Credit spread derivatives

- Options, forwards and swaps that are linked to credit spread.

  Credit spread = yield of debt − risk-free or reference yield

- Investors gain protection from any degree of credit deterioration resulting from ratings downgrade, poor earnings etc.

  (This is unlike default swaps which provide protection against defaults and other clearly defined 'credit events'.)

- To isolate the reference on credit spreads, some credit spread derivatives are even knocked out if a default happens on the reference asset.
- The reference asset must be a liquidly traded bond to allow a meaningful observation of the credit spread.

Credit spread forward

- One party pays at time T a pre-agreed fixed payment and receives the credit spread of the reference asset at time T
- Can be structured around the relative credit spread between two different defaultable bonds.
- Several credit spread forwards can be combined to a credit spread swap.
- Similar to a forward rate agreement (spread instead of an interest rate).
- Make a position in the reference asset neutral against credit spread movements.

Credit spread options

A credit spread put gives counterparty A the right to sell the reference asset (referenced bond or asset swap) to counterparty B at a pre-specified strike spread over default free interest rates — an exchange option that gives A the right to exchange one defaultable bond for a certain number (< 1) of default free bonds.

- Contingent event: credit spread of the reference asset is above the strike spread at maturity
- Contingent payment: price that the reference asset at strike spread minus the market price of the reference asset

Applications of credit spread options

- Funds are restricted by their investment mandate. Allow the funds a switch out of the defaultable bond investment if the credit quality of the reference asset deteriorates. The contingent payment can be an exchange of the reference asset against an index of investment grade debt.

- Exposure management of committed lines of credit (A committed line of credit is similar to an overdraft on a bank account.)

By paying a regular fee to the committed bank A, the debtor C has the right to enter a pre-agreed loan contract at any time he chooses.

The bank A hedges with a credit spread option (American style) which enables bank A to put the loan to B as soon as the line of credit is drawn.

This type of option is also called a synthetic lending facility.

- target the future purchase of assets at favorable prices.

The holder of the put spread option has the right to sell the bond at the strike spread (say, spread = 330 bps) when the spread moves above the strike spread (corresponding to a drop of the bond price).

The investor intends to purchase the bond below current market price (300 bps above US Treasury) in the next year and has targeted a forward purchase price corresponding to a spread of 350 bps. She sells for 20 bps a one-year credit spread put struck at 330 bps to a counterparty who is currently holding the bond and would like to protect the market price against spread above 330 bps.

- spread < 330; investor earns the option premium
- spread > 330; investor acquires the bond at 350 bps

Put option on a defaultable bond

- Right to sell the defaultable bond at the strike price
- Offers protection against

  (i) rising default free interest rate

  (ii) rising credit spread

  (iii) default

$$\text{Payoff at } T_1 = \left(\underset{\uparrow \atop \text{strike}}{K} - \underset{\uparrow \atop T_2\text{-maturity defaultable bond}}{\bar{B}(T_1,T_2)}\right)^+$$

If the put is knocked out at default, then

$$\text{payoff} = \mathbb{1}_{\{\underset{\uparrow \atop \text{random default time}}{\tau} > T_1\}} \left(K - \bar{B}(T_1,T_2)\right)^+$$

Survival probability and density of default time τ

The process $P(t,T)$

$$P(t,T) = \mathbf{E}\left[e^{-\int_t^T \lambda(s)ds} \,\middle|\, \mathcal{F}_t \right].$$

as the *survival probability* from time t until time T.

In general

$$\mathbf{1}_{\{\tau>t\}} P(t,T) = \mathbf{E}\left[\mathbf{1}_{\{\tau>T\}} \,\middle|\, \mathcal{F}_t \right].$$

For $\tau > t$ (survival up to time t),

the density of the time of the first default as seen from t is for $T > t$

$$p(t,T) = \mathbf{E}\left[\lambda(T) \exp\{-\int_t^T \lambda(s)ds\} \,\middle|\, \mathcal{F}_t \right].$$

<u>Assumption</u> (equivalent recovery model)

At the time of default τ, one defaulttable bond $\overline{B}(\tau,T)$ has the payoff of c default free bonds $B(\tau,T)$ (as recovery value).

