MAFS525 — Computational Methods for Pricing Structural Products

Homework One

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1. Suppose the underlying asset is paying a continuous dividend yield at the rate q, the two governing equations for u, d and p are modified as

$$pu + (1-p)d = e^{(r-q)\Delta t}$$
$$pu^2 + (1-p)d^2 = e^{2(r-q)\Delta t}e^{\sigma^2 \Delta t}.$$

Show that the parameter values in the binomial model are modified by replacing the growth factor of the asset price $e^{r\Delta t}$ (under the risk neutral measure) by the new factor $e^{(r-q)\Delta t}$ while the discount factor in the binomial formula remains to be $e^{-r\Delta t}$.

2. (a) Show that the risk neutral probabilities in a Cox–Ross–Rubinstein binomial tree may become negative when time steps are so large that

$$\sigma < |(r-q)\sqrt{\Delta t}|.$$

(b) Suppose we relax the constraint that $u = 1/d$ and instead we always set $p = 0.5$. Show that a solution to u and d when terms of order higher than Δt are ignored is

$$u = e^{\left(r-q-\frac{\sigma^2}{2}\right)\Delta t + \sigma\sqrt{\Delta t}} \quad \text{and} \quad d = e^{\left(r-q-\frac{\sigma^2}{2}\right)\Delta t - \sigma\sqrt{\Delta t}}$$

3. Show that the total number of multiplications and additions in performing n steps of numerical calculations using the trinomial and binomial schemes are given by

Scheme	*Number of multiplications*	*Number of additions*
trinomial	$3n^2$	$2n^2$
binomial	$n^2 + n$	$\frac{1}{2}(n^2+n)$

4. Suppose we let $p_2 = 0$ and write $p_1 = 1 - p_3 = p$ in the trinomial scheme. By matching the mean and variance of $\zeta(t)$ and $\zeta^a(t)$ accordingly

$$\begin{aligned} E[\zeta^a(t)] &= 2pv - v = \left(r - \frac{\sigma^2}{2}\right)\Delta t \\ \text{var}(\zeta^a(t)) &= v^2 - E[\zeta^a(t)]^2 = \sigma^2 \Delta t, \end{aligned}$$

show that the parameters v and p obtained by solving the above pair of equations are found to be

$$v = \sqrt{\left(r - \frac{\sigma^2}{2}\right)^2 \Delta t^2 + \sigma^2 \Delta t}$$
$$p = \frac{1}{2}\left[1 + \frac{\left(r - \frac{\sigma^2}{2}\right)\Delta t}{\sqrt{\sigma^2 \Delta t + \left(r - \frac{\sigma^2}{2}\right)^2 \Delta t^2}}\right].$$

5. Boyle proposes the following three-jump process for the approximation of the asset price process over one period:

nature of jump	*probability*	*asset price*
up	p_1	uS
horizontal	p_2	S
down	p_3	dS

where S is the current asset price. The middle jump ratio m is chosen to be 1. There are five parameters in Boyle's trinomial model: u, d and the probability values. The governing equations for the parameters can be obtained by

(i) setting sum of probabilities to be 1

$$p_1 + p_2 + p_3 = 1,$$

(ii) equating the first two moments of the approximating discrete distribution and the corresponding continuous lognormal distribution

$$p_1 u + p_2 + p_3 d = e^{r\triangle t} = R$$

$$p_1 u^2 + p_2 + p_3 d^2 - (p_1 u + p_2 + p_3 d)^2 = e^{2r\triangle t}\,(e^{\sigma^2 \triangle t} - 1).$$

The last equation can be simplified as

$$p_1 u^2 + p_2 + p_3 d^2 = e^{2r\triangle t}\, e^{\sigma^2 \triangle t}.$$

The remaining two conditions can be chosen freely. They are chosen by Boyle (1988) to be

$$ud = 1$$

and

$$u = e^{\lambda\sigma\sqrt{\triangle t}}, \qquad \lambda \text{ is a free parameter.}$$

By solving the five equations together, show that

$$p_1 = \frac{(W - R)u - (R - 1)}{(u-1)(u^2-1)}, \qquad p_3 = \frac{(W - R)u^2 - (R - 1)u^3}{(u-1)(u^2-1)},$$

where $W = R^2 e^{\sigma^2 \Delta t}$. Also show that Boyle's trinomial model reduces to the Cox–Ross–Rubinstein binomial scheme when $\lambda = 1$.

