MAFS525 — Computational Methods for Pricing Structural Products

Computer Assignment Two

* Work as a group of two. **Course instructor:** *Prof. Y.K. Kwok*

On Pricing of Accumulators

Product nature

The "accumulator" or "accumulative forward" is a daily accumulated and knock-out structured product linked to the performance of an underlying asset. It can be considered as a portfolio of forward contracts with the "occupation time" feature. The accumulated amount of assets depends on the total excursion time of the asset price below the strike price. This leads to an enhanced downside loss. The upside gain is limited by the knock-out feature with an upside barrier.

A typical equality-linked accumulator contract obligates an investor to buy a preset amount of underlying stocks at the strike price X, if the closing stock price on a trading day is higher than X. However, when the stock closes lower than X, the investor has to buy twice the amount of stocks at X. Normally, the strike price X is set at a discount of the original spot price S_0. This explains why the accumulator is also called "discounted stock" among public. On the other hand, the profit from an accumulator contract is capped by an knock-out barrier H which is set higher than S_0.

A sample contract

Underlying Shares	SEMBCORP INDUSTRIES LTD
Start Date	05 November 2007
Final Accumulation Date	03 November 2008
Maturity Date	06 November 2008
Number of Business Days	250 (subject to adjustment if knocked-out)
Strike Price	\$4.7824
Initial Price	\$5.70
Knock-out Price	\$6.20

Knock-Out Event: A Knock-Out Event occurs if the official closing price of the Underlying Share on any Scheduled Trading Day is greater than or equal to the Knock-Out Price. Under such event, there will be no further daily accumulation of Shares from that day onward. The aggregate number of shares accumulated will be settled on the Early Termination Date, which is the third business day following the occurrence of Early Termination Event.

Shares Accumulation: On each Scheduled Trading Day prior to the occurrence of Early Termination Event, the number of shares accumulated will be 1 when Official Closing Price for the day is higher than or equal to the Strike Price; 2 when Official Closing Price for the day is lower than the Strike Price.

Monthly Settlement Date: The Shares accumulated for each Accumulation Period will be delivered to the investor on the third business day following the end of each monthly Accumulation Period.

Total Number of Shares: Up to the maximum of 500 shares.

Accumulation Period and Delivery Schedule:

Accumulation Period	Number of days	Delivery Date
05 Nov 07 to 03 Dec 07	20	06 Dec 07
04 Dec 07 to 02 Jan 08	19	07 Jan 08
03 Jan 08 to 04 Feb 08	23	11 Feb 08
05 Feb 08 to 03 Mar 08	18	06 Mar 08
04 Mar 08 to 02 Apr 08	21	07 Apr 08
03 Apr 08 to 02 May 08	21	07 May 08
05 May 08 to 02 Jun 08	20	05 Jun 08
03 Jun 08 to 02 Jul 08	22	07 Jul 08
03 Jul 08 to 04 Aug 08	23	07 Aug 08
05 Aug 08 to 02 Sep 08	21	05 Sep 08
03 Sep 08 to 02 Oct 08	21	07 Oct 08
03 Oct 08 to 03 Nov 08	21	06 Nov 08

12 accumulation periods in total.

Replication of barrier-type derivatives

To derive the analytic formula based on discrete settlement of stock transaction on each business day, we assume continuous monitoring of the knock-out barrier H.

(i) With immediate transaction of the stock, the payoff at date t_i is given by

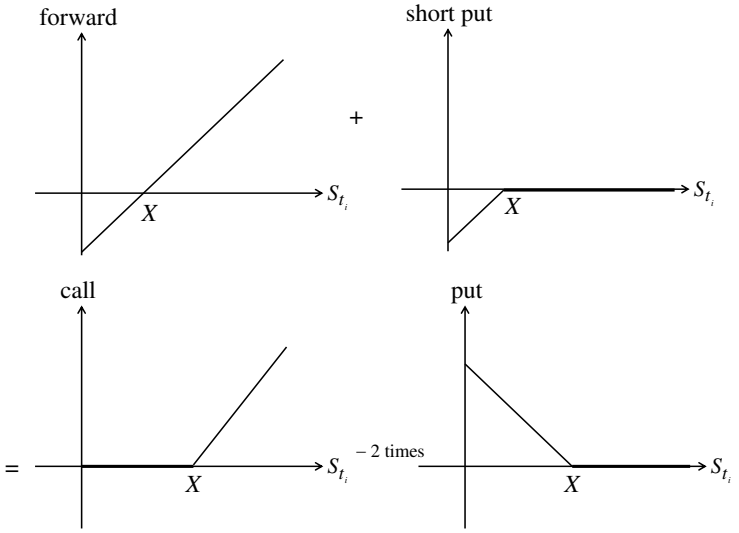
$$\begin{cases} 0 & \text{if } \max_{0\le\tau\le t_i} S_\tau \ge H \\ S_{t_i} - X & \text{if } \max_{0\le\tau\le t_i} S_\tau < H \text{ and } S_{t_i} \ge X \\ 2(S_{t_i} - X) & \text{if } \max_{0\le\tau t_i} S_\tau < H \text{ and } S_{t_i} < X \end{cases}.$$

Assuming that there are n business days, the fair value of the accumulator is given by

$$V = \sum_{i=1}^{n} c_{uo}(t_i, X, H) - 2p_{uo}(t_i, X, H),$$

where c_{uo} and p_{uo} denote the price function of the up-and-out call and up-and-out put, respectively.