With constant c and $\tau > t$, then

$$\begin{aligned}
\overline{B}(t,T) &= \mathbf{E}\left[\, \beta_{t,T}\mathbf{1}_{\{\tau>T\}} + c\beta_{t,\tau}B(\tau,T)\mathbf{1}_{\{\tau\le T\}} \;\middle|\; \mathcal{F}_t \,\right] \\
&= \mathbf{E}\left[\, \beta_{t,T}\mathbf{1}_{\{\tau>T\}} \;\middle|\; \mathcal{F}_t \,\right] + c\mathbf{E}\left[\, \beta_{t,T} \;\middle|\; \mathcal{F}_t \,\right] - c\mathbf{E}\left[\, \beta_{t,T}\mathbf{1}_{\{\tau>T\}} \;\middle|\; \mathcal{F}_t \,\right] \\
&= (1-c)\overline{B}_0(t,T) + cB(t,T),
\end{aligned}$$

$\beta_{t,T}$ is the discount factor over $(t,T]$, and $\overline{B}_0(t,T)$ is the price of a defaultable bond under zero recovery:

$$\mathbf{E}\left[\, \beta_{t,T}\mathbf{1}_{\{\tau>T\}} \;\middle|\; \mathcal{F}_t \,\right] = \mathbf{1}_{\{\tau>t\}}\mathbf{E}\left[\, e^{-\int_t^T r(s)+\lambda(s)ds} \;\middle|\; \mathcal{F}_t \,\right].$$

Assumptions

The following data is required for all maturities, $T > 0$.

- default free term structure of bond prices

$$B(0,T) = E\left[e^{-\int_0^T r(s)\,ds}\right]$$

- defaultable bond prices

$$\bar{B}(0,T)$$

- defaultable bond prices under zero recovery

$$\bar{B}_0(0,T) = E\left[e^{-\int_0^T [r(s)+\lambda(s)]\,ds}\right]$$

The term structures of interest rates and credit spreads are independent. This independence gives

$$P(0,T) = E\left[e^{-\int_0^T \lambda(s)\,ds}\right] = \frac{\bar{B}_0(0,T)}{B(0,T)}.$$

European default digital put

- pays off \$1 at T iff there has been a default by T.

$$
\begin{aligned}
D =& \mathbf{E}\left[e^{-\int_0^T r(s)ds} \mathbf{1}_{\{\tau<T\}} \right] \\
=& \mathbf{E}\left[\mathbf{E}\left[e^{-\int_0^T r(s)ds} \mathbf{1}_{\{\tau<T\}} \,\middle|\, \{\lambda(s)\}_{s\leq T} \right] \right] \\
=& \mathbf{E}\left[e^{-\int_0^T r(s)ds} (1 - e^{-\int_0^T \lambda(s)ds}) \right] \\
=& B(0,T) - \mathbf{E}\left[e^{-\int_0^T r(s)+\lambda(s)ds} \right] \\
=& B(0,T) - \overline{B}_0(0,T).
\end{aligned}
$$

This is obvious since this digital put replicates the payoff of a portfolio that is long one default free bond and short one zero recovery defaultable bond.

Payoff at Default

In the case when the payoff takes place *at* default, the expectation to calcula

$$D = \mathbf{E}\left[e^{-\int_0^\tau r(s)ds} \mathbf{1}_{\{\tau<T\}} \right].$$

Conditioning on the realization of λ yields

$$D = \mathbf{E}\left[e^{-\int_0^\tau r(s)ds} \mathbf{1}_{\{\tau<T\}} \right] = \mathbf{E}\left[\mathbf{E}\left[e^{-\int_0^\tau r(s)ds} \mathbf{1}_{\{\tau<T\}} \,\middle|\, \lambda \right] \right].$$

Now note that the probability distribution of τ (given λ) is

$$\mathbf{P}[\, \tau \leq T \,] = 1 - \mathbf{P}[\, \tau > T \,] = 1 - \exp\{-\int_0^T \lambda(s)ds\},$$

so the density of τ is

$$\lambda(t)\exp\{-\int_0^t \lambda(s)ds\}.$$

The price of the default digital put

$$D = \mathbf{E}\Big[\,\mathbf{E}\Big[\,e^{-\int_0^T r(s)ds}\mathbf{1}_{\{\tau<T\}}\,\Big|\,\lambda\Big]\Big]$$

$$= \mathbf{E}\Big[\int_0^T \lambda(t)e^{-\int_0^t \lambda(s)ds}e^{-\int_0^t r(s)ds}dt\Big]$$

$$= \int_0^T \mathbf{E}\Big[\,\lambda(t)e^{-\int_0^t \lambda(s)ds}e^{-\int_0^t r(s)ds}\,\Big]dt.$$ [Interchanging order of E and ∫]

