MAFS5250 - Computational Methods for Pricing Structured Products

Homework One

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1. Suppose the underlying asset is paying a continuous dividend yield at the rate q, the two governing equations for u, d and p are modified as

$$\begin{aligned} pu + (1-p)d &= e^{(r-q)\Delta t} \\ pu^2 + (1-p)d^2 &= e^{2(r-q)\Delta t} e^{\sigma^2 \Delta t}. \end{aligned}$$

Show that the parameter values in the binomial model are modified by replacing the growth factor of the asset price $e^{r\Delta t}$ (under the risk neutral measure) by the new factor $e^{(r-q)\Delta t}$ while the discount factor in the binomial formula remains to be $e^{-r\Delta t}$.

2. (a) Show that the risk neutral probabilities in the Cox–Ross–Rubinstein binomial tree may become negative when the time steps are so large such that

$$\sigma < |(r-q)\sqrt{\Delta t}|.$$

(b) Suppose we relax the constraint that $u = 1/d$ and instead we always set $p = 0.5$. Show that a solution to u and d when terms of order higher than Δt are ignored is

$$u = e^{\left(r-q-\frac{\sigma^2}{2}\right)\Delta t + \sigma\sqrt{\Delta t}} \quad \text{and} \quad d = e^{\left(r-q-\frac{\sigma^2}{2}\right)\Delta t - \sigma\sqrt{\Delta t}}$$

Hint: Approximate $\ln \frac{S_{t+\Delta t}}{S_t}$ by ζ^a, where

$$\zeta^a = \begin{cases} v_1 & \text{with probability equals } 0.5 \\ v_2 & \text{with probability equals } 0.5 \end{cases}.$$

3. Show that the total number of multiplications and additions in performing n steps of numerical calculations using the trinomial and binomial schemes are given by

Scheme	*Number of multiplications*	*Number of additions*
trinomial	$3n^2$	$2n^2$
binomial	$n^2 + n$	$\frac{1}{2}(n^2+n)$

4. Suppose we let $p_2 = 0$ and write $p_1 = 1 - p_3 = p$ in the trinomial scheme. By matching the mean and variance of $\zeta(t)$ and $\zeta^a(t)$ accordingly, we obtain

$$\begin{aligned} E[\zeta^a(t)] &= 2pv - v = \left(r - \frac{\sigma^2}{2}\right)\triangle t \\ \operatorname{var}(\zeta^a(t)) &= v^2 - E[\zeta^a(t)]^2 = \sigma^2 \triangle t. \end{aligned}$$

Show that the parameters v and p obtained by solving the above pair of equations are found to be

$$v = \sqrt{\left(r - \frac{\sigma^2}{2}\right)^2 \Delta t^2 + \sigma^2 \Delta t}$$

$$p = \frac{1}{2}\left[1 + \frac{\left(r - \frac{\sigma^2}{2}\right)\Delta t}{\sqrt{\sigma^2 \Delta t + \left(r - \frac{\sigma^2}{2}\right)^2 \Delta t^2}}\right].$$

5. Boyle (1988) proposes the following three-jump process for the approximation of the jump ratio of the asset price process over one period:

nature of jump	*probability*	*asset price*
up	p_1	uS
horizontal	p_2	S
down	p_3	dS

Here S is the current asset price and the middle jump ratio m is chosen to be 1. There are five parameters in Boyle's trinomial model: u, d and the probability values. The governing equations for the parameters can be obtained by

(i) setting the sum of probabilities to be 1

$$p_1 + p_2 + p_3 = 1;$$

(ii) equating the first two moments of the approximating discrete distribution and the corresponding continuous lognormal distribution

$$p_1 u + p_2 + p_3 d = e^{r\triangle t} = R$$

$$p_1 u^2 + p_2 + p_3 d^2 - (p_1 u + p_2 + p_3 d)^2 = e^{2r\triangle t}\left(e^{\sigma^2 \triangle t} - 1\right).$$

The last equation can be simplified as

$$p_1 u^2 + p_2 + p_3 d^2 = e^{2r\triangle t}\, e^{\sigma^2 \triangle t}.$$

The remaining two conditions can be chosen freely. They are chosen by Boyle to be

$$ud = 1$$

and

$$u = e^{\lambda\sigma\sqrt{\triangle t}}, \qquad \lambda \text{ is a free parameter.}$$

By solving the five equations together, show that

$$p_1 = \frac{(W - R)u - (R - 1)}{(u - 1)(u^2 - 1)}, \qquad p_3 = \frac{(W - R)u^2 - (R - 1)u^3}{(u - 1)(u^2 - 1)},$$

where $W = R^2 e^{\sigma^2 \Delta t}$. Also, show that Boyle's trinomial model reduces to the Cox–Ross–Rubinstein binomial scheme when $\lambda = 1$.