Provided that there is no knock-out up to time t_i, the payoff at t_i can be decomposed as



The value of the forward is significantly capped by the upper knock-out barrier H while the value of the put sold to the issuer is not much affected by the upper knock-out barrier. The compensation to the buyer is the sale of the stocks at a discount to the prevailing stock price.

(ii) Let T denote the settlement date of the stocks fixed at the observation date $t_i, T > t_i, i = 1, 2, \cdots, n$. Typically, T is 3 business days after the last monitoring date of the given accumulation period. Let $c^F_{uo}(t_i, X, H, T)$ denote the price of a t_i-maturity up-and-out barrier call option on a forward contract. The corresponding forward contract is an agreement to buy the underlying stock at time T at a price equals X, and a similar definition for $p^F_{uo}(t_i, X, H, T)$ as the price of the corresponding put option. We then have

$$V_{delay} = \sum_{i=1}^{n} c^F_{uo}(t_i, X, H, T) - 2p^F_{uo}(t_i, X, H, T).$$

Monte Carlo simulation

We assume the usual Black-Scholes framework where

$$\frac{dS_t}{S_t} = r\,dt + \sigma\,dZ_t.$$

The stock prices at successive time step t_j can be generated as

$$S_j = S_{j-1} e^{\left(r - \frac{\sigma^2}{2}\right)\Delta t + \sigma\epsilon\sqrt{\Delta t}},$$

where $\epsilon \sim N(0, 1)$. We take $\Delta t = \frac{T}{N}$, where N is the number of trading days until maturity date T.

For each trading day i, the following computation is implemented:

1. If $S_i \geq H, V_i = V_{i-1} + (1+K)e^{-ir\delta t}(S_i - e^{-3r\delta t}X)$. Terminate the loop and return $V_N = V_i$; the accumulator is terminated.
2. If $X \leq S_i < H, K = K + 1, V_i = V_{i-1}$; the number of stock sold is one.
3. If $S_i < X, K = K + 2, V_i = V_{i-1}$; the number of stock sold is two.
4. If day i is a settlement day, $V_i = V_{i-1} + (1+K)e^{-ir\delta t}(S_i - e^{-3r\delta t}X), K = 0$; after settlement, the count of stocks accumulated is set to be zero.
5. Go to next day.

Generate M stock price paths do the above computations to obtain $\{V_{N,j}\}_{j=1}^{M}$. The price of the accumulator is:

$$\overline{V} = \frac{1}{M}\sum_{j=1}^{M} V_{N,j}.$$

The standard error of the estimate is

$$SE_{\overline{V}} = \sqrt{\frac{\text{var}\left(\{V_{N,j}\}_{j=1}^{M}\right)}{N}},$$

where $\text{var}\left(\{V_{N,j}\}_{j=1}^{M}\right)$ is the variance of the sample result set. The standard error of the estimate is expected to be inversely proportional to the square root of the number of samples M.

Work elements

Compute the fair value of the accumulator using

(i) forward shooting grid method;

(ii) Monte Carlo simulation.

Based on the previous experiences by the students of an earlier class, the numerical value of the accumulator is highly dependent on the choices of the time steps and stepwidth in the numerical calculations.