Under independence of r and λ and observing

$$\mathbf{E}\Big[\,\lambda(t)e^{-\int_0^t \lambda(s)ds}\,\Big] = P(0,t)\mathbf{E}^{P_t}[\,\lambda(t)\mid \mathcal{F}_t] = P(0,t)h(0,t),$$

we have

$$\mathbf{E}\Big[\,\lambda(t)e^{-\int_0^t \lambda(s)ds}e^{-\int_0^t r(s)ds}\,\Big] = B(0,t)P(0,t)h(0,t) = \overline{B}_0(0,t)h(0,t).$$

$P(0,t) = \overline{B}_0(0,t) / B(0,t)$ = survival probability $= E\left[e^{-\int_0^t \lambda(s)\,ds} \right]$

$h(0,t)$ = associated "forward rate" of the spreads of the zero recovery bond

$$= -\frac{\partial}{\partial T} \ln P(0,T) \Big|_{T=t}.$$

Analogy.
$$\mathbf{E}\left[r(t) e^{-\int_0^t r(s)ds} \right] = B(0,t) f(0,t).$$

Under the equivalent recovery model

$$\overline{B}_0(t,T) = \tfrac{1}{1-c}(\overline{B}(t,T) - cB(t,T)),$$

so that under independence assumption

$$h(t,T) = -\frac{(1-c)\,B(t,T)}{\bar{B}(t,T) - c\,B(t,T)} \frac{\partial}{\partial T}\left(\frac{\bar{B}(t,T)}{B(t,T)}\right).$$

Alternatively,

$$\bar{B}_0(t,T)\, h(t,T) = \bar{B}(t,T)\left[\bar{f}(t,T) - f(t,T)\right],$$

where $\bar{f}(t,T) = -\frac{\partial}{\partial T} \ln \bar{B}(t,T)$

= defaultable forward rate based on defaultable bond prices with positive recovery

Summing together, assuming that r and λ are independent and the recovery rate is at constant value c, then

$$\text{price of a default digital put} = \int_0^T \bar{B}(0,t)\left[\bar{f}(0,t) - f(0,t)\right] dt.$$

Without independence assumption, then one requires joint dynamics:

Two-factor Gaussian model

$$\begin{aligned} dr(t) &= [k(t) - ar]\,dt + \sigma(t)\,dZ(t) \\ d\lambda(t) &= [\overline{k}(t) - \overline{a}\lambda]\,dt + \overline{\sigma}(t)\,d\overline{Z}(t) \end{aligned}$$

with $dZ\,d\overline{Z} = \rho\,dt$. For default-free bonds, the discount bond price is

$$\frac{dB(t,T)}{B(t,T)} = r(t)\,dt - \frac{\sigma(t)}{a}\left[1 - e^{-a(T-t)}\right]\,dZ(t).$$

Recall that

$$B(t,T) = E\left[e^{-\int_t^T r(s)\,ds}|\mathcal{F}_t\right] = e^{\widehat{A}(t,T)-r(t)\widehat{B}(t,T)}$$

where

$$\widehat{B}(t,T;a) = \frac{1}{a}\left[1 - e^{-a(T-t)}\right]$$

$$\widehat{A}(t,T;a,k,\sigma) = \frac{1}{2}\int_t^T \sigma^2(s)\widehat{B}(t,s;a)^2\,ds - \int_t^T \widehat{B}(t,s;a)k(s)\,ds.$$

Recall

$$\overline{B}_0(t,T) = E\left[e^{-\int_t^T [r(s)+\lambda(s)]\,ds}\right] = B(t,T)E_{Q_T}\left[e^{-\int_t^T \lambda(s)\,ds}\right].$$

The dynamics of default intensity under the T-forward measure is

$$d\lambda(t) = [\widetilde{k}(t) - \overline{a}\lambda]\,dt + \overline{\sigma}(t)\,d\overline{Z}_T(t)$$

where

$$\widetilde{k}(t) = \overline{k}(t) - \rho\overline{\sigma}(t)\sigma(t)\widehat{B}(t,T).$$