6. Show that the width of the domain of dependence of the trinomial scheme with reference to x, where $x = \ln S$, increases as $\sqrt{n}$, where n is the number of time steps to expiry.

7. Consider a three-state option model where the logarithmic return processes of the underlying assets are given by

$$\ln \frac{S_i^{\Delta t}}{S_i} = \zeta_i, \quad i = 1, 2, 3.$$

Here, ζ_i denotes the normal random variable with mean $\left(r - \frac{\sigma_i^2}{2}\right)\Delta t$ and variance $\sigma_i^2 \Delta t, i = 1, 2, 3$. Let ρ_{ij} denote the instantaneous correlation coefficient between ζ_i and $\zeta_j, i, j = 1, 2, 3, i \neq j$. Suppose the approximating multi-variate distribution $\xi_i^a, i = 1, 2, 3$, is taken to be

ζ_1^a	ζ_2^a	ζ_3^a	probability
v_1	v_2	v_3	p_1
v_1	v_2	$-v_3$	p_2
v_1	$-v_2$	v_3	p_3
v_1	$-v_2$	$-v_3$	p_4
$-v_1$	v_2	v_3	p_5
$-v_1$	v_2	$-v_3$	p_6
$-v_1$	$-v_2$	v_3	p_7
$-v_1$	$-v_2$	$-v_3$	p_8
0	0	0	p_9

where $v_i = \lambda\sigma_i\sqrt{\Delta t}, i = 1, 2, 3$. Following the Kamrad-Ritchken approach, find the probability values so that the approximating discrete distribution converges to the continuous multi-variate distribution as $\Delta t \to 0$.

Hint: The first and last probability values are given by

$$\begin{aligned} p_1 &= \frac{1}{8}\left\{\frac{1}{\lambda^2} + \frac{\sqrt{\Delta t}}{\lambda}\left(\frac{r - \frac{\sigma_1^2}{2}}{\sigma_1} + \frac{r - \frac{\sigma_2^2}{2}}{\sigma_2} + \frac{r - \frac{\sigma_3^2}{2}}{\sigma_3}\right),\right. \\ &\qquad \left. + \frac{\rho_{12} + \rho_{13} + \rho_{23}}{\lambda^2}\right\} \\ p_9 &= 1 - \frac{1}{\lambda^2}. \end{aligned}$$

8. Consider the 5-point multinomial scheme and the corresponding 4-point scheme (obtained by setting $\lambda = 1$), show that the total number of multiplications and additions in performing n steps of the schemes are given by

Scheme	*Number of multiplications*	*Number of additions*
5-points	$\frac{5}{3}(2n^3 + n)$	$\frac{4}{3}(2n^3 + n)$
4-point	$\frac{2}{3}(2n^3 + 3n^2 + n)$	$\frac{1}{2}(2n^3 + 3n^2 + n)$

9. Consider the window Parisian feature, associated with each time point, a moving window is defined with $\widehat{m}$ consecutive monitoring instants before and including that time point. The option is knocked out at a given time when the asset price has already stayed within the knock-out region exactly m times, $m \leq \widehat{m}$, within the moving window. Under what condition does the window Parisian feature reduce to the consecutive Parisian feature? How to construct the corresponding discrete grid function g_{win} in the forward shooting grid (FSG) algorithm?

Hint: We define a binary string $A = a_1a_2 \cdots a_{\widehat{m}}$ to represent the history of asset price path falling inside or outside the knock-out region within the moving window. The augmented path dependence state vector has binary strings as elements.

10. Construct the FSG scheme for pricing the European style fixed strike lookback call option under (i) continuous monitoring, (ii) discrete monitoring, where the terminal payoff at maturity date T is given by $\max(S_T^{\max} - X, 0)$, X is the strike price. How do you modify the FSG scheme in order to incorporate the American early exercise feature?