special assumptions are made about the behavior of the stock price on default, and recovery after default

No ad-hoc decisions are required about which part of the convertible is discounted at the risky rate, and which part is discounted at the risk free rate.

— results in a coupled system of linear complementarity problems.

Example of total default

The stock price jumps to zero upon default. The equity component of the convertible bond is zero. A fraction of the bond component is recovered.

- Assume no put provision, taking $B_p = -\infty$
- $V = E + B$, $X = B$ and $B = F$ initially; in this case, B is simply the value of a risky debt with face value F:

$$B = F \exp\left(-\int_t^T [r(u) + \lambda(u)(1-R)]\, du\right),$$

independent of S.

Governing equation for the value of the equity component E

$$\mathcal{M} E - \lambda \max(kS(1-\eta) - RB, 0) = 0$$

Subject to the constraints

$$E + B \leq \max(B_c, kS)$$

$$E + B \geq kS.$$

Recall

$$V(S,T) = \max(F, kS) = F + \max(kS - F, 0)$$

so that

$$E(S,T) = \max(kS - F, 0)$$

$$B(S,T) = F.$$