MAFS5250 – Computational Methods for Pricing Structured Products

Topic 1 – Lattice tree methods

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- Continuous limit of the binomial model
- Multiperiod extension
- Dynamic programming procedure
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1.2 Trinomial schemes

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1.1 Binomial option pricing models

From replication to risk neutral valuation

Discrete model of the dynamics of the underlying price process

Under the binomial random walk model, the asset price after one period Δt will be either uS or dS with probability q and $1 - q$, respectively.

We assume $u > 1 > d$ so that uS and dS represent the up-move and down-move of the asset price, respectively. The proportional jump parameters u and d will be related to the asset price dynamics.

Let R denote the growth factor of riskless investment over one period so that \$1 invested in a riskfree money market account will grow to \$$R$ after one period. In order to avoid riskless arbitrage opportunities, we must have $u > R > d$.

Construction of a replicating portfolio

By buying the asset and borrowing cash (in the form of riskfree money market account) in appropriate proportions, one can replicate the position of a call.

Suppose we form a portfolio which consists of α units of asset and cash amount M in the form of riskless investment (money market account). After one period of time $\triangle t$, the value of the portfolio becomes

$$\begin{cases} \alpha uS + RM & \text{with probability } q \\ \alpha dS + RM & \text{with probability } 1-q. \end{cases}$$

We have the fortunate coincidence: 2 investment instruments: risky asset and money market account and 2 states of the world: up and down in the binomial model. Under this special scenario, the replication approach of deriving the binomial pricing formula works.

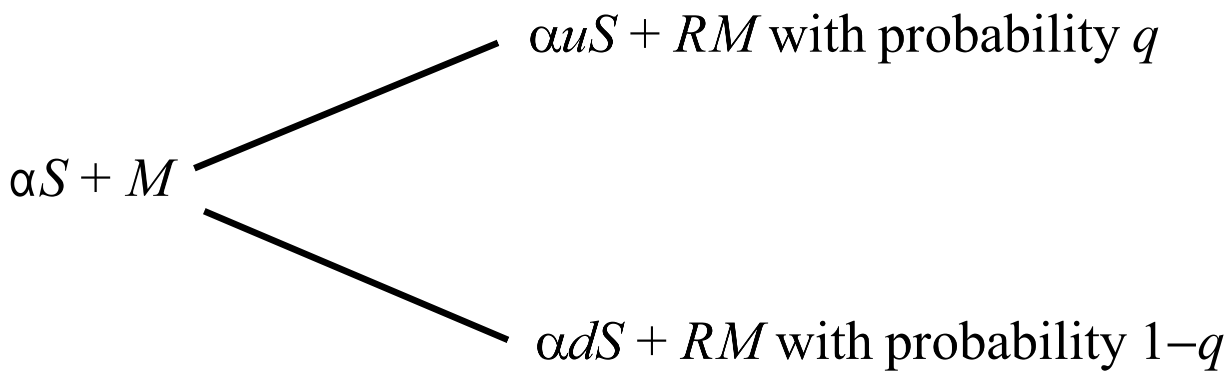
The portfolio is used to replicate the long position of a call option on a non-dividend paying asset.

As there are two possible states of the world: asset price goes up or down, the call price is dependent on the asset price, thus it is a contingent claim.

Suppose the current time is only one period $\triangle t$ prior to expiration. Let c denote the current call price, and c_u and c_d denote the call price after one period (which is the expiration time in the present context) corresponding to the up-move and down-move of the asset price, respectively.

Let X denote the strike price of the call. The payoff of the call at expiry is given by

$$\begin{cases} c_u = \max(uS - X, 0) & \text{with probability } q \\ c_d = \max(dS - X, 0) & \text{with probability } 1 - q. \end{cases}$$



Evolution of the asset price S and the money market account M after one time period under the binomial model. The risky asset value may either go up to uS or go down to dS, while the riskless investment amount M grows to RM with certainty.

Replicating procedure

The above portfolio containing the risky asset and money market account is said to replicate the long position of the call if and only if the values of the portfolio and the call option match for each possible outcome, that is,

$$\alpha u S + RM = c_u \quad \text{and} \quad \alpha d S + RM = c_d.$$

Solving the pair of equations, we obtain

$$\alpha = \frac{c_u - c_d}{(u-d)S} \geq 0, \qquad M = \frac{uc_d - dc_u}{(u-d)R} \leq 0.$$

Apparently, we have 2 states of the world that generate 2 equations via matching the outcomes. There are two unknowns α and M to be determined, so we have equal number of states and unknowns.

- The uninteresting case occurs when $c_u = c_d = 0$. This leads to $\alpha = M = 0$. If the call is surely to be out-of-the-money under the two possible states, then its present value is zero.

- Since α is always non-negative and M is always non-positive, the replicating portfolio involves buying the asset and borrowing cash in the corresponding proportions (excluding the degenerate case of $\alpha = M = 0$).
- The number of units of asset held is seen to be the ratio of the difference of call values $c_u - c_d$ to the difference of asset values $uS - dS$. This is called the hedge ratio.

The call option can be replicated by a portfolio of the two basic securities: risky asset and riskfree money market account. By invoking the law of one price, the call value is identical to the value of the replicating portfolio.

Query: Can we adopt the above replicating procedure if the discrete asset price process follows the trinomial random walk model (3 states of the world in the next move)? One has to use the risk neutral valuation principle for deriving the trinomial pricing formula, which holds under the assumption of absence of arbitrage. Recall that absence of arbitrage $\Leftrightarrow$ existence of risk neutral measure under discrete models.

Binomial option pricing formula

The current value of the call is given by the current value of the replicating portfolio, that is,

$$\begin{aligned} c &= \alpha S + M = \frac{\frac{R-d}{u-d}c_u + \frac{u-R}{u-d}c_d}{R} \\ &= \frac{pc_u + (1-p)c_d}{R} \qquad \text{where} \quad p = \frac{R-d}{u-d}. \end{aligned}$$

Note that the probability q, which is the subjective probability of up-move or down-move of the asset price, does not appear in the call value.

The parameter p can be shown to be $0 < p < 1$ since $u > R > d$ and so p can be interpreted as a probability.

