MAFS5250 - Computational Methods for Pricing Structured Products

Homework Two

Course instructor: *Prof. Y.K. Kwok*

---

1. Suppose we would like to approximate $\frac{df}{dx}$ at x_0 up to $O(\Delta x^2)$ using function values of f at $x_0, x_0 - \Delta x$ and $x - 2\Delta x$, the one-sided backward difference formula takes the form:

$$\left.\frac{df}{dx}\right|_{x_0} = \alpha_{-2} f(x_0 - 2\Delta x) + \alpha_{-1} f(x_0 - \Delta x) + \alpha_0 f(x_0) + O(\Delta x^2),$$

where α_{-2}, α_{-1} and α_0 are unknown coefficients to be determined. Show that these coefficients can be obtained by solving

$$\begin{pmatrix} 1 & 1 & 1 \\ -2 & -1 & 0 \\ 4 & 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha_{-2} \\ \alpha_{-1} \\ \alpha_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1/\Delta x \\ 0 \end{pmatrix}.$$

Deduce the corresponding one-sided forward difference formula for $\frac{df}{dx}$ at x_0.

2. Show that the leading local truncation error terms of the following Crank-Nicolson scheme

$$\begin{aligned} \frac{V_j^{n+1} - V_j^n}{\Delta\tau} &= \frac{\sigma^2}{4}\left(\frac{V_{j+1}^n - 2V_j^n + V_{j-1}^n}{\Delta x^2} + \frac{V_{j+1}^{n+1} - 2V_j^{n+1} + V_{j-1}^{n+1}}{\Delta x^2}\right) \\ &\quad + \frac{1}{2}\left(r - \frac{\sigma^2}{2}\right)\left(\frac{V_{j+1}^n - V_{j-1}^n}{2\Delta x} + \frac{V_{j+1}^{n+1} - V_{j-1}^{n+1}}{2\Delta x}\right) \\ &\quad - \frac{r}{2}(V_j^n + V_j^{n+1}) \end{aligned}$$

are $O(\Delta\tau^2, \Delta x^2)$.

Hint: Perform the Taylor expansion at $\left(j\Delta x, \left(n + \frac{1}{2}\right)\Delta\tau\right)$.

3. Consider the pricing of a bond using the Cox-Ingersoll-Ross (CIR) model, where the short rate r_t under the risk neutral measure Q is governed by

$$dr_t = \alpha(\beta - r_t)\, dt + \sigma\sqrt{r_t}\, dZ_t,$$

where α, β and σ are constants, Z_t is the standard Brownian process. The corresponding governing equation for the bond price $B(r,t)$ takes the form:

$$\frac{\partial B}{\partial \tau} = \frac{\sigma^2 r}{2}\frac{\partial^2 B}{\partial r^2} + \alpha(\beta - r)\frac{\partial B}{\partial r} - rB.$$

We perform the finite difference calculations using the explicit Forward-Time-Centered-Difference shceme

$$\frac{B_j^{n+1} - B_j^n}{\Delta\tau} = \frac{\sigma^2 r_j}{2}\frac{B_{j+1}^{n+1} - 2B_j^{n+1} + B_{j-1}^{n+1}}{\Delta r^2} + \alpha(\beta - r_j)\frac{B_{j+1}^{n+1} - B_{j-1}^{n+1}}{2\Delta r} - r_j B_j^n,$$

where $\Delta\tau$ is the time step, Δr is the step width, and B_j^n is the numerical bond price at the $(j,n)^{th}$ node. We choose the computational domain to be $[r_{\min}, r_{\max}] \times [0,T]$, where T is the maturity date. Here, $r_{\max}$ is some sufficiently large value for the short rate (say, $r_{\max} = 10\%$). Also we may choose $r_{\min} = 0.2\%$.

Note that the boundary conditions for the bond price at the upper boundary $r = r_{\max}$ and the lower boundary $r = r_{\min}$ are unknown. How to deal with the absence of boundary condition in the construction of the finite difference scheme, specifically along the boundary nodes on the right side of the computational domain? Write down the corresponding explicit schemes for the boundary nodes at $r = r_{\max}$ and $r = r_{\min}$.

4. Let $p(S,M,t)$ denote the price function of the European floating strike lookback put option. Define $x = \ln \dfrac{M}{S}$ and $V(x,t) = \dfrac{p(S,M,t)}{S}$. The pricing formulation of $V(x,t)$ is given by

$$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2}\frac{\partial^2 V}{\partial x^2} + \left(q - r - \frac{\sigma^2}{2}\right)\frac{\partial V}{\partial x} - qV = 0, \quad x > 0,\ 0 < t < T.$$

The final and boundary conditions are

$$V(x,T) = e^x - 1 \quad \text{and} \quad \frac{\partial V}{\partial x}(0,t) = 0,$$

respectively. By writing $\alpha = \dfrac{1}{2} + \dfrac{\Delta x}{2}\left(\dfrac{r-q}{\sigma^2} + \dfrac{1}{2}\right)$, show that the binomial scheme takes the form

$$V_j^n = \frac{1}{1+q\Delta t}\left[\alpha V_{j-1}^{n+1} + (1-\alpha)V_{j+1}^{n+1}\right], \quad j \geq 1.$$

Suppose the boundary condition at $x = 0$ is approximated by

$$V_{-1}^{n+1} = V_0^{n+1},$$

show that the numerical boundary value is given by

$$V_0^n = \frac{1}{1+q\Delta t}\left[\alpha V_0^{n+1} + (1-\alpha)V_1^{n+1}\right].$$