Monte Carlo Simulation Code

```
function [value,se] = acc_mc(S,X,H,r,sigma,settlement,T,M,ngrid)
%
% Li Lewei, 12 Mar 2010
%
% MATLAB program for Final Year Project
% Monte Carlo simulation for SEMBCORP accumulator
%
% S: initial price
% X: strike price
% H: barrier level
% r: risk-free rate
% sigma: volatility
% settlement: array of settlement days
% T: number of trading days per year
% M: number of samples
% ngrid: number of barrier observation per day
%
% value: the estimated value from Monte Carlo simulation
% se: the standard error of the estimate
%
% Sample call: [price,stderr] = acc_mc(5.7, 4.7824, 6.1425,
  0.02, 0.3, [20,39,62,80,101,122,142,164,187,208,229,250], 250,
   1000000, 1)
%
dt = 1/T;
npath = 1000;

nsample = M/npath;
price = zeros(1,nsample);
strikepath = X*ones(1,T);

tic
for i = 1 : nsample
        p = 0;
        for j = 1 : npath
                settle = [0 settlement];
                accumulation = ones(1,T);
                [spath,knockout,discount] = stockpath(S,H,r,
                  sigma,T,ngrid);
                if knockout > 0
                        accumulation(knockout:T) = 0;
%                       Change settlement day as the knockout
  event occurs.
                        for ix = 1:length(settle)-1
                                if settle(ix) >= knockout
                                        settle(ix) = knockout;
                                        settle = settle(1:ix);
                                        break
                                end
                        end
                end
%               Compute the accumulation amount for each day:
  {0,1,2}
                accumulation = (1*(spath>strikepath)+2*(spath<=
                  strikepath)).*accumulation;
%               Sum up the PV of payoff for each settlement day
                for k = 1 : length(settle)-1
                        p = p + sum(accumulation((settle(k)+1):
                          settle(k+1)))*(spath(settle(k+1))-X)*
                          discount(settle(k+1));
                end
        end
        price(i) = p/npath;
end
value = mean(price);
se = sqrt(var(price)/nsample);
toc
return

function [path,knockout,discount] = stockpath(S,H,r,sigma,T,
  ngrid)
% This function generates random stock paths.
% It returns 'path' as the stock price series, 'knockout' as the
   knockout day, 'discount' as the discount factor series.
% If the stock path doesn't knock out, knockout = 0.
N = T*ngrid;
dt = 1/N;
s = zeros(1,N+1);
discount = zeros(1,N);
s(1) = S;

eps = exp((r-0.5*sigma^2)*dt+sigma*sqrt(dt)*randn(1,N));
```

```
knockout = 0;
stayin = 1;
for i = 2 : (N+1)
    s(i) = s(i-1)*eps(1,i-1);
    if ((s(i) > H) && (stayin==1))
            knockout = floor((i-1)/ngrid);
            stayin = 0;
    end
end
path = s(1+ngrid:ngrid:N+1);
discount = exp(-r*dt*(1:N));
return
```

Forward Shooting Grid

```
function price = acc_fsgm(S0,X,H,r,sigma,settlement,T,mesh)
%
% Li Lewei, 12 Apr 2010
%
% MATLAB program for Final Year Project
% Forward Shooting Grid Method for SEMBCORP accumulator
%
% S0: initial price
% X: strike price
% H: barrier level
% r: risk-free rate
% sigma: volatility
% settlement: array of settlement days
% T: number of trading days per year
% mesh: number of time steps per day
%
% Sample call: [price,stderr] = acc_fsgm(5.7, 4.7824, 6.1425,
  0.02, 0.3, [20,39,62,80,101,122,142,164,187,208,229,250], 250,
   1)
%
% This is the master function
settlement = [0,settlement];
NT = T;
AccValue = zeros(1,2*NT*mesh+1);
tic
for i = length(settlement):-1:2
        AccValue = fsAccMonthPartmesh(S0,X,H,r,sigma,NT,
          settlement(i),settlement(i)-settlement(i-1),AccValue,
          mesh);
end
price = AccValue;
toc
return

function price = fsAccMonthPartmesh(S0,X,H,r,sigma,NT,Nt,Nday,
  AccValue,mesh)
% FSGM for one settlement period
dt = 1/(NT*mesh);
discount = exp(-r*dt);
u = exp(sigma*sqrt(dt));
d = 1/u;
p = (exp(r*dt)-d)/(u-d);
S = S0*exp((sigma*sqrt(dt))*(-Nt*mesh:Nt*mesh));
payoff = (S-X)'*(0:2*Nday) + repmat(AccValue',1,2*Nday+1);
V = payoff;
Vtemp = payoff;
indS = Nt*mesh+1;
indj = 1;
for j = Nday-1:-1:0
% Last time step of each day, daily fixings
```

```
46          for i = (Nday-Nt-j-1)*mesh+1+indS:(Nt-Nday+j+1)*mesh-1+
              indS
47                  if S(i-1) > X
48                          Vu = V(i+1,j+1+indj:2*j+1+indj);
49                          Vd = V(i-1,j+1+indj:2*j+1+indj);
50                          if S(i+1) >= H
51                                  Vu = (S(i)-X)*(j+indj:2*j+indj);
52                          end
53                          if S(i-1) > H
54                                  Vd = (S(i)-X)*(j+indj:2*j+indj);
55                          end
56                  else if S(i+1) > X
57                                  Vu = V(i+1,j+1+indj:2*j+1+indj);
58                                  Vd = V(i-1,j+2+indj:2*j+2+indj);
59                          else
60                                  Vu = V(i+1,j+2+indj:2*j+2+indj);
61                                  Vd = V(i-1,j+2+indj:2*j+2+indj);
62                          end
63                  end
64                  Vtemp(i,j+indj:2*j+indj) = discount*(p*Vu + (1-p
                      )*Vd);
65          end
66          V = Vtemp;
67  % Other time steps for each day, trivial binomial expectation
68          if mesh > 1
69                  for m = 2 : mesh
70                          for i = (Nday-Nt-j-1)*mesh+m+indS:(Nt-
                              Nday+j+1)*mesh-m+indS
71                                  Vtemp(i,j+indj:2*j+indj) =
                                      discount*(p*V(i+1,j+indj:2*j+
                                      indj)+(1-p)*V(i-1,j+indj:2*j+
                                      indj));
72                          end
73                          V = Vtemp;
74                  end
75          end
76  end
77  price = V((Nday-Nt)*mesh+indS:(Nt-Nday)*mesh+indS,indj)';
78  return
```