Risk neutral pricing measure

From the relation

$$puS + (1-p)dS = \frac{R-d}{u-d}\,uS + \frac{u-R}{u-d}\,dS = RS,$$

one can interpret the result as follows: the expected rate of returns on the asset with p as the probability of upside move is just equal to the riskless interest rate. In other words, we observe

$$S = \frac{1}{R}E^*[S^{\Delta t}|S],$$

where E^* is expectation under this probability measure. We may view p as the *risk neutral probability* that the asset price goes up in the next move.

Since $E^*\left[\frac{S^{\Delta t}}{R}\middle|S\right]$ equals the current asset value S, we say that the discounted asset value process is a martingale under the risk neutral pricing measure.

Discounted expectation of the terminal payoff

The call price can be interpreted as the expectation of the payoff of the call option at expiry under the risk neutral probability measure discounted at the riskless interest rate.

The binomial call value formula can be expressed as

$$c = \frac{1}{R} E^*[c^{\Delta t}|S],$$

where c denotes the call value at the current time, and $c^{\Delta t}$ denotes the random variable representing the call value one period later.

Besides applying the *principle of replication of claims*, the binomial option pricing formula can also be derived using the *riskless hedging principle* (similar to the derivation of the continuous Black-Scholes equation) or finding the *state prices* of the up-state and down-state.

Determination of the jump parameters (u and d) that respects the asset price dynamics

- The jump parameters are related to the variance of the continuous asset value process under the risk neutral measure.
- For the continuous asset price dynamics of Geometric Brownian motion under the risk neutral measure, we have

$$d \ln S_t = \left(r - \frac{\sigma^2}{2}\right) dt + \sigma \; dZ_t$$

  so that $\ln \frac{S_{t+\triangle t}}{S_t}$ becomes normally distributed with mean $\left(r - \frac{\sigma^2}{2}\right)\triangle t$ and variance $\sigma^2 \triangle t$, where r is the riskless interest rate and σ^2 is the variance rate.
- Expressed in the form of the exponentiation of a normal random variable, the mean and variance of $\frac{S_{t+\triangle t}}{S_t}$ are R and $R^2(e^{\sigma^2 \triangle t} - 1)$, respectively, where $R = e^{r\triangle t}$.

- For the one-period binomial option model under the risk neutral measure, the mean and variance of the asset price ratio $\dfrac{S_{t+\triangle t}}{S_t}$ are

$$pu + (1-p)d \quad \text{and} \quad pu^2 + (1-p)d^2 - [pu + (1-p)d]^2,$$

respectively.

- By equating the mean and variance of the asset price ratio in both the continuous and discrete models, we obtain

$$E[\frac{S^{\Delta t}}{S}] = pu + (1-p)d = R$$
$$E[(\frac{S^{\Delta t}}{S})^2] - \{E[\frac{S^{\Delta t}}{S}]\}^2 = pu^2 + (1-p)d^2 - R^2 = R^2(e^{\sigma^2 \triangle t} - 1).$$

The first equation leads to $p = \dfrac{R-d}{u-d}$, the usual risk neutral probability.

We have 3 unknowns: u, d and p, but only two equations. How to find the third condition? It is superfluous to match moments beyond the second order since the mean and variance fully specify a normal random variable.

A convenient choice of the third condition is the *tree-symmetry condition*

$$u = \frac{1}{d},$$

so that the lattice nodes associated with the binomial tree are symmetrical.

Writing $\widetilde{\sigma}^2 = R^2 e^{\sigma^2 \triangle t}$, the solution is found to be

$$u = \frac{1}{d} = \frac{\widetilde{\sigma}^2 + 1 + \sqrt{(\widetilde{\sigma}^2 + 1)^2 - 4R^2}}{2R}, \qquad p = \frac{R - d}{u - d}.$$

How to obtain a nice approximation function to u instead of using the above daunting expression?

By expanding u in Taylor series in powers of $\sqrt{\triangle t}$, we obtain

$$u = 1 + \sigma\sqrt{\triangle t} + \frac{\sigma^2}{2}\triangle t + \frac{4r^2 + 4\sigma^2 r + 3\sigma^4}{8\sigma} \triangle t^{\frac{3}{2}} + O(\triangle t^2).$$

Observe that the first three terms in the above Taylor series agree with those of $e^{\sigma\sqrt{\triangle t}}$ up to $O(\triangle t)$ term.

This suggests the judicious choice of the following set of parameter values

$$u = e^{\sigma\sqrt{\triangle t}}, \quad d = e^{-\sigma\sqrt{\triangle t}}, \qquad p = \frac{R-d}{u-d}.$$

With this new set of parameters, the variance of the price ratio $\dfrac{S_{t+\triangle t}}{S_t}$ in the continuous and discrete models agree up to $O(\triangle t)$.

We define $x_t = \ln S_t$. If we write $\Delta x = \ln S_{t+\Delta t} - \ln S_t$ as the discrete change in the log asset price over Δt, then the proportional upward jump $u = e^{\sigma\sqrt{\Delta t}}$ is equivalent to $\Delta x = \ln u = \sigma\sqrt{\Delta t}$. This is consistent with the Brownian increment Δx of the log asset price (without drift rate though) over the differential time interval Δt. Note that the increment due to drift rate $r - \dfrac{\sigma^2}{2}$ over Δt is $O(\Delta t)$ while the random Brownian increment is $O(\sqrt{\Delta t})$. The higher order term $\left(r - \dfrac{\sigma^2}{2}\right)\Delta t$ arising from the drift can be dropped compared to $O(\sqrt{\Delta t})$ term arising from Brownian diffusion.

Continuous limit of the binomial model

We consider the asymptotic limit $\triangle t \to 0$ of the binomial formula

$$c = [pc_u^{\Delta t} + (1-p)c_d^{\Delta t}]\, e^{-r\triangle t}.$$

In the continuous analog, the binomial formula can be written as

$$c(S, t - \triangle t) = [pc(uS, t) + (1-p)c(dS, t)]\, e^{-r\triangle t}.$$

Instead of choosing $c(uS, t+\Delta t)$ and $c(dS, t+\Delta t)$ in the formula, the above form is more convenient for subsequent analytic derivation since cross derivative terms do not appear in the later Taylor expansion procedure.

