MAFS5250 - Computational Methods for Pricing Structured Products

Solution to Homework Two

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1. Expand $f(x_0 - 2\Delta x)$ and $f(x_0 - \Delta x)$ at x_0 into Taylor series, where

$$\begin{aligned} f(x_0 - 2\Delta x) &= f(x_0) - 2\Delta x f'(x_0) + 4\Delta x^2 f''(x_0) + O(\Delta x^3) \\ f(x_0 - \Delta x) &= f(x_0) - \Delta x f'(x_0) + \Delta x^2 f''(x_0) + O(\Delta x^3). \end{aligned}$$

We determine α_{-2}, α_{-1} and α_0 such that

$$\alpha_{-2} f(x_0 - 2\Delta x) + \alpha_{-1} f(x_0 - \Delta x) + \alpha_0 f(x_0) = f'(x_0) + O(\Delta x^2).$$

Collecting like terms, we obtain

$$\begin{aligned} &(\alpha_{-2} + \alpha_{-1} + \alpha_0) f(x_0) + (-2\alpha_{-2} - \alpha_{-1})\Delta x f'(x_0) \\ &+ (4\alpha_{-2} + \alpha_{-1})\Delta x^2 f''(x_0) = f'(x_0) + O(\Delta x^2). \end{aligned}$$

The corresponding linear system of equations for α_{-2}, α_{-1} and α_0 is

$$\begin{pmatrix} 1 & 1 & 1 \\ -2 & -1 & 0 \\ 4 & 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha_{-2} \\ \alpha_{-1} \\ \alpha_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1/\Delta x \\ 0 \end{pmatrix}.$$

The solution of the system gives $\alpha_{-2} = \dfrac{1}{2\Delta x}, \alpha_{-1} = -\dfrac{2}{\Delta x}$ and $\alpha_0 = \dfrac{3}{2\Delta x}$. The corresponding finite difference formula is the one-sided backward difference formula for the first order derivative, namely

$$f'(x_0) \approx \frac{f(x_0 - 2\Delta x) - 4f(x_0 - \Delta x) + 3f(x_0)}{2\Delta x}.$$

By changing Δx to $-\Delta x$, we deduce that the one-sided forward difference formula for the first order derivative is given by

$$f'(x_0) \approx \frac{-f(x_0 + 2\Delta x) + 4f(x_0 + \Delta x) - 3f(x_0)}{2\Delta x}.$$

2. Local truncation error of the Crank-Nicolson scheme

$$\begin{aligned} = \; & \frac{V(j\Delta x, (n+1)\Delta\tau) - V(j\Delta x, n\Delta\tau)}{\Delta\tau} \\ & - \frac{\sigma^2}{4} \left[\frac{V((j+1)\Delta x, (n+1)\Delta\tau) - 2V(j\Delta x, (n+1)\Delta\tau) + V((j-1)\Delta x, (n+1)\Delta\tau)}{\Delta x^2} \right. \\ & \qquad \left. + \frac{V((j+1)\Delta x, n\Delta\tau) - 2V(j\Delta x, n\Delta\tau) + V((j-1)\Delta x, n\Delta\tau))}{\Delta x^2} \right] \\ & - \frac{1}{2}\left(r - \frac{\sigma^2}{2}\right) \left[\frac{V((j+1)\Delta x, (n+1)\Delta\tau) - V((j-1)\Delta x, (n+1)\Delta\tau)}{2\Delta x} \right. \\ & \qquad \left. + \frac{V((j+1)\Delta x, n\Delta\tau) - V((j-1)\Delta x, n\Delta\tau)}{2\Delta x} \right] \\ & + \frac{r}{2}\left[V(j\Delta x, (n+1)\Delta\tau) + V(j\Delta x, n\Delta\tau) \right]. \end{aligned}$$