6. Consider a three-state option model where the logarithmic return processes of the three underlying assets are given by

$$\ln \frac{S_i^{\Delta t}}{S_i} = \zeta_i, \quad i = 1, 2, 3.$$

Here, ζ_i denotes the normal random variable with mean $\left(r - \frac{\sigma_i^2}{2}\right)\Delta t$ and variance $\sigma_i^2 \Delta t, i = 1, 2, 3$. Let ρ_{ij} denote the instantaneous correlation coefficient between ζ_i and $\zeta_j, i, j = 1, 2, 3, i \neq j$. Suppose the discrete multi-variate distribution ζ_i^a that approximates $\zeta_i, i = 1, 2, 3$, is taken to be

ζ_1^a	ζ_2^a	ζ_3^a	probability
v_1	v_2	v_3	p_1
v_1	v_2	$-v_3$	p_2
v_1	$-v_2$	v_3	p_3
v_1	$-v_2$	$-v_3$	p_4
$-v_1$	v_2	v_3	p_5
$-v_1$	v_2	$-v_3$	p_6
$-v_1$	$-v_2$	v_3	p_7
$-v_1$	$-v_2$	$-v_3$	p_8
0	0	0	p_9

where $v_i = \lambda \sigma_i \sqrt{\Delta t}, i = 1, 2, 3$. Following the Kamrad-Ritchken approach, find the probability values so that the approximating discrete distribution converges to the continuous multi-variate distribution as $\Delta t \to 0$.

Hint: There are 3 equations that can be derived by equating the respective mean values, 3 equations from covariances, only 1 equation from variances (though there are 3 variances, only one equation is effective), and 1 equation from summing the probability values to be one. It is necessary to derive one additional equation so that we have 9 equations for the 9 unknowns. Since $E[\zeta_1^a\ \zeta_2^a\ \zeta_3^a] = O((\Delta t)^{3/2})$, so an ingenious choice is to set $E[\zeta_1^a\ \zeta_2^a\ \zeta_3^a] = 0$. For reference, the first and last probability values are given below:

$$p_1 = \frac{1}{8}\left\{\frac{1}{\lambda^2} + \frac{\sqrt{\Delta t}}{\lambda}\left(\frac{r - \frac{\sigma_1^2}{2}}{\sigma_1} + \frac{r - \frac{\sigma_2^2}{2}}{\sigma_2} + \frac{r - \frac{\sigma_3^2}{2}}{\sigma_3}\right)\right.$$

$$
\left. + \frac{\rho_{12} + \rho_{13} + \rho_{23}}{\lambda^2} \right\},
$$

$$
p_9 = 1 - \frac{1}{\lambda^2}.
$$

7. Cheuk and Vorst (1997) mention in their paper that their binomial scheme with adjusted strike cannot be adopted to price an American fixed strike lookback call option. Why?

8. Consider the construction of the binomial model for pricing the discretely monitored fixed strike lookback call option using the Cheuk-Vorst approach. The terminal payoff of the fixed strike lookback option with strike price K is given by

$$
\max\left(\max_{0 \le iZ \le N} S(t_{iZ}) - K, 0 \right) = \max(\overline{M}^Z(t_N) - K, 0),
$$