Assuming sufficient continuity of $c(S, t)$, we perform the Taylor expansion of the binomial scheme at (S, t) as follows:

$$
\begin{aligned}
& -c(S, t - \triangle t) + [pc(uS, t) + (1-p)c(dS, t)]e^{-r\triangle t} \\
= \; & \frac{\partial c}{\partial t}(S,t)\triangle t - \frac{1}{2}\frac{\partial^2 c}{\partial t^2}(S,t)\triangle t^2 + \cdots - (1 - e^{-r\triangle t})c(S,t) \\
& + \; e^{-r\triangle t}\left\{[p(u-1) + (1-p)(d-1)]S\frac{\partial c}{\partial S}(S,t)\right. \\
& + \; \frac{1}{2}[p(u-1)^2 + (1-p)(d-1)^2]S^2\frac{\partial^2 c}{\partial S^2}(S,t) \\
& + \; \left.\frac{1}{6}[p(u-1)^3 + (1-p)(d-1)^3]S^3\frac{\partial^3 c}{\partial S^3}(S,t) + \cdots\right\}.
\end{aligned}
$$

First, we observe that

$$
1 - e^{-r\triangle t} = r\triangle t + O(\triangle t^2),
$$

and also

$$
\begin{aligned}
e^{-r\triangle t}\,[p(u-1) + (1-p)(d-1)] &= \; r\triangle t + O(\triangle t^2), \\
e^{-r\triangle t}\,[p(u-1)^2 + (1-p)(d-1)^2] &= \sigma^2\triangle t + O(\triangle t^2), \\
e^{-r\triangle t}\,[p(u-1)^3 + (1-p)(d-1)^3] &= O(\triangle t^2).
\end{aligned}
$$

Combining the results, we obtain

$$\begin{aligned} & -c(S,t-\triangle t)+[pc(uS,t)+(1-p)c(dS,t)]\,e^{-r\triangle t} \\ = \quad & \left[\frac{\partial c}{\partial t}(S,t)+rS\frac{\partial c}{\partial S}(S,t)+\frac{\sigma^2}{2}S^2\,\frac{\partial^2 c}{\partial S^2}(S,t)-rc(S,t)\right]\triangle t+O(\triangle t^2). \end{aligned}$$

Since $c(S,t)$ satisfies the binomial formula, so we obtain

$$0=\frac{\partial c}{\partial t}(S,t)+rS\frac{\partial c}{\partial S}(S,t)+\frac{\sigma^2}{2}S^2\frac{\partial^2 c}{\partial S^2}(S,t)-rc(S,t)+O(\triangle t).$$

In the limit $\Delta t \to 0$, the binomial call value $c(S,t)$ satisfies the Black-Scholes equation.

Multiperiod extension

Let c_{uu} denote the call value at two periods beyond the current time with two consecutive upward moves of the asset price and similar notational interpretation for c_{ud} and c_{dd}. The call values c_u and c_d are related to c_{uu}, c_{ud} and c_{dd} as follows:

$$c_u = \frac{pc_{uu} + (1-p)c_{ud}}{R} \text{ and } c_d = \frac{pc_{ud} + (1-p)c_{dd}}{R}.$$

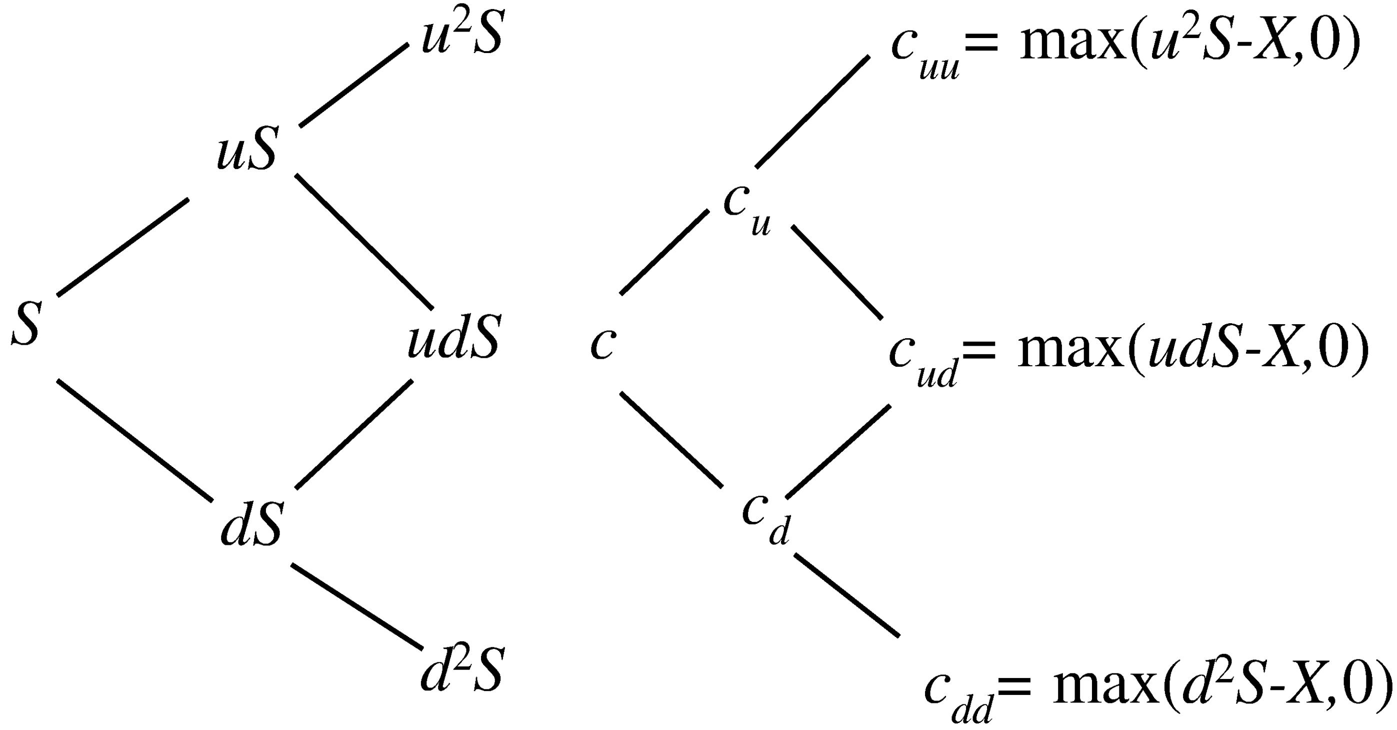
The call value at the current time which is two periods from expiry is found to be

$$c = \frac{p^2 c_{uu} + 2p(1-p)c_{ud} + (1-p)^2 c_{dd}}{R^2},$$

where the corresponding terminal payoff values are given by

$$c_{uu} = \max(u^2 S - X, 0), c_{ud} = \max(udS - X, 0), c_{dd} = \max(d^2 S - X, 0).$$

The coefficients $p^2, 2p(1-p)$ and $(1-p)^2$ represent the respective risk neutral probability of having two up jumps, one up jump and one down jump, and two down jumps in the two consecutive moves of the binomial process.