5. Suppose we use the FTCS scheme to solve the Black-Scholes equation so that

$$\frac{V_j^{n+1} - V_j^n}{\Delta\tau} = \frac{\sigma^2}{2}S_j^2\frac{V_{j+1}^n - 2V_j^n + V_{j-1}^n}{\Delta S^2} + rS_j\frac{V_{j+1}^n - V_{j-1}^n}{2\Delta S} - rV_j^n.$$

Show that the sufficient conditions for non-appearance of spurious oscillations in the numerical scheme are given by

$$\Delta S < \frac{\sigma^2 S_i}{r} \quad \text{and} \quad \Delta\tau < \frac{1}{r + \frac{\sigma^2 S_j^2}{\Delta S^2}}.$$

6. A sequential barrier option has two-sided barriers. Unlike the usual double barrier options, the order of breaching of the barrier is specified. The second barrier is activated only after the first barrier has been hit earlier, and the option is knocked out only if both barriers have been hit in the pre-specified order. Construct the explicit finite difference scheme for pricing this sequential barrier option under the Black-Scholes pricing framework.

7. The penalty method is characterized by the replacement of the linear complementarity formulation of the American option model by appending a non-linear penalty term in the Black-Scholes equation (Forsyth and Vetzal, 2002). Let $h(S)$ denote the exercise payoff of an American option. The non-linear penalty term takes the form $\rho \max(h-V, 0)$, where ρ is a positive penalty parameter and $V(S,\tau)$ is the option price function. It can be shown that when $\rho \to \infty$, the solution of the following equation

$$\frac{\partial V}{\partial \tau} = \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} + (r-q)S\frac{\partial V}{\partial S} - rV + \rho \max(h-V, 0)$$

gives the solution of the American option price function. Discuss the construction of the Crank-Nicolson scheme for solving the above non-linear differential equation. Can we apply the Thomas algorithm to solve for the numerical option values at the nodes at the new time level?

8. Show how to use the inverse transform method to generate the exponential distribution with mean θ, whose cumulative distribution function is

$$F(x) = 1 - e^{-x/\theta}, \quad x \geq 0.$$

9. The correlation matrix between 3 unit-variance correlated normal variables ϵ_1, ϵ_2 and ϵ_3 is given by

$$\Sigma = \begin{pmatrix} 1 & 0.6 & 0.5 \\ 0.6 & 1 & 0.7 \\ 0.5 & 0.7 & 1 \end{pmatrix}.$$

Suppose 3 uncorrelated standard normal variables x_1, x_2 and x_3 have been generated. How to generate the 3 unit-variance correlated normal variables using a linear transformation between the two set of random variables?

10. Considering the antithetic variates method, explain why

$$\text{var}\left(\frac{c_i + \widetilde{c}_i}{2}\right) = \frac{1}{2}\left[\text{var}(c_i) + \text{cov}(c_i, \widetilde{c}_i)\right].$$

The amount of computational work to generate $\overline{c}_{AV} = (\widehat{c}+\widetilde{c})/2$ is about twice the work to generate $\widetilde{c}$. Show that the antithetic variates method improves computational efficiency provided that

$$\text{cov}(c_i, \widetilde{c}_i) < 0.$$

Explain why the above negative correlation property is in general valid.

11. By performing transformation of two independent standard normal distributed random variables $Z_i \sim N(0,1), i = 1, 2$, the two new random variables are obtained by

$$\Delta\widehat{W} = Z_1\sqrt{\Delta t}, \quad \Delta\widehat{Y} = \frac{1}{2}\sqrt{\Delta t}\left(Z_1 + \frac{1}{\sqrt{3}}Z_2\right).$$

Show that $\Delta\widehat{W}$ and $\Delta\widehat{Y}$ have their respective moment as given by

$$E[\Delta\widehat{Y}] = 0, E[\Delta\widehat{Y}^2] = \frac{\Delta t}{3} \text{ and } E[\Delta\widehat{Y}\Delta\widehat{W}] = \frac{\Delta t}{2}.$$

Find the correlation coefficient between $\Delta\widehat{Y}$ and $\Delta\widehat{W}$.

12. For a Wiener process W_t, consider the Brownian bridge defined by

$$X_t = W_t - \frac{t}{T} W_T \quad \text{for } 0 \leq t \leq T.$$

Calculate $\mathrm{var}(X_t)$ and show that

$$\sqrt{t\left(1 - \frac{t}{T}\right)} Z \quad \text{with } Z \sim N(0,1)$$

is a realization of X_t.

13. If we want to compute $\mathrm{E}[X]$, we should try to compute as much as possible exactly and should only compute that part by Monte Carlo simulation that we cannot avoid. If we know a random variable Y which is close to X and for which we can compute $\mathrm{E}[Y]$ exactly, then this random variable can be chosen as a control variable. We consider the relation

$$\mathrm{E}[X] = \mathrm{E}[X - Y] + \mathrm{E}[Y],$$

which motivates the following control variate Monte Carlo estimator

$$\bar{X}_Y = \frac{1}{N} \sum_{i=1}^{N} (X_i - Y_i) + \mathrm{E}[Y]$$

for $\mathrm{E}[X]$ with X_i, Y_i being independent copies of X and Y.

(a) Explain why a reduction of the variance for the control variate estimator compared to the crude one if we have

$$2\mathrm{cov}(X, Y) > \mathrm{var}(Y).$$

(b) Find the confidence interval for the control variate Monte Carlo estimator.