To price the defaultable claim of paying \$1 at T if a default happens at T,

$$e(t,T) = E_t\left[\lambda(T)e^{-\int_t^T [r(s)+\lambda(s)]\,ds}\right].$$

We use $\overline{B}_0(t,T)$ as the numeraire so that

$$e(t,T) = \overline{B}_0(t,T)E^{\overline{B}_0}[\lambda(T)]$$

and the dynamics under the new measure

$$\begin{aligned} d\lambda(t) &= [\overline{k}(t) - \sigma(t)\overline{\sigma}(t)\rho\widehat{B}(t,T;a) - \overline{\sigma}^2(t)\widehat{B}(t,T;\overline{a}) - \overline{a}(t)\lambda(t)]\,dt \\ &\quad + \overline{\sigma}(t)\,dZ^{\overline{B}_0}(t). \end{aligned}$$

The evaluation of the expectation gives

$$e(t,T) = \overline{B}_0(t,T)\left[\lambda(t)e^{-\overline{a}(T-t)} + \int_t^T e^{-a(T-s)}\widetilde{k}'(s)\,ds\right]$$

where

$$\widetilde{k}'(t) = \overline{k}(t) - \rho\overline{\sigma}(t)\sigma(t)\widehat{B}(t,T;a) - \overline{\sigma}^2(t)\widehat{B}(t,T;\overline{a}).$$

Default digital put with maturity T

Payoff of \$1 at default, if default occurs before T. Its price is

$$\int_0^T e(0,t)\,dt = \int_0^T E\left[\lambda(t)e^{-\int_0^t \lambda(s)\,ds}e^{-\int_0^t r(s)\,ds}\right]dt.$$

Credit spread forwards

- Adoption of the classical forward rate agreement to credit spreads.
- Payoff function is

$$\underbrace{\frac{1/\overline{B}(T_1,T_2)-1}{T_2-T_1}}_{\substack{\text{simply compounded} \\ \text{defaultable rate} \\ \text{based on } \overline{B}(t,T_1) \\ \& \ \overline{B}(t,T_2)}} - \underbrace{\frac{1/B(T_1,T_2)-1}{T_2-T_1}}_{\substack{\text{classical FRA} \\ \text{can be replicated} \\ \text{by } B(t,T_1)\ \&\ B(t,T_2)}} - \bar{s} \qquad \text{at } T_2.$$

This is equivalent to

$$\frac{1}{T_2-T_1}\left(\frac{B(T_1,T_2)}{\overline{B}(T_1,T_2)}-1\right)-\bar{s}B(T_1,T_2) \qquad \text{at } T_1.$$

Financial interpretation:

The payoff $\frac{1}{\overline{B}(T_1,T_2)}$ at T_2 is unaffected by defaults in $[T_1,T_2]$,

thus one needs to invest $\frac{1}{\overline{B}(T_1,T_2)}$ default-free bonds $B(T_1,T_2)$

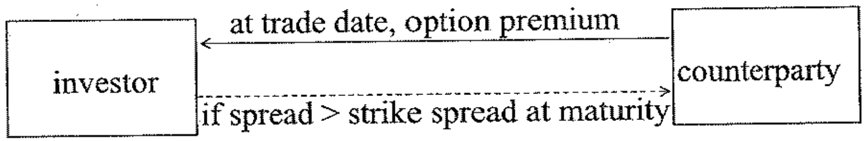
at T_1 to replicate $\frac{1}{\overline{B}(T_1,T_2)}$ at T_2.

The key term for the pricing of the credit spread forward is

$$\mathbf{E}\left[\,\beta_{T_1}\frac{B(T_1,T_2)}{\overline{B}(T_1,T_2)}\,\right]=\mathbf{E}\left[\,\beta_{T_1}\frac{B(T_1,T_2)}{\overline{B}(T_1,T_2)}\mathbf{1}_{\{\tau>T_1\}}+\beta_{T_1}\frac{1}{c}\mathbf{1}_{\{\tau\le T_1\}}\,\right],$$

under the equivalent recovery model.

Credit spread options



$$\text{Payout} = \text{notional} \times (\text{final spread} - \text{strike spread})^{+}$$

The terminal payoff is given by

$$P_{sp}(r, s, T) = \max(s - K, 0)$$

where $r =$ riskless interest rate

$s =$ credit spread

$K =$ strike spread

Discrete models

- Follows the HJM term structure approach that models the forward rate process and forward spread process for riskless and risky bonds.

- The model takes the observed term structures of riskfree forward rates and credit spreads as input information.