Dynamics of asset price and call price in a two-period binomial model

With n binomial steps, the risk neutral probability of having j up jumps and $n-j$ down jumps is given by $C_j^n p^j (1-p)^{n-j}$, $j = 0, 1, \ldots, n$, where $C_j^n = \frac{n!}{j!(n-j)!}$ is the binomial coefficient.

The corresponding terminal payoff when j up jumps and $n-j$ down jumps occur is seen to be $\max(u^j d^{n-j} S - X, 0)$.

By the risk neutral valuation principle, the call value obtained from the n-period binomial model is given by

$$c = \frac{\displaystyle\sum_{j=0}^{n} C_j^n p^j (1-p)^{n-j} \max(u^j d^{n-j} S - X, 0)}{R^n}.$$

How to get rid of the "max" function in the option type payoff function?

We define k to be the smallest non-negative integer such that $u^k d^{n-k} S \geq X$, that is, $k \geq \dfrac{\ln \frac{X}{Sd^n}}{\ln \frac{u}{d}}$. It is seen that

$$\max(u^j d^{n-j} S - X, 0) = \begin{cases} 0 & \text{when} \quad j < k \\ u^j d^{n-j} S - X & \text{when} \quad j \geq k \end{cases}.$$

The integer k gives the minimum number of upward moves required for the asset price in the multiplicative binomial process in order that the call expires in-the-money.

The call price formula can be simplified as

$$c = S \sum_{j=k}^{n} C_j^n p^j (1-p)^{n-j} \frac{u^j d^{n-j}}{R^n} - XR^{-n} \sum_{j=k}^{n} C_j^n p^j (1-p)^{n-j}.$$

Interpretation of the call price formula

The last term in above equation can be interpreted as the expectation value of the payment made by the holder at expiration discounted by the factor R^{-n}, and $\sum_{j=k}^{n} C_j^n p^j (1-p)^{n-j}$ is seen to be the probability under the risk neutral measure that the call expires in-the-money.

The above probability is related to the *complementary binomial distribution function* defined by

$$\Phi(n,k,p) = \sum_{j=k}^{n} C_j^n p^j (1-p)^{n-j}.$$

Note that $\Phi(n,k,p)$ gives the probability for achieving at least k successes in n trials of a binomial experiment, where p is the probability of success in each trial.

Further, if we write $p' = \frac{up}{R}$ so that $1 - p' = \frac{d(1-p)}{R}$, then the call price formula for the n-period binomial model can be expressed as

$$c = S\Phi(n,k,p') - XR^{-n}\Phi(n,k,p).$$

- The first term gives the discounted expectation of the terminal asset price given that the call expires in-the-money.
- The second term gives the present value of the expected cost incurred by exercising the call.
- In the replicating portfolio, we require long holding of $\Phi(n,k,p')$ units of the underlying asset and short holding of $XR^{-n}\Phi(n,k,p)$ dollars of the money market account.
- The parameter n is related to time to expiry. The another parameter k is related to expectation of being in-the-money at expiry, which exhibits implicit dependence on volatility (via u and d) and strike price X.

Mathematical representation

The call price for the n-period binomial model can be expressed as the discounted expectation of the terminal payoff under the risk neutral measure

$$c = \frac{1}{R^n} E^* [c_T] = \frac{1}{R^n} E^* [\max(S_T - X, 0)], \quad T = t + n\Delta t,$$

where c_T is the terminal payoff, $\max(S_T - X, 0)$, of the call at expiration time T and $\frac{1}{R^n}$ is the discount factor over n periods. That is,

$$\begin{aligned} S\Phi(n, k, p') &= \frac{1}{R^n} E^*[S_T \mathbf{1}_{\{S_T > X\}}] \\ \Phi(n, k, p) &= E^*[\mathbf{1}_{\{S_T > X\}}] = P^*[S_T > X]. \end{aligned}$$

The expectation operator E^* is taken under the risk neutral measure rather than the subjective probability measure associated with the actual (physical) asset price process.

Dynamic programming procedure for pricing an American option

How to price the early exercise premium in an American option?

Without the early exercise privilege, risk neutral valuation principle leads to the usual binomial formula

$$V_{cont} = \frac{pV_u^{\Delta t} + (1-p)V_d^{\Delta t}}{R}.$$

To reflect the optimal decision of either continuing to hold the American option or exercising the option, the following dynamic programming procedure is applied at each binomial node

$$V = \max(V_{cont}, h(S)),$$

where $h(S)$ is the exercise payoff when the asset price assumes the value S. The stochastic optimization of the optimal stopping rule (early exercise feature) associated with an American option can be realized by the dynamic programming procedure applied at each binomial node.

American put option

The intrinsic value of a vanilla put option is $X - S_j^n$ at the (n, j) node, where X is the strike price. Here, n is the number of time steps from the tip of the binomial tree and j is the number of up-moves among the n steps. The dynamic programming procedure applied at each node is given by

$$P_j^n = \max\left(\frac{pP_{j+1}^{n+1} + (1-p)P_j^{n+1}}{R}, X - S_j^n\right),$$

where $n = N-1, \cdots, 0$, and $j = 0, 1, \cdots, n$. Here, N is the total number of time steps in the binomial tree.

Example 1 – Pricing an American put option

Consider a 5-month American put option on a non-dividend-paying stock when the stock price is \$50, the strike price is \$50, the risk-free interest rate is 10% per annum, and the volatility is 40% per annum. That is, $S = 50, X = 50, r = 0.10, \sigma = 0.40, T = 0.4167$.