Expanding each term using the Taylor series expansion at the intermediate time level $\left(j\Delta x, \left(n+\frac{1}{2}\right)\Delta\tau\right)$, we obtain

$$\begin{aligned}
&\frac{V(j\Delta x,(n+1)\tau) - V(j\Delta x, n\Delta\tau)}{\Delta\tau} \\
= \;& \left\{\left[V + \frac{\partial V}{\partial\tau}\frac{\Delta\tau}{2} + \frac{1}{2}\frac{\partial^2 V}{\partial\tau^2}\left(\frac{\Delta\tau}{2}\right)^2 + \frac{1}{6}\frac{\partial^3 V}{\partial\tau^3}\left(\frac{\Delta\tau}{2}\right)^3 + \cdots\right]\right. \\
- \;& \left.\left[V - \frac{\partial V}{\partial\tau}\frac{\Delta\tau}{2} + \frac{1}{2}\frac{\partial^2 V}{\partial\tau^2}\left(\frac{\Delta\tau}{2}\right)^2 - \frac{1}{6}\frac{\partial^3 V}{\partial\tau^3}\left(\frac{\Delta\tau}{2}\right)^3 + \cdots\right]\right\} \Big/ \Delta\tau \\
= \;& \frac{\partial V}{\partial\tau} + \frac{1}{24}\frac{\partial^3 V}{\partial\tau^3}(\Delta\tau)^2 + O(\Delta\tau^4).
\end{aligned}$$

Here, we adopt the convention that any derivative with no argument specified would implicitly implies that the derivative is evaluated at $\left(j\Delta x, \left(n+\frac{1}{2}\right)\Delta\tau\right)$. Note that

$$\begin{aligned}
&[V((j+1)\Delta x, n\Delta\tau) - 2V(j\Delta x, n\Delta\tau) + V((j-1)\Delta x, n\Delta\tau)]/(\Delta x)^2 \\
= \;& \frac{\partial^2 V}{\partial x^2}(j\Delta x, n\Delta\tau) + \frac{\Delta x^2}{12}\frac{\partial^4 V}{\partial x^4}(j\Delta x, n\Delta\tau) + O(\Delta x^4) \\
= \;& \frac{\partial^2 V}{\partial x^2} - \frac{\Delta\tau}{2}\frac{\partial^3 V}{\partial x^2\partial\tau} + \frac{1}{2}\left(\frac{\Delta\tau}{2}\right)^2\frac{\partial^4 V}{\partial x^2\partial\tau^2} + \cdots \\
+ \;& \frac{\Delta x^2}{12}\left[\frac{\partial^4 V}{\partial x^4} - \frac{\Delta\tau}{2}\frac{\partial^5 V}{\partial x^4\partial\tau} + \frac{1}{2}\left(\frac{\Delta\tau}{2}\right)^2\frac{\partial^6 V}{\partial x^4\partial\tau^2} + \cdots\right] + O(\Delta x^4),
\end{aligned}$$

and

$$\begin{aligned}
&[V((j+1)\Delta x, (n+1)\Delta\tau) - 2V(j\Delta x, (n+1)\Delta\tau) + V((j-1)\Delta x, (n+1)\Delta\tau)]/(\Delta x)^2 \\
= \;& \frac{\partial^2 V}{\partial x^2} + \frac{\Delta\tau}{2}\frac{\partial^3 V}{\partial x^2\partial\tau} + \frac{1}{2}\left(\frac{\Delta\tau}{2}\right)^2\frac{\partial^4 V}{\partial x^2\partial\tau^2} + \cdots \\
+ \;& \frac{\Delta x^2}{12}\left[\frac{\partial^2 V}{\partial x^4} + \frac{\Delta\tau}{2}\frac{\partial^5 V}{\partial x^4\partial\tau} + \frac{1}{2}\left(\frac{\Delta\tau}{2}\right)^2\frac{\partial^6 V}{\partial x^4\partial\tau^2} + \cdots\right] + O(\Delta x^4).
\end{aligned}$$