- Find the risk neutral drifts of the stochastic processes such that all discounted security prices are martingales.

Example Price a one-year put spread option on a two-year risky zero-coupon bond struck at the strike spread $K = 0.01$.

Let the current observed term structure of riskless interest rates as obtained from the spot rate curve for Treasury bonds be

$$\boldsymbol{r} = \begin{pmatrix} 0.07 \\ 0.08 \end{pmatrix}.$$

The riskless forward rate between year one and year two is

$$f_{12} = \frac{1.08^2}{1.07} - 1 \approx 0.09.$$

The market one-year and two-year spot spreads are

$$\boldsymbol{s} = \begin{pmatrix} 0.010 \\ 0.012 \end{pmatrix}.$$

The two-year risky rate is $0.08 + 0.012 = 0.092$. The current price of a risky two-year zero coupon bond with face value \$100 is

$$\overline{B}(0) = \$100/(1.092)^2 = \$83.86.$$

- The discrete stochastic process for the spread under the true measure is assumed to take the form of a square-root process where the volatility depends on $\sqrt{s(0)}$

$$s(\Delta t) = s(0) + k[\theta - s(0)]\Delta t \pm \sigma\sqrt{s(0)\Delta t}$$

where $k = 0.3, \theta = 0.02$ and $\sigma = 0.04, \Delta t = 1, s(0) = 0.01$.

- We need to add an adjustment term γ in the drift term in order to risk-adjust the stochastic forward spread process

$$s(t) = s(0) + k[\theta - s(0)]\Delta t + \gamma \pm \sigma\sqrt{s(0)\Delta t}.$$

The adjustment term γ is determined by requiring the discounted bond prices to be martingales.

- Let $\overline{B}(1)$ denote the price at $t = 1$ of the risky bond maturing at $t = 2$. The forward defaultable discount factor over year one and year two is $\dfrac{1}{1 + f_{12} + s(1)}$, where $s(1)$ is the forward spread over the period.

$$s(1) = \begin{cases} \gamma + 0.017 \\ \gamma + 0.009 \end{cases} \text{ so that } \overline{B}(1) = \begin{cases} \frac{100}{1+f_{12}+\gamma+0.017} \\ \\ \frac{100}{1+f_{12}+\gamma+0.009}, \end{cases}$$

with equal probabilities for assuming the high and low values.

We determine γ such that the bond price is a martingale.

$$\overline{B}(0) = 83.86 = \frac{1}{1 + 0.07 + 0.01} \times \frac{1}{2}\left(\frac{100}{1.107 + \gamma} + \frac{100}{1.099 + \gamma}\right), \ f_{12} = 0.09.$$

The first term is the risky defaultable discount factor and the last term is the expected value of $\overline{B}(1)$. We obtain $\gamma = 0.0012$ so that

$$s(1) = \begin{cases} 0.0182 \\ 0.0102 \end{cases}.$$

The current value of put spread option is

$$\frac{1}{1.07} \times \frac{1}{2}[(0.0182 - 0.01) + (0.0102 - 0.01)]L = 0.00393L,$$

where L is the notional value of the put spread option. Note that the default free discount factor $1/1.07$ is used in the option value calculation.

Continuous model

Under the risk neutral measure Q, the stochastic processes followed by the short rate r and the short spread rate s are

$$dr = [\phi(t) - \alpha r]\, dt + \sigma_r\, dZ_r$$

$$ds = [\theta(t) - \beta s]\, dt + \sigma_s\, dZ_s,$$

where $\phi(t)$ and $\theta(t)$ are time dependent parameter functions, α and β are mean reversion parameters, $dZ_r\, dZ_s = \rho\, dt$. The sum $r(t)+s(t)$ is considered as the risky short rate.