Suppose that we divide the life of the option into five intervals of length of 1 month (= 0.0833 year) for the purpose of constructing a binomial tree.

With $\Delta t = 0.0833$, we have

$$u = e^{\sigma\sqrt{\Delta t}} = 1.1224, \quad d = e^{-\sigma\sqrt{\Delta t}} = 0.8909, \quad R = e^{r\Delta t} = 1.0084,$$

$$p = \frac{R-d}{u-d} = 0.5073, \quad 1 - p = 0.4927.$$

At each node:

Upper value = Underlying Asset Price

Lower value = Option Price

Shading indicates the node at which the option is exercised

Strike price = 50

Discount factor per step $= 1/R = e^{-r\Delta t} = 0.9917$

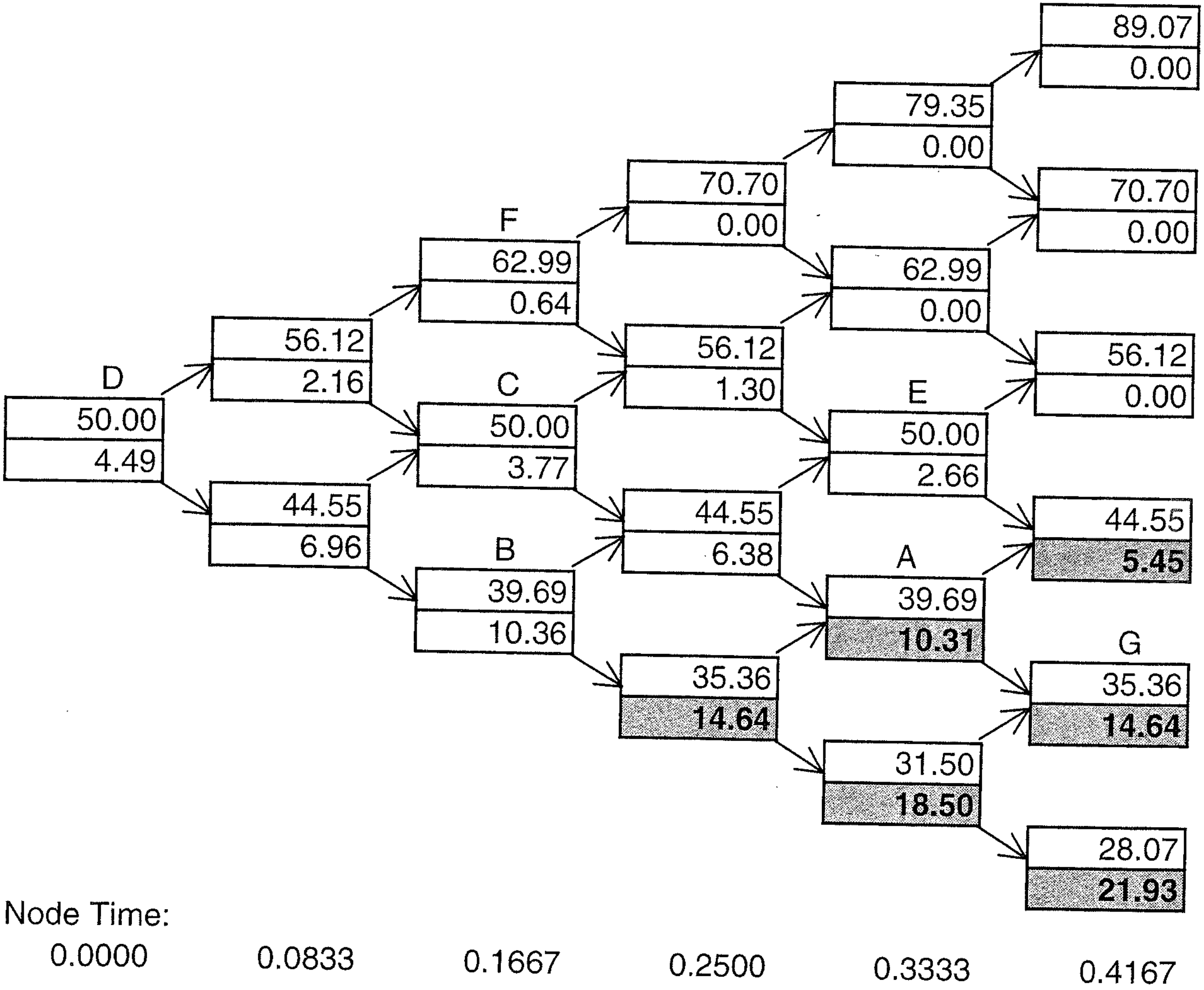
Time step, $\Delta t = 0.0833$ years, 30.42 days

Growth factor per step, $R = 1.0084$

Risk neutral probability of up-move, $p = 0.5073$

Proportional up-jump factor, $u = 1.1224$

Proportional down-jump factor, $d = 1/u = 0.8909$


D
50.00
4.49
56.12
2.16
44.55
6.96
F
62.99
0.64
C
50.00
3.77
B
39.69
10.36
70.70
0.00
56.12
1.30
44.55
6.38
35.36
14.64
79.35
0.00
62.99
0.00
E
50.00
2.66
A
39.69
10.31
31.50
18.50
89.07
0.00
70.70
0.00
56.12
0.00
44.55
5.45
G
35.36
14.64
28.07
21.93
Node Time:
0.0000
0.0833
0.1667
0.2500
0.3333
0.4167

- The stock price at the j^{th} node ($j = 0, 1, \cdots, n$) at time $n\Delta t$ ($n = 0, 1, \cdots, 5$) is calculated as $S_0 u^j d^{n-j}$. For example, the stock price at node A ($n = 4, j = 1$) (i.e., the second node up at the end of the fourth time step) is $50 \times 1.1224 \times 0.8909^3 = \$39.69$.

- The option prices at the final nodes are calculated as $\max(X - S_T, 0)$. For example, the option price at node G is $50.00 - 35.36 = 14.64$.