Combining the results, we have

$$\begin{aligned}
&\frac{\sigma^2}{4}\left[\frac{V((j+1)\Delta x, (n+1)\Delta\tau) - 2V(j\Delta x, (n+1)\Delta\tau) + V((j-1)\Delta x, (n+1)\Delta\tau)}{\Delta x^2}\right. \\
&\quad \left. + \frac{V((j+1)\Delta x, n\Delta\tau) - 2V(j\Delta x, n\Delta\tau) + V((j-1)\Delta x, n\Delta\tau)}{\Delta x^2}\right] \\
= \;& \frac{\sigma^2}{2}\left[\frac{\partial^2 V}{\partial x^2} + O(\Delta\tau^2) + O(\Delta x^2)\right].
\end{aligned}$$

Similarly, we obtain

$$
\begin{aligned}
&\frac{1}{2}\left(r-\frac{\sigma^2}{2}\right)\left[\frac{V((j+1)\Delta x,(n+1)\Delta\tau)-V((j-1)\Delta x,(n+1)\Delta\tau)}{2\Delta x}\right.\\
&\qquad\qquad\left.+\ \frac{V((j+1)\Delta x,n\Delta\tau)-V((j-1)\Delta x,n\Delta\tau)}{2\Delta x}\right]\\
=\ &\frac{1}{2}\left(r-\frac{\sigma^2}{2}\right)\left[\frac{\partial V}{\partial x}+O(\Delta\tau^2)+O(\Delta x^2)\right],
\end{aligned}
$$

and

$$
\begin{aligned}
&r\left[V(j\Delta x,(n+1)\Delta\tau)+V(j\Delta x,n\Delta\tau)\right]\\
=\ &rV+O(\Delta\tau)^2.
\end{aligned}
$$

By noting that $V\left(j\Delta x,\left(n+\frac{1}{2}\right)\Delta\tau\right)$ satisfies the Black-Scholes equation, we obtain the local truncation error of the Crank-Nicholson scheme $=O(\Delta\tau^2)+O(\Delta x^2)$.

3. Instead of imposing artificial boundary conditions for the bond price at $r=r_{\min}$ and $r=r_{\max}$, we enforce the condition that the option values along the boundary nodes of the computational domain remain to be governed by the bond price equation. We apply backward difference operators to approximate the continuous differential operators:

$$
\begin{aligned}
\left.\frac{\partial^2 B}{\partial r^2}\right|_{r_{\max}=(N+1)\Delta r} &\approx \frac{B_{N+1}-5B_N+4B_{N-1}-B_{N-2}}{\Delta r^2},\\
\left.\frac{\partial B}{\partial r}\right|_{r_{\max}=(N+1)\Delta r} &\approx \frac{3B_{N+1}-4B_N+B_{N-1}}{2\Delta r}.
\end{aligned}
$$

The corresponding explicit FTCS becomes

$$
\begin{aligned}
\frac{B_{N+1}^{n+1}-B_{N+1}^n}{\Delta\tau} =\ &\frac{\sigma^2 r_{\max}}{2}\frac{B_{N+1}^n-5B_N^n+4B_{N-1}^n-B_{N-2}^n}{\Delta r^2}\\
&+\alpha(\beta-r_{\max})\frac{3B_{N+1}^n-4B_N^n+B_{N-1}^n}{2\Delta r}-r_{\max}B_{N+1}^n.
\end{aligned}
$$

In a similar manner, we now apply the forward difference operators to approximate the differential operators:

$$
\begin{aligned}
\left.\frac{\partial^2 B}{\partial r^2}\right|_{r_{\min}} &\approx \frac{B_0-5B_1+4B_2-B_3}{\Delta r^2},\\
\left.\frac{\partial B}{\partial r}\right|_{r_{\min}} &\approx \frac{-3B_0+4B_1-B_2}{2\Delta r}.
\end{aligned}
$$

We obtain the following explicit FTCS at $j=0$:

$$
\begin{aligned}
\frac{B_0^{n+1}-B_0^n}{\Delta\tau} =&\frac{\sigma^2 r_{\min}}{2}\frac{B_0^n-5B_1^n+4B_2^n-B_3^n}{\Delta r^2}\\
&+\alpha(\beta-r_{\min})\frac{-3B_0^n+4B_1^n-B_2^n}{2\Delta r}-r_{\min}B_0^n.
\end{aligned}
$$