Crank-Nicolson Scheme Code

```
function price = acc_cn(S0,X,H,r,sigma,settlement,NT,Ns,Mesht)
%
% Li Lewei, 15 Apr 2010
%
% MATLAB program for Final Year Project
% Crank-Nicolson scheme for SEMBCORP accumulator
%
% S0: initial price
% X: strike price
% H: barrier level
% r: risk-free rate
% sigma: volatility
% settlement: array of settlement days
% NT: total number of days per year
% Ns: number of steps in S mesh
% Mesht: number of time steps per day
%
% Sample call: price = acc_cn(5.7, 4.7824, 6.1425, 0.02, 0.3,
  [20,39,62,80,101,122,142,164,187,208,229,250], 250, 1000, 32)
%

Smax = 1.25*H;
% settlement = [20,39,62,80,101,122,142,164,187,208,229,250];
NT = settlement(length(settlement));
AccValue = zeros(Ns-1,1);
tic
if length(settlement)>1
        for i = length(settlement):-1:2
                AccValue = acc_cn_month(S0,X,H,r,sigma,NT,
                  settlement(i),settlement(i)-settlement(i-1),
                  AccValue,Smax,Ns,Mesht);
        end
end
AccValue = acc_cn_month(S0,X,H,r,sigma,NT,settlement(1),
  settlement(1),AccValue,Smax,Ns,Mesht);
price = AccValue(floor(Ns*S0/Smax));
toc
return

function price = acc_cn_month(S0,X,H,r,sigma,NT,Nt,Nday,AccValue
  ,Smax,Ns,Mesht)
% CN for one settlement period

h = Smax / Ns;
dt = 1 / (NT*Mesht);
indX = floor(X/h);
indH = floor(H/h);
S = [1:Ns-1]*h;
payoff = (S-X)'*(0:2*Nday) + repmat(AccValue,1,2*Nday+1);

% boundary conditions
```

```
u = payoff;
u0 = (-X)*exp(-r*[Nday*Mesht:-1:0]*dt)'*(0:2*Nday)+((-X)*exp(-r*[Nday*Mesht:-1:0]*dt).*floor((Nday*Mesht:-1:0)/Mesht))'*ones(1,2*Nday+1);
u1 = (H-X)*exp(0*[Nday*Mesht:-1:0]*dt)'*(0:2*Nday);

% matrix build-up
i = [1:Ns-1]';
alpha = sigma^2 / 2 * (i.^2);
beta = r / 2 * i;
B = spdiags([alpha + beta, -alpha * 2 - r + 2/dt, alpha - beta], [-1:1], Ns-1, Ns-1)';
A = spdiags([-alpha - beta, alpha * 2 + r + 2/dt, -alpha + beta], [-1:1], Ns-1, Ns-1)';

uu0 = (alpha(1) - beta(1)) * (u0(1:Nday*Mesht,:) + u0(2:Nday*Mesht+1,:));
uu1 = (alpha(Ns-1) + beta(Ns-1)) * (u1(1:Nday*Mesht,:) + u1(2:Nday*Mesht+1,:));

count = 0;
for n = 1:Nday
        u(1:indX,(1+Nday-n):(1+2*(Nday-n))) = u(1:indX,(3+Nday-n):(3+2*(Nday-n)));
        u(indX+1:indH,(1+Nday-n):(1+2*(Nday-n))) = u(indX+1:indH,(2+Nday-n):(2+2*(Nday-n)));
        if n>1
                u(indH+1:Ns-1,(1+Nday-n):(1+2*(Nday-n))) = (S(indH+1:Ns-1)-X)'*((Nday-n):(2*(Nday-n)));
        end
        for m = Mesht:-1:1
                w = B * u;
                w(1,:) = w(1,:) + uu0(1+n*Mesht-m,:);
                w(Ns-1,:) = w(Ns-1,:) + uu1(1+n*Mesht-m,:);
                u = A\ w;
        end
end
price = u(:,1);
return
```