Backward induction procedure

- First, we assume no exercise of the option at the nodes. This means that the option price is calculated as the present value of the expected option price one time step later. For example, at node E, the option price is calculated as

$$(0.5073 \times 0 + 0.4927 \times 5.45)e^{-0.10 \times 0.0833} = 2.66$$

whereas at node A it is calculated as

$$(0.5073 \times 5.45 + 0.4927 \times 14.64)e^{-0.10 \times 0.0833} = 9.90.$$

Check to see if early exercise is preferable to waiting

- At node E, early exercise would give a value for the option of zero because both the stock price and strike price are \$50. Clearly it is best to wait. The correct value for the option at node E, therefore, is \$2.66.

- At node A, it is a different story. If the option is exercised, it is worth \$50.00 – \$39.69, or \$10.31. This is more than \$9.90. If node A is reached, then the option should be exercised and the correct value for the option at node A is \$10.31.

- Option prices at earlier nodes are calculated in a similar way. Note that it is **not always** best to exercise an option early even when it is in the money at that node.

- Consider node B, the American put is in-the-money since the asset price \$39.69 is below \$50. If the option is exercised, it is worth \$50.00 − \$39.69, or \$10.31. However, if it is held, it is worth

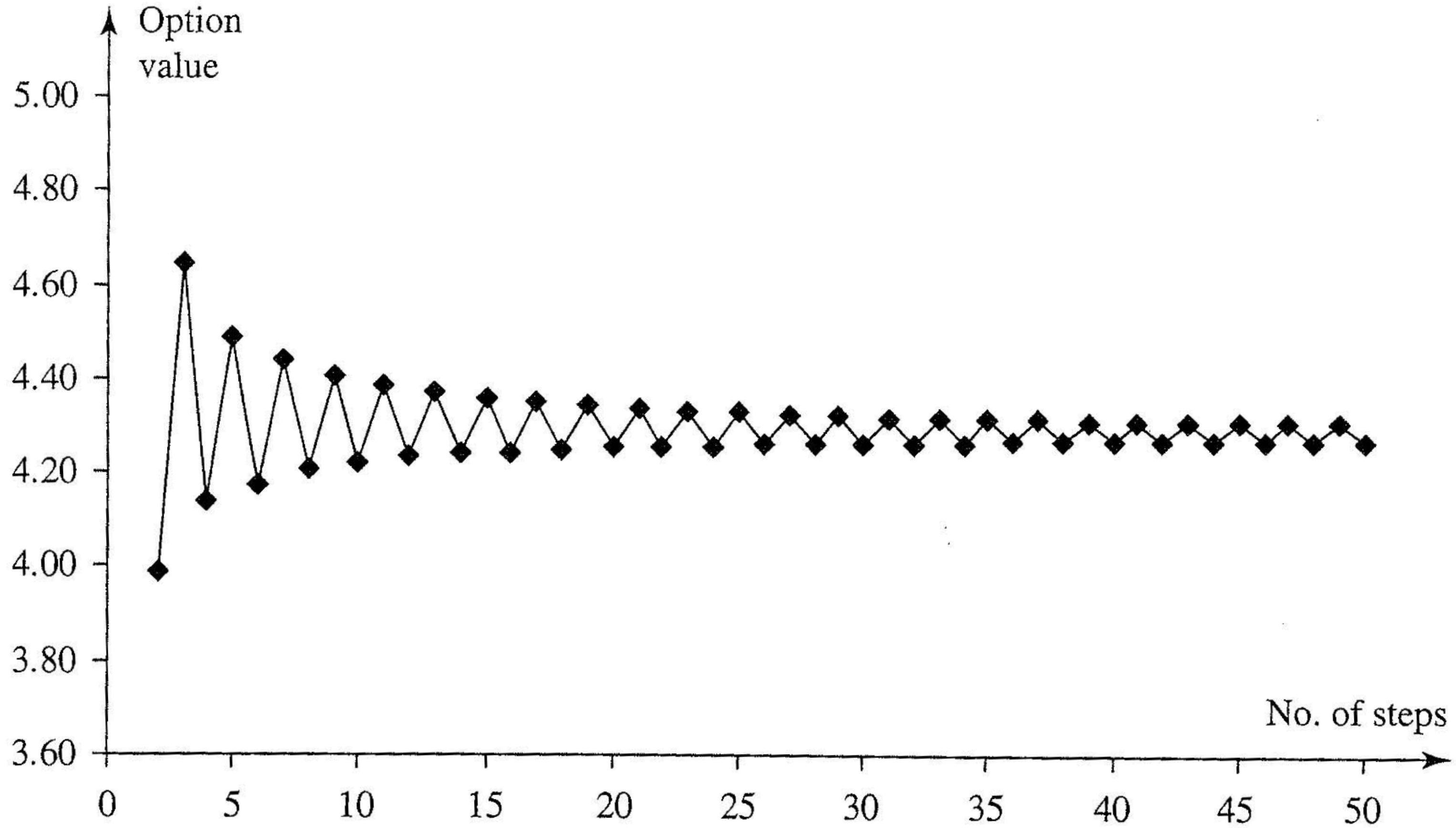
$$(0.5073 \times 6.38 + 0.4927 \times 14.64)e^{-0.10\times 0.0833} = 10.36.$$

The option should not be exercised at this node, and the correct option value at the node is \$10.36.

- Working back through the tree, the value of the option at the initial node is \$4.49. This is our numerical estimate for the option's current value.

- In practice, a smaller value of Δt, and many more nodes, would be used. With $30, 50, 100$, and 500 time steps we obtain values for the option of $4.263, 4.272, 4.278$, and 4.283.