- "No-arbitrage" refers to the determination of the time dependent drift terms in the mean reversion stochastic processes of r and s by fitting the current term structures of default free and defaultable bond prices. This is a *calibration* procedure.

$$\begin{aligned} B(t,T) &= E_Q\left[\exp\left(-\int_t^T r(u)\, du\right)\right] \\ \overline{B}(t,T) &= E_Q\left[\exp\left(-\int_t^T [r(u)+s(u)]\, du\right)\right]. \end{aligned}$$

$B(r,t;T)$ admits solution of the form $a(r,t;T)e^{-rb(r,t;T)}$. The process followed by $B(t,T)$ is given by

$$\frac{dB}{B} = r\,dt - \sigma_B(t,T)\,dZ_r$$

where

$$\sigma_B(t,T) = \frac{\sigma_r}{\alpha}[1 - e^{-\alpha(T-t)}].$$

In terms of the forward measure Q^T where $B(t,T)$ is used as the numeraire, the defaultable bond price is

$$\overline{B}(t,T) = B(t,T)E_{Q^T}\left[\exp\left(-\int_t^T s(u)\,du\right)\right].$$

In general, $\dfrac{V(t)}{B(t,T)} = E_{Q^T}\left[\dfrac{V(T)}{B(T,T)}\right]$ so that $V(t) = B(t,T)E_{Q^T}[V(T)]$.

The process followed by the credit spread s under Q^T is given by

$$ds = [\theta(t) - \beta s + \rho\sigma_s\sigma_B(t,T)]\,dt + \sigma_s\,dZ^T$$

where Z^T is the standard Wiener process under Q^T.

The stochastic quantity $\int_t^T s(u)\,du$ has

$$\begin{aligned}
\text{mean} \;= E_{Q^T}\left[\int_t^T s(u)\,du\right] &= \frac{s(t)}{\beta}[1 - e^{-\beta(T-t)}] \\
&\quad + \int_t^T [\theta(u) + \rho\sigma_s\sigma_B(u,T)]\frac{1 - e^{-\beta(T-u)}}{\beta}\,du \\
\text{variance} \;= \text{var}\left(\int_t^T s(u)\,du\right) &= \int_t^T \frac{\sigma_s^2}{\beta^2}\left[1 - e^{-\beta(T-u)}\right]^2\,du.
\end{aligned}$$

Hence,

$$\begin{aligned}\frac{\overline{B}(t,T)}{B(t,T)} &= E_{Q^T}\left[\exp\left(-\int_t^T s(u)\,du\right)\right]\\ &= \exp\left(-\frac{s(t)\sigma_s}{\beta}[1-e^{-\beta(T-t)}]\right.\\ &\quad -\int_t^T [\theta(u)+\rho\sigma_s\sigma_B(u,T)]\frac{1-e^{-\beta(T-u)}}{\beta}\,du\\ &\quad \left.+\int_t^T \frac{\sigma_s^2}{2\beta^2}[1-e^{-\beta(T-u)}]^2\,du\right).\end{aligned}$$

Here, we make use of the formula:

$X(t)$ is a Brownian process with drift μ and variance σ^2.

$$Y(t) = e^{-X(t)}, \quad t \geq 0.$$

$$E[Y(t) \mid Y(0) = y_0] = y_0 \exp\left(-\mu t + \frac{\sigma^2 t}{2}\right).$$

By solving the integral equation

$$\theta(T) = \frac{\sigma_s^2}{2\beta}[1 - e^{-2\beta(T-t)}] - \beta\frac{\partial}{\partial T}\left[\ln \frac{\overline{B}(t,T)}{B(t,T)}\right] - \frac{\partial^2}{\partial T^2}\left[\ln \frac{\overline{B}(t,T)}{B(t,T)}\right]$$
$$+ \rho\sigma_s\sigma_r\left[\frac{1 - e^{-\alpha(T-t)}}{\alpha} + e^{-\alpha(T-t)}\frac{1 - e^{-\beta(T-t)}}{\beta}\right].$$

Trick to solve for $\theta(T)$

Differentiate the integral involving $\theta(u)$ with respect to T and subtract those terms involving $\int_t^T \theta(u)e^{-\beta(T-u)}\,du$ so as to obtain an explicit expression for $\int_t^T \theta(u)\,du$.

Pricing of the credit spread option

$$p_{sp}(r,s,t) = B(t,T)E_{Q^T}[\max(s-K,0)].$$

Note the presence of the mean reversion term $-\beta s$ in the drift term. Define

$$\xi(t) = e^{-\beta(T-t)}s(t)$$

so that

$$d\xi = [\theta(t) + \rho\sigma_s\sigma_B(t,T)]e^{-\beta(T-t)}\,dt + \sigma_s e^{-\beta(T-t)}\,dZ^T.$$