4. For the given pricing formulation of the floating strike lookback put, the binomial parameters are determined by equating the mean and variance of the discrete random walk and the continuous price process up to $O(\Delta t)$:

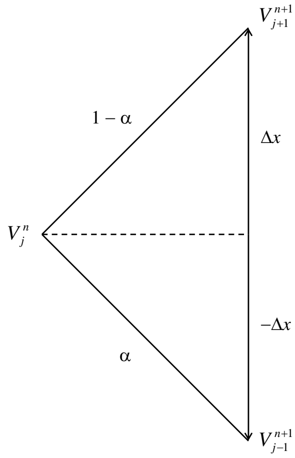
$$(1-\alpha)\Delta x - \alpha\Delta x = \left(q - r - \frac{\sigma^2}{2}\right)\Delta t,$$
$$(1-\alpha)\Delta x^2 + \alpha\Delta x^2 = \sigma^2\Delta t.$$

We obtain $\Delta x = \sigma\sqrt{\Delta t}$ and

$$(1-2\alpha)\Delta x = \left(q - r - \frac{\sigma^2}{2}\right)\Delta t = \left(q - r - \frac{\sigma^2}{2}\right)\frac{\Delta x^2}{\sigma^2}.$$

This gives

$$\alpha = \frac{1}{2} + \frac{\Delta x}{2}\left(\frac{r-q}{\sigma^2} + \frac{1}{2}\right).$$



Here, α denotes the probability of down-move in the binomial tree. The binomial scheme takes the form:

$$V_j^n = \alpha V_{j-1}^{n+1} + (1-\alpha)V_{j+1}^{n+1} - qV_j^n\Delta t,$$
$$V_j^n = \frac{1}{1+q\Delta t}\left[\alpha V_{j-1}^{n+1} + (1-\alpha)V_{j+1}^{n+1}\right], \quad j \geq 1.$$

The term $-qV$ in the governing equation for V is similar to the discount term $-rV$ in the usual Black-Scholes equation. This gives rise to the discount factor $\frac{1}{1+q\Delta t}$ in the above binomial pricing formula. When $j = 0$, the binomial formula involves the grid point at $j = -1$, which is outside the computational domain. We then approximate the Neumann boundary condition: $\frac{\partial V}{\partial x}(0,t) = 0$ using the one-sided finite difference formula, that is, $V_{-1}^{n+1} = V_0^{n+1}$. The numerical boundary value is given by

$$V_0^n = \frac{1}{1+q\Delta t}\left[\alpha V_0^{n+1} + (1-\alpha)V_1^{n+1}\right].$$

This binomial scheme is similar to the Cheuk-Vorst scheme. To complete the construction of the binomial scheme, we adopt the terminal payoff condition:

$$V_j^N = e^{j\Delta x} - 1, \; j \geq 0.$$

5. We write the FTCS scheme in the form of the standard 2-level-4-point scheme:

$$\begin{aligned} V_j^{n+1} &= \left(\frac{\sigma^2}{2}S_j^2\frac{\Delta\tau}{\Delta S^2} + rS_j\frac{\Delta\tau}{2\Delta S}\right)V_{j+1}^n \\ &+ \left(1 - \sigma^2 S_j^2\frac{\Delta\tau}{\Delta S^2} - r\Delta\tau\right)V_j^n \\ &+ \left(\frac{\sigma^2}{2}S_j^2\frac{\Delta\tau}{\Delta S^2} - rS_j\frac{\Delta\tau}{2\Delta S}\right)V_{j-1}^n. \end{aligned}$$

In order to avoid spurious oscillations, it suffices to have all the coefficients to be positive. That is,

$$\begin{aligned} &\frac{\sigma^2}{2}S_j^2\frac{\Delta\tau}{\Delta S^2} + rS_j\frac{\Delta\tau}{2\Delta S} > 0 \\ &1 - \sigma^2 S_j^2\frac{\Delta\tau}{\Delta S^2} - r\Delta\tau > 0 \\ &\frac{\sigma^2}{2}S_j^2\frac{\Delta\tau}{\Delta S^2} - rS_j\frac{\Delta\tau}{2\Delta S} > 0. \end{aligned}$$