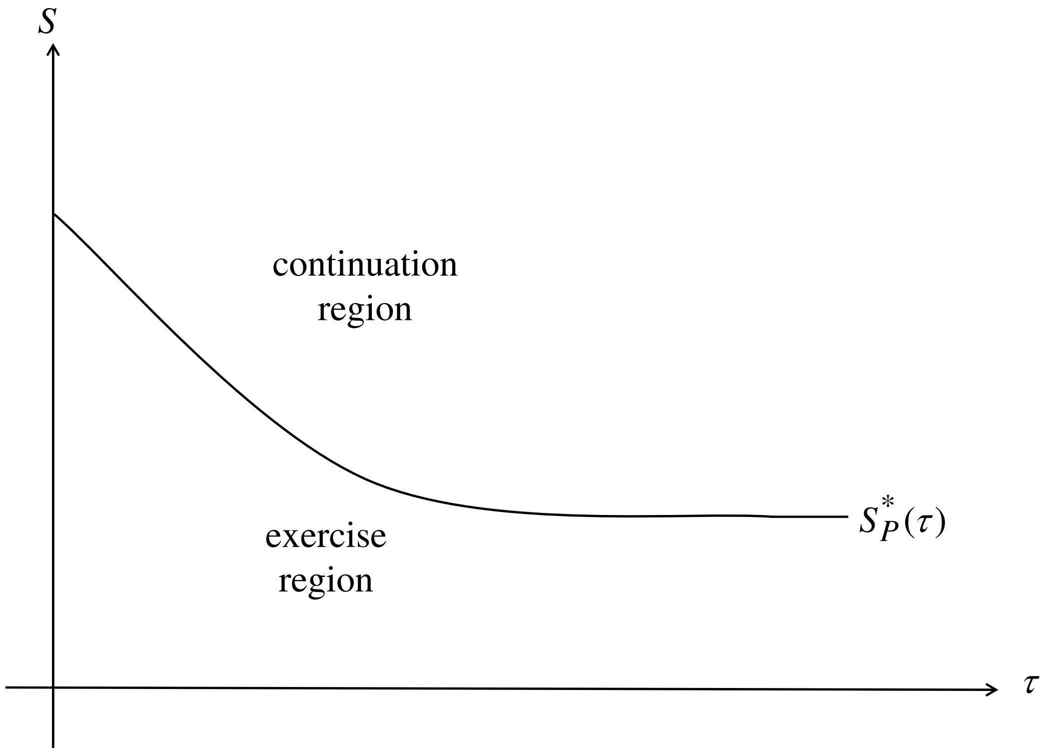
Convergence of the price of the option with respect to increasing number of time steps



The convergence trend is oscillatory. Overshooting the theoretical true value at the current choice of the time step becomes undershooting when the number of time steps is increased by one.

Early exercise boundary $S_P^*(\tau)$

The optimal exercise policy is characterized by the early exercise curve $S_P^*(\tau)$, where the American put option should be exercised when the stock price falls below the critical threshold value $S_P^*(\tau)$ for a given time to expiry τ.



- The numerical approximation of $S_P^*(\tau)$ can be deduced from the binomial tree calculations by recording the node points at which the American put value assumes the exercise payoff.

- The early exercise boundary is approximated by taking the (arithmetic or geometric) average of the asset prices at neighboring nodes at the same time level at which continuation value is taken at the upper node while exercise value is taken at the lower node.

- As a numerical example (refer to the numerical results shown on p.29), we approximate the early exercise boundary at $t = 0.25, t = 0.3333$ and $t = 0.4167$ as $\frac{44.55+35.36}{2}$ or $\sqrt{44.55 \times 35.36}$, $\frac{50.00+39.69}{2}$ or $\sqrt{50.00 \times 39.69}$, $\frac{56.12+44.55}{2}$ or $\sqrt{56.12 \times 44.55}$, respectively.

Callable American call – game option between the issuer and holder

- The callable feature entitles the issuer to buy back the American option at any time at a predetermined call price.
- Upon call, the holder can choose either to exercise the call or receive the call price as cash.
- Let the call price be K. The dynamic programming procedure applied at each node to model the game between the issuer and holder can be constructed as follows:

$$C_j^n = \min\left(\max\left(\frac{pC_{j+1}^{n+1} + (1-p)C_j^{n+1}}{R}, S_j^n - X\right),\right.$$
$$\left.\max(K, S_j^n - X)\right).$$

Justification of the dynamic programming procedure

- The first term max $\left(\frac{pC_{n+1}^{n+1} + (1-p)C_j^{n+1}}{R}, S_j^n - X\right)$ represents the optimal strategy of the holder, given no call of the option by the issuer.
- Upon call by the issuer, the payoff is given by the second term max$(K, S_j^n - X)$ since the holder can either receive cash amount K or exercise the option.
- From the perspective of the issuer, he chooses to call or restrain from calling so as to minimize the option value with reference to the possible actions of the holder. The value of the callable call is given by taking the minimum value of the above two terms.

Recall the well known distributive rule: $\alpha x + \alpha y = \alpha(x + y)$. In the current context, we may treat "taking max" as multiplication and "taking min" as addition. An equivalent dynamic programming procedure can be constructed as follows:

$$C_j^n = \max\left(S_j^n - X, \min\left(\frac{pC_{j+1}^{n+1} + (1-p)C_j^{n+1}}{R}, K\right)\right).$$

- From financial intuition, the option will be called when the continuation value is above the call price K. Independent of whether the option to be either called or not called, the holder can always choose to exercise to receive $S_j^n - X$ if the exercise payoff has a higher value.

Estimating delta and other Greek letters

- The delta (Δ) of an option is the rate of change of its price with respect to the underlying stock price. It can be calculated as
$$\frac{\Delta f}{\Delta S}$$
where ΔS is a small change in the stock price and Δf is the corresponding small change in the option price.

- At time Δt, we have an estimate f_{11} for the option price when the stock price is $S_0 u$ and an estimate f_{10} for the option price when the stock price is $S_0 d$.

- When $\Delta S = S_0 u - S_0 d, \Delta f = f_{11} - f_{10}$. Therefore an estimate of delta at time Δt is
$$\Delta = \frac{f_{11} - f_{10}}{S_0 u - S_0 d}.$$

Gamma calculations

To determine gamma (Γ), note that we have two estimates of Δ at time $2\Delta t$.

When $S = (S_0u^2 + S_0)/2$ (halfway between the second and third node), delta is $(f_{22} - f_{21})/(S_0u^2 - S_0)$; when $S = (S_0 + S_0d^2)/2$ (halfway between the first and second node), delta is $(f_{21} - f_{20})/(S_0 - S_0d^2)$.

The difference between the two values of S is h, where

$$h = 0.5(S_0u^2 - S_0d^2).$$

Gamma is the change in delta divided by h:

$$\Gamma = \frac{[(f_{22} - f_{21})/(S_0u^2 - S_0)] - [(f_{21} - f_{20})/(S_0 - S_0d^2)]}{h}.$$

Theta calculations

Theta is the rate of change of the option price with time when all else is kept constant. If the tree starts at time zero, an estimate of theta is

$$\Theta = \frac{f_{21} - f_{00}}{2\Delta t}.$$

Note that f_{21} is the option value at two time steps from time zero and with the same asset price.