Now, ξ_T is normally distributed with mean μ_ξ and σ_ξ^2, where

$$\begin{aligned}\mu_\xi &= s(t)e^{-\beta(T-t)} + \frac{\rho\sigma_s\sigma_r}{\alpha}\left[\frac{1-e^{-(\alpha+\beta)(T-t)}}{\alpha+\beta} - \frac{1-e^{-\beta(T-t)}}{\beta}\right] \\ &\quad + \int_t^T \theta(u)e^{-\beta(T-u)}\,du\end{aligned}$$

and

$$\sigma_\xi^2 = \frac{\sigma_s^2}{2\beta}[1-e^{2\beta(T-t)}].$$

Spread option value

$$p_{sp}(r,s,t) = B(r,t)\left[\frac{\sigma_\xi}{\sqrt{2\pi}}e^{-(K-\mu_\xi)^2/2\sigma_s^2} + (\mu_\xi - K)N\left(\frac{\mu_\xi - K}{\sigma_\xi}\right)\right].$$

Appendix: Relevant mathematical results

1. Expectation of r_T under Q_T is the forward rate

$$
\begin{aligned}
B(t,T)E^t_{Q_T}[r_T] &= E^t_Q\left[e^{-\int_t^T r_u\,du} r_T\right] \\
&= E^t_Q\left[-\frac{\partial}{\partial T} e^{-\int_t^T r_u\,du}\right] \\
&= -\frac{\partial}{\partial T}\left\{E^t_Q\left[e^{-\int_t^T r_u\,du}\right]\right\} \\
&= -\frac{\partial B}{\partial T}(t,T),
\end{aligned}
$$

$$
E^t_{Q_T}[r_T] = -\frac{1}{B(t,T)}\frac{\partial B}{\partial T}(t,T) = F(t,T).
$$

Change of Measure from Q to Q_T

We would like to illustrate how to effect the change of measure from the risk neutral measure Q to the T-forward measure Q_T. Let the dynamics of the T-maturity discount bond price under Q be governed by

$$\frac{dB(t,T)}{B(t,T)} = r(t)\,dt - \sigma_B(t,T)\,dZ(t), \tag{8.1.4}$$

where $Z(t)$ is Q-Brownian. By integrating the above equation and observing $\frac{M(t)}{M(0)} = \int_0^t r(u)\,du$, we obtain

$$\frac{B(t,T)}{M(t)} = \frac{B(0,T)}{M(0)} \exp\left(-\int_0^t \sigma_B(u,T)\,dZ(u) - \frac{1}{2}\int_0^t \sigma_B(u,T)^2\,du\right).$$

The Radon–Nikodym derivative $\frac{dQ_T}{dQ}$ conditional on $\mathcal{F}_T$ is found to be [see (3.2.4)]

$$\begin{aligned}\frac{dQ_T}{dQ} &= \frac{B(T,T)}{B(0,T)} \Big/ \frac{M(T)}{M(0)} \\ &= \exp\left(-\int_0^T \sigma_B(u,T)\,dZ(u) - \frac{1}{2}\int_0^T \sigma_B(u,T)^2\,du\right). \end{aligned} \tag{8.1.5}$$

For a fixed T, we define the process

$$\xi_t^T = E_Q\left[\frac{dQ_T}{dQ}\middle|\mathcal{F}_t\right] \tag{8.1.6}$$

and since $M(0) = 1$ and $B(0, T)$ is known at time t, we obtain

$$\xi_t^T = \frac{1}{B(0,T)} E_Q\left[\frac{B(T,T)}{M(T)}\bigg|\mathcal{F}_t\right] = \frac{B(t,T)}{B(0,T)M(t)}$$
$$= \exp\left(-\int_0^t \sigma_B(u,T)\,dZ(u) - \frac{1}{2}\int_0^t \sigma_B(u,T)^2\,du\right). \qquad (8.1.7)$$

By virtue of the Girsanov Theorem and observing the result in (8.1.7), we deduce that the process

$$Z^T(t) = Z(t) + \int_0^t \sigma_B(u,T)\,du \qquad (8.1.8)$$

is Q_T-Brownian.