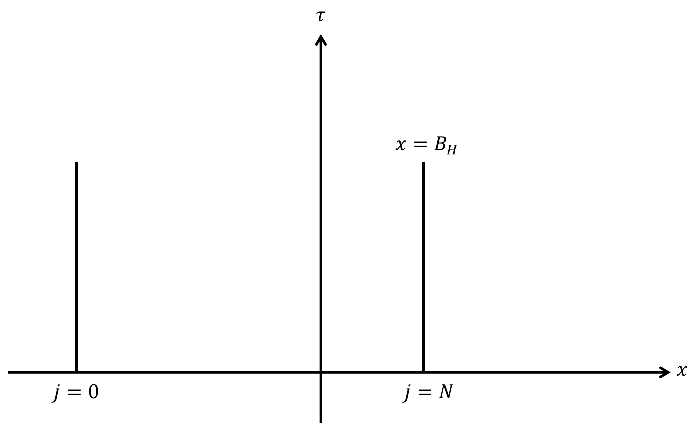
The first inequality is always satified. The last two inequalities are satisfied provided that

$$\Delta S < \frac{\sigma^2 S_j}{r} \quad \text{and} \quad \Delta\tau < \frac{1}{r + \frac{\sigma^2 S_j^2}{\Delta S^2}}.$$

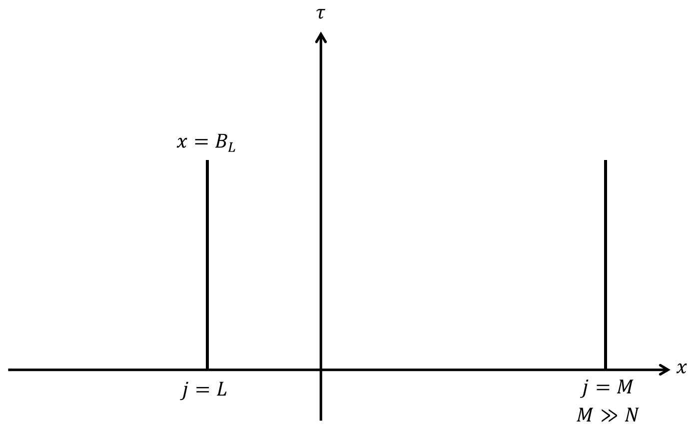
6. Let the first barrier be an up-stream barrier B_H while the second barrier be the down-stream barrier B_L. The sequential barrier option is equivalent to a one-sided up-and-out barrier with a rebate paid upon knock-out. The rebate is a one-sided down-and-out barrier option. We assume a call payoff of the sequential barrier option. Let B_j^n and R_j^n denote the numerical option value of the sequential barrier option and the rebate barrier option at the $(j, n)^{\text{th}}$ node, respectively.

Design of the computational domain

For the sequential barrier option, we set the right boundary to coincide with the upstream barrier B_H. The left boundary must lie sufficiently far to the left end.



For the down-and-out barrier option (treated as the rebate upon breaching the up-barrier in the sequential barrier option), we set the left boundary to coincide with the down-stream barrier B_L. The right boundary must lie sufficiently far to the right end. Here, we choose M such that $M >> N$.



Boundary conditions

(i) At $j = 0$, the sequential barrier option is deep out-of-the-money so that the option value is close to zero. We set

$$B_0^n = 0, \quad \text{for all } n.$$

(ii) At $j = N$, which corresponds to the up-barrier B_H, the sequential barrier option is apparently "knocked" out, receiving the down-and-out barrier option as the rebate. Therefore, we have

$$B_N^n = R_N^n, \quad \text{for all } n.$$

(iii) At $j = M$, the (rebate) down-and-out barrier option is deep in-the-money so that it is almost like a forward contract. Accordingly, the second order derivative of the price function with respect to the stock price is close to zero. That is,

$$R_M^n = \frac{5R_{M-1}^n - 4R_{M-2}^n + R_{M-3}^n}{2}, \quad \text{for all } n.$$