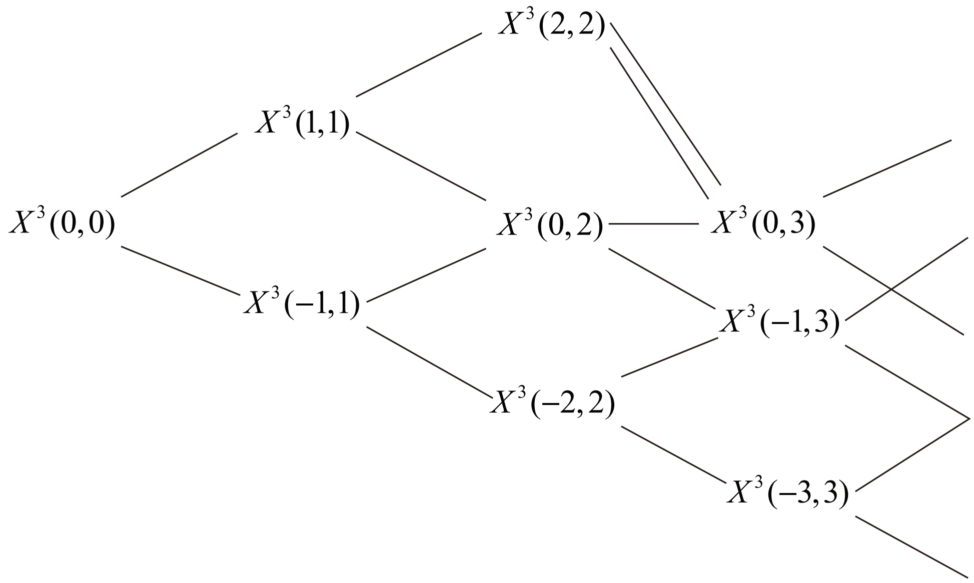
where Z is the number of time steps between successive monitoring instants and N is the total number of time steps in the binomial tree. We define the call value function normalized by the stock price by

$$
X^Z(k, t_j) = \frac{C_X(S(t_j), K', t_j)}{S(t_j)},
$$

where $C_X(S(t_j), K', t_j)$ is the fixed strike lookback call value with adjusted strike price $K' = \max(\overline{M}(t_j), K)$ at time level t_j. We define the index k by

$$
k = \left[\ln \frac{S(t_j)}{K'} \right] \Big/ \ln u.
$$

As an illustration, the binomial tree with $Z = 3$ is shown below. Note that $k \le 2$ for $t_j \ne iZ$ and $k \le 0$ for $t_j = iZ$.



Explain why the backward induction procedure in the binomial calculations under various cases are constructed as below. Give details on how each term enters into the respective binomial formula, like the choice of the indexes $k-1$ and $k+1$, the appearance of the multipliers u and d, the payoff terms that correspond to deferred payments at maturity, etc.

(a) When $t_{j+1} \neq iZ$, for all i, we have

$$X^Z(k, t_j) = [pX^Z(k+1, t_{j+1})u + (1-p)X^Z(k-1, t_{j+1})d]e^{-r\Delta t},$$

where p is the probability of upward move (under the risk neutral measure), $e^{-r\Delta t}$ is the discount factor over one time step, u is the proportional upward jump in the asset price, and $d = 1/u$.

(b) When $k \geq 1$ and $t_{j+1} = iZ$, we have

$$\begin{aligned} X^Z(k, t_{iZ-1}) &= X^Z(0, t_{iZ}) \\ &\quad + [p(u - u^{-k}) + (1-p)(u^{-1} - u^{-k})]e^{-(N-iZ+1)\Delta t}. \end{aligned}$$

(c) When $k = 0$ and $t_{j+1} = iZ$, we have

$$\begin{aligned} X^Z(0, t_{iZ-1}) &= [pX^Z(0, t_{iZ})u + (1-p)X^Z(-1, t_{iZ})d]e^{-r\Delta t} \\ &\quad + p(u-1)e^{-(N-iZ+1)\Delta t}. \end{aligned}$$

9. Consider the window Parisian feature, associated with each time point, a moving window is defined with $\widehat{m}$ consecutive monitoring instants before and including that time point. The option is knocked out at a given time when the asset price has already stayed within the knock-out region exactly m times, $m \leq \widehat{m}$, within the moving window. Under what condition does the window Parisian feature reduce to the consecutive Parisian feature? How to construct the corresponding discrete grid function g_{win} in the forward shooting grid (FSG) algorithm?

   Hint: We define a binary string $A = a_1 a_2 \cdots a_{\widehat{m}}$ to represent the history of asset price path falling inside or outside the knock-out region within the moving window. The augmented path dependence state vector has binary strings as elements.

10. Construct the FSG scheme for pricing the American style floating strike lookback call option under (i) continuous monitoring, (ii) discrete monitoring, where the payoff at maturity date t is given by $\max_{I \in [0,t]} S_\tau - S_t$?

11. Consider the decomposition of the accumulator into a portfolio of up-and-out calls and up-and-out puts, where K is the strike price and H is the up-and-out barrier, and $H > K$. Derive the price formulas of the up-and-out call and put if the settlement date T_i is some period behind the observation date t_i.