As an example, consider the Vasicek model where the short rate is modeled by

$$dr(t) = \alpha[\gamma - r(t)]\,dt + \rho\,dZ(t),$$

where $Z(t)$ is Q-Brownian. The corresponding volatility function $\sigma_B(t, T)$ of the discount bond price process is known to be

$$\sigma_B(t,T) = \frac{\rho}{\alpha}[1 - e^{-\alpha(T-t)}].$$

Fitting Term Structures of Bond Prices

We consider a special form of the Hull–White model, where $\phi(t)$ in the drift term is the only time dependent function in the model. Under the risk neutral measure Q, the short rate is assumed to follow

$$dr_t = [\phi(t) - \alpha r_t]\,dt + \sigma\,dZ_t, \tag{7.2.36}$$

where α and σ are constant parameters. The model possesses the mean reversion property and exhibits nice analytic tractability. We illustrate the analytic procedure for the determination of $\phi(t)$ using the information of the current term structure of bond prices.

The governing equation for the bond price $B(r, t; T)$ is given by

$$\frac{\partial B}{\partial t} + \frac{\sigma^2}{2}\frac{\partial^2 B}{\partial r^2} + [\phi(t) - \alpha r]\frac{\partial B}{\partial r} - rB = 0. \tag{7.2.37}$$

The bond price function assumes the affine form shown in (7.2.14). Solving the pair of ordinary differential equations for $a(t, T)$ and $b(t, T)$, we obtain

$$b(t, T) = \frac{1}{\alpha}\left[1 - e^{-\alpha(T-t)}\right] \tag{7.2.38a}$$

$$a(t, T) = \frac{\sigma^2}{2}\int_t^T b^2(u, T)\,du - \int_t^T \phi(u)b(u, T)\,du. \tag{7.2.38b}$$

Our goal is to determine $\phi(T)$ in terms of the current term structure of bond prices $B(r,t;T)$. From (7.2.38b) and applying the relation

$$\ln B(r,t;T) + rb(t,T) = a(t,T),$$

we have

$$\int_t^T \phi(u)b(u,T)\,du = \frac{\sigma^2}{2}\int_t^T b^2(u,T)\,du - \ln B(r,t;T) - rb(t,T). \quad (7.2.39)$$

To solve for $\phi(u)$, the first step is to obtain an explicit expression for $\int_t^T \phi(u)\,du$. This can be achieved by differentiating $\int_t^T \phi(u)b(u,T)\,du$ with respect to T and subtracting the terms involving $\int_t^T \phi(u)e^{-\alpha(T-t)}\,du$. The derivative of the left-hand side of (7.2.39) with respect to T gives

$$\begin{aligned}\frac{\partial}{\partial T}\int_t^T \phi(u)b(u,T)\,du &= \phi(u)b(u,T)\Big|_{u=T} + \int_t^T \phi(u)\frac{\partial}{\partial T}b(u,T)\,du \\ &= \int_t^T \phi(u)e^{-\alpha(T-u)}\,du. \end{aligned} \quad (7.2.40)$$

Next, we equate the derivatives on both sides to obtain

$$\begin{aligned}\int_t^T \phi(u)e^{-\alpha(T-u)}\,du &= \frac{\sigma^2}{\alpha}\int_t^T [1-e^{-\alpha(T-u)}]e^{-\alpha(T-u)}\,du \\ &\quad - \frac{\partial}{\partial T}\ln B(r,t;T) - re^{-\alpha(T-t)}. \end{aligned} \quad (7.2.41)$$

We multiply (7.2.39) by α and add it to (7.2.41) to obtain

$$\int_t^T \phi(u)\, du = \frac{\sigma^2}{2\alpha} \int_t^T [1 - e^{-2\alpha(T-u)}]\, du - r$$
$$- \frac{\partial}{\partial T} \ln B(r, t; T) - \alpha \ln B(r, t; T).$$

Finally, by differentiating the above equation with respect to T again, we obtain $\phi(T)$ in terms of the current term structure of bond prices $B(r, t; T)$ as follows:

$$\phi(T) = \frac{\sigma^2}{2\alpha}[1 - e^{-2\alpha(T-t)}] - \frac{\partial^2}{\partial T^2} \ln B(r, t; T)$$
$$- \alpha \frac{\partial}{\partial T} \ln B(r, t; T). \qquad (7.2.42a)$$

Alternatively, one may express $\phi(T)$ in terms of the current term structure of forward rates $F(t, T)$. Recall that $-\frac{\partial}{\partial T} \ln B(r, t; T) = F(t, T)$ so that we may rewrite $\phi(T)$ in the form

$$\phi(T) = \frac{\sigma^2}{2\alpha}[1 - e^{-2\alpha(T-t)}] + \frac{\partial}{\partial T} F(t, T) + \alpha F(t, T). \qquad (7.2.42b)$$