(iv) At $j = L$, the rebate barrier option is knocked out at $x = B_L$ with zero value. That is

$$R_L^n = 0, \quad \text{for all } n.$$

Both price functions of the sequential barrier option and the rebate barrier option satisfy the Black-Scholes equation. We start with computation of the rebate barrier option, then continue with the sequential barrier option. The explicit finite difference schemes in both option calculations take an identical form:

$$\begin{aligned} R_j^{n+1} &= \left[\frac{\mu + c}{2} R_{j+1}^n + (1-\mu) R_j^n + \frac{\mu - c}{2} R_{j-1}^n\right] e^{-r\Delta\tau}, \\ &\qquad j = L+1, \cdots, M-1, \quad n = 0, 1, 2, \cdots. \\ B_j^{n+1} &= \left[\frac{\mu + c}{2} B_{j+1}^n + (1-\mu) B_j^n + \frac{\mu - c}{2} B_{j-1}^n\right] e^{-r\Delta\tau}, \\ &\qquad j = 1, \cdots, N-1, \quad n = 0, 1, 2, \cdots, \end{aligned}$$

where $\mu = \frac{\sigma^2 \Delta\tau}{\Delta x^2}$ and $c = \left(r - q - \frac{\sigma^2}{2}\right) \frac{\Delta\tau}{\Delta x}$.

7. In the continuation region where $V > h$, the penalty term vanishes. However, if the option remains unexercised when it should be optimally exercised (in the region of optimal stopping), we have $V < h$. As a result, the penalty term becomes highly dominant since the penalty parameter ρ is typically chosen to be a very large positive parameter. Intuitively, we observe that V should assume the exercise payoff h in the limit $\rho \to \infty$.

   Suppose we use S as the state variable and let V_j^n denote the numerical option value at the node $(j\Delta S, n\Delta\tau)$, where $S_j = j\Delta S + S_0$ and S_0 is the initial stock price. The Crank-Nicolson scheme takes the form:

$$\begin{aligned}\frac{V_j^{n+1} - V_j^n}{\Delta\tau} &= \frac{\sigma^2 S_j^2}{2}\left[\frac{V_{j+1}^{n+1} - 2V_j^{n+1} + V_{j-1}^{n+1}}{\Delta S^2} + \frac{V_{j+1}^n - 2V_j^n + V_{j-1}^n}{\Delta S^2}\right] \\ &\quad + (r-q)S_j\left[\frac{V_{j+1}^{n+1} - V_{j-1}^{n+1}}{2\Delta S} + \frac{V_{j+1}^n - V_{j-1}^n}{2\Delta S}\right] \\ &\quad - r\frac{V_j^{n+1} + V_j^n}{2} + \rho\max\{h(S_j) - \frac{V_j^{n+1} + V_j^n}{2}, 0\}.\end{aligned}$$

   Due to the non-linearity exhibited in the penalty term, we cannot apply the Thomas algorithm. Indeed, we need to use an iteration method to solve the resulting non-linear algebraic system of equations.

8. The exponential distribution is the distribution of the random time between two successive jumps of a Poisson process with rate $1/\theta$. Inverting the exponential distribution gives

$$X = -\theta\ln(1 - U).$$

   This can be implemented as

$$X = -\theta\ln(U)$$

   since U and $1 - U$ have the same distribution. As a summary, with $U \sim \mathbb{U}(0,1)$, we have $X = -\theta\ln U \sim \exp(\theta)$.

9. Write

$$\begin{pmatrix}\epsilon_1\\ \epsilon_2\\ \epsilon_3\end{pmatrix} = \begin{pmatrix}\alpha_{11} & 0 & 0\\ \alpha_{21} & \alpha_{22} & 0\\ \alpha_{31} & \alpha_{32} & \alpha_{33}\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix}$$

   where

$$M = \begin{pmatrix}\alpha_{11} & 0 & 0\\ \alpha_{21} & \alpha_{22} & 0\\ \alpha_{31} & \alpha_{32} & \alpha_{33}\end{pmatrix}.$$

   The entries in M are determined by the relation:

$$MM^T = \Sigma.$$

   This yields

$$MM^T = \begin{pmatrix}\alpha_{11}^2 & \alpha_{11}\alpha_{21} & \alpha_{31}\alpha_{11}\\ \alpha_{11}\alpha_{21} & \alpha_{21}^2 + \alpha_{22}^2 & \alpha_{21}\alpha_{31} + \alpha_{22}\alpha_{32}\\ \alpha_{31}\alpha_{11} & \alpha_{21}\alpha_{31} + \alpha_{22}\alpha_{32} & \alpha_{31}^2 + \alpha_{32}^2 + \alpha_{33}^2\end{pmatrix} = \begin{pmatrix}1 & 0.6 & 0.5\\ 0.6 & 1 & 0.7\\ 0.5 & 0.7 & 1\end{pmatrix},$$

and accordingly,

$$\begin{aligned}&\alpha_{11}^2 = 1, \quad \alpha_{11}\alpha_{21} = 0.6, \quad \alpha_{31}\alpha_{11} = 0.5\\ &\alpha_{21}^2 + \alpha_{22}^2 = 1, \quad \alpha_{21}\alpha_{31} + \alpha_{22}\alpha_{32} = 0.7, \quad \alpha_{31}^2 + \alpha_{32}^2 + \alpha_{33}^2 = 1.\end{aligned}$$

We take $\alpha_{11} = 1$ (we choose the positive root without loss of generality), then

$$\alpha_{21} = 0.6, \quad \alpha_{22} = \sqrt{1 - 0.6^2} = 0.8 \quad \text{(choosing the positive root)}.$$

Also, $\alpha_{31} = 0.5$, then

$$\begin{aligned}&(0.6)(0.5) + 0.8\alpha_{32} = 0.7 \text{ so that}\\ &\alpha_{32} = \frac{0.7 - 0.3}{0.8} = 0.5.\end{aligned}$$

Lastly, $\alpha_{33} = \sqrt{1 - \alpha_{31}^2 - \alpha_{32}^2} = 1/\sqrt{2}$ (choosing the positive root). Hence

$$\begin{pmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0.6 & 0.8 & 0 \\ 0.5 & 0.5 & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

10. Since $\overline{c}_{AV} = \dfrac{c + \widetilde{c}}{2}$ so that

$$\operatorname{var}(\overline{c}_{AV}) = \operatorname{var}\left(\frac{c + \widetilde{c}}{2}\right) = \frac{1}{4}\operatorname{var}(c) + \frac{1}{4}\operatorname{var}(\widetilde{c}) + \frac{1}{2}\operatorname{cov}(c, \widetilde{c}).$$

As $\widetilde{c}$ is generated using $-\epsilon^{(i)}$ while c is generated using $\epsilon^{(i)}$, we expect to have

$$\operatorname{var}(c) = \operatorname{var}(\widetilde{c})$$

so that

$$\operatorname{var}(\overline{c}_{AV}) = \frac{1}{2}\operatorname{var}(c) + \frac{1}{2}\operatorname{cov}(c, \widetilde{c}). \qquad (A)$$

Now, we apply the following criterion of determining the trade-off between computational work units and variances: $\sigma_1^2/\sigma_2^2 < W_2/W_1$. Since the amount of computational work to compute $\overline{c}_{AV}$ is about twice that of c, the control variate is preferred in terms of computational efficiency provided that

$$\operatorname{var}(\overline{c}_{AV}) < \frac{\operatorname{var}(c)}{2}. \qquad (B)$$

Based on Eq. (A), Ineq (B) is equivalent to

$$\operatorname{cov}(c, \widetilde{c}) < 0.$$

Since we have chosen $-\epsilon^{(i)}$ for computing $\widetilde{c}_i$, the chances are high that c_i and $\widetilde{c}_i$ are negatively correlated. Hence, the antithetic variates method improves computational efficiency.

11. $E[\Delta\widehat{Y}] = E\left[\frac{\sqrt{\Delta t}}{2}\left(Z_1 + \frac{1}{\sqrt{3}}Z_2\right)\right] = 0;$

$E[\Delta\widehat{Y}^2] = \frac{\Delta t}{4}E\left[\left(Z_1 + \frac{1}{\sqrt{3}}Z_2\right)^2\right] = \frac{\Delta t}{4}\left(1 + \frac{1}{3}\right) = \frac{\Delta t}{3};$

$E[\Delta\widehat{Y}\Delta\widehat{W}] = \frac{\Delta t}{2}E\left[Z_1\left(Z_1 + \frac{1}{\sqrt{3}}Z_3\right)\right] = \frac{\Delta t}{2}.$

Correlation coefficient between $\Delta\widehat{Y}$ and $\Delta\widehat{W}$
$= \frac{E[\Delta\widehat{Y}\Delta\widehat{W}] - E[\Delta\widehat{Y}]E[\Delta\widehat{W}]}{\sqrt{\text{var}(\Delta\widehat{Y})}\sqrt{\text{var}(\Delta\widehat{W})}} = \frac{\frac{\Delta t}{2}}{\sqrt{\frac{\Delta t}{3}}\sqrt{\Delta t}} = \frac{\sqrt{3}}{2}.$

12. We consider

$$\begin{aligned}\text{var}(X_t) &= E\left[\left(W_t - \frac{t}{T}W_T\right)^2\right] - E\left[W_t - \frac{t}{T}W_T\right]^2 \\ &= E[W_t^2] - \frac{2t}{T}E[W_tW_T] + \frac{t^2}{T^2}E[W_T^2] \\ &= t - \frac{2t}{T}E[W_t(W_T - W_t) + W_t^2] + \frac{t^2}{T^2}T.\end{aligned}$$

Since the Brownian increments over non-overlapping time intervals are independent, so

$$E[W_t(W_T - W_t)] = 0.$$

Hence

$$\text{var}(X_t) = t - \frac{2t}{T}t + \frac{t^2}{T^2}T = t - \frac{t^2}{T} = t\left(1 - \frac{t}{T}\right).$$

Let $Y_t = \sqrt{t\left(1 - \frac{t}{T}\right)}Z$, $Z \sim N(0,1)$, we have

$$E[Y_t] = E[X_t] = 0 \text{ and } \text{var}(Y_t) = t\left(1 - \frac{t}{T}\right) = \text{var}(X_t).$$

Also, $Y_0 = X_0 = 0$ and $Y_T - X_T = 0$; so $\sqrt{t\left(1 - \frac{t}{T}\right)}\,Z$, with $Z \sim N(0,1)$, is a realization of X_t.

13. (a) Note that

$$\begin{aligned}\text{var}(\bar{X}_Y) &= \frac{1}{N}\text{var}(X - Y) \\ &= \frac{1}{N}\left[\text{var}(X) + \text{var}(Y) - 2\text{cov}(X,Y)\right].\end{aligned}$$

We achieve a reduction of the variance for the control variate estimator provided that $\text{var}(X) > \text{var}(X - Y)$, which is equivalent to $2\text{cov}(X,Y) > \text{var}(Y)$. If Y is very close to X, then we have the satisfaction of the inequality. This would imply the significant reduction of the variance of the crude Monte Carlo estimator by using its control variate variable.

(b) The efficiency of the control variate hinges on the fact that we can directly simulate the difference $X_i - Y_i$ as one random variable. It is supposed that we know its exact distribution, so one needs to generate one random variable.

To obtain the confidence interval for the control variate Monte Carlo estimator, one starts with that of the crude Monte Carlo estimator for $\mathrm{E}[X - Y]$ and add $\mathrm{E}[Y]$ to the interval. For example, we obtain an approximate 95%-confidence interval by

$$\left[\bar{X}_Y - 1.96\frac{\hat{\sigma}_{X-Y}}{\sqrt{N}},\ \bar{X}_Y + 1.96\frac{\hat{\sigma}_{X-Y}}{\sqrt{N}}\right],$$

where

$$\hat{\sigma}^2_{X-Y} = \frac{1}{N-1}\sum_{i=1}^{N}\left[X_i - Y_i - \frac{1}{N}\sum_{j=1}^{N}(X_j - Y_j)\right]^2.$$