Vega calculations

Vega can be calculated by making a small change, $\Delta\sigma$, in the volatility and constructing a new tree to obtain a new value of the option. The time step Δt should be kept the same. The estimate of vega is

$$\nu = \frac{f^* - f}{\Delta\sigma},$$

where f and f^* are the estimates of the option price from the original and the new tree, respectively.

Example 2

- Consider again Example 1. We have $f_{1,0} = 6.96$ and $f_{1,1} = 2.16$. An estimate for delta is given by

$$\frac{2.16 - 6.96}{56.12 - 44.55} = -0.41.$$

- An estimate of the gamma of the option can be obtained from the values at nodes B, C, and F as

$$\frac{[(0.64 - 3.77)/(62.99 - 50.00)] - [(3.77 - 10.36)/(50.00 - 39.66)]}{11.65} = 0.03.$$

- An estimate of the theta of the option can be obtained from the values at nodes D and C as

$$\frac{3.77 - 4.49}{0.1667} = -4.3 \text{ per year}$$

  or -0.012 per calendar day.

- These are only rough estimates. They become progressively better as the number of time steps on the tree is increased.

Discrete dividend models

Let S be the asset price at the current time which is $n\triangle t$ from expiry, and suppose a discrete dividend of amount D is paid at time between one time step and two time steps from the current time.

Consider the naive construction of the binomial tree. The nodes in the binomial tree at two time steps from the current time would correspond to asset prices

$$u^2S - D, \quad S - D \quad \text{and} \quad d^2S - D,$$

since the asset price drops by the same amount as the dividend right after the dividend payment.

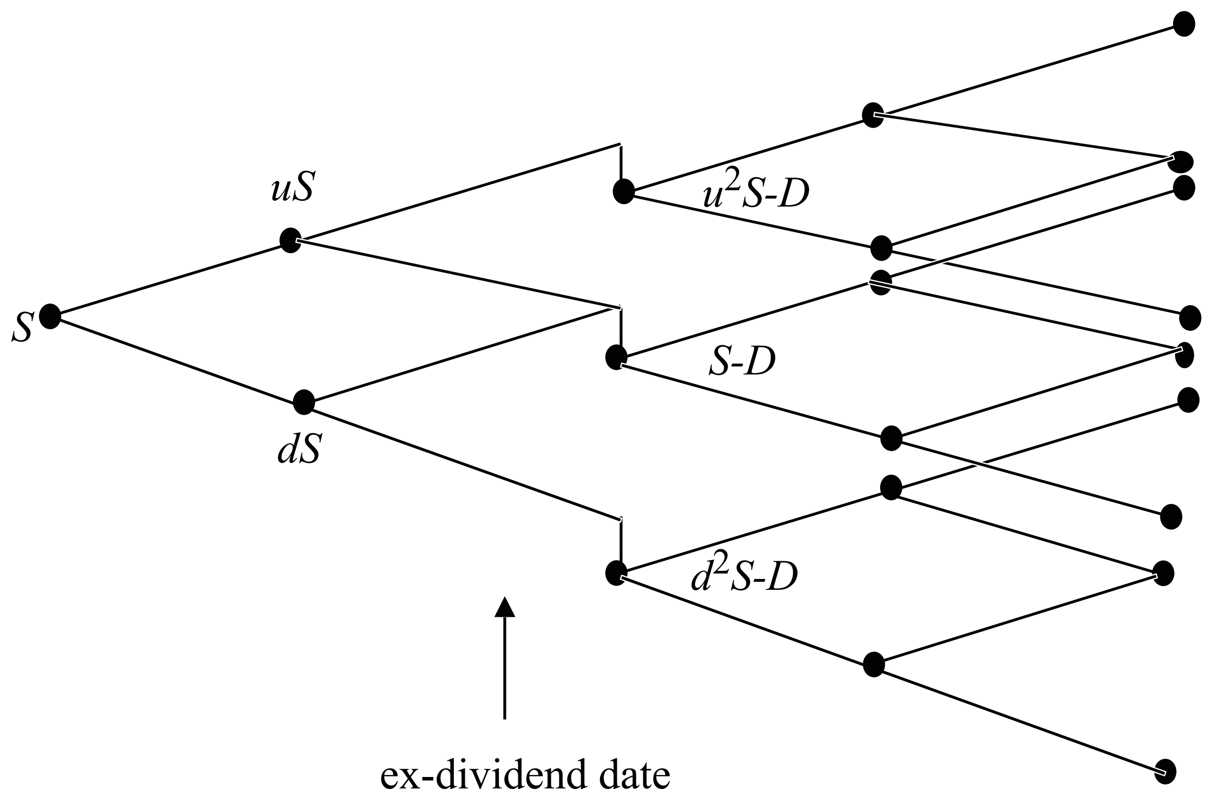
- Extending one time step further, there will be six nodes

$$(u^2S - D)u, (u^2S - D)d, (S - D)u, (S - D)d, (d^2S - D)u, (d^2S - D)d$$

  instead of four nodes as in the usual binomial tree without discrete dividend.

- This is because $(u^2S - D)d \neq (S - D)u$ and $(S - D)d \neq (d^2S - D)u$, so the interior nodes do not recombine.

- In general, suppose a discrete dividend is paid in the future between $(k-1)^{\text{th}}$ and k^{th} time step, then at the $(k+m)^{\text{th}}$ time step, the number of nodes would be $(m+1)(k+1)$ rather than $k+m+1$ nodes as in the usual reconnecting binomial tree.



Binomial tree with single discrete dividend

In this pictorial representation, we have $m = n = 2$, so that there are $(2+1)(2+1) = 9$ nodes after 4 time steps.

Splitting the asset price into the deterministic dividends component and risky component

- Splitting the asset price S_t into two parts: the risky component $\widetilde{S}_t$ that is stochastic and the remaining part that will be used to pay the discrete dividend (assumed to be deterministic) in the future.
- Suppose the dividend date is t^*, then at the current time t, the risky component $\widetilde{S}_t$ is given by

$$\widetilde{S}_t = \begin{cases} S_t - De^{-r(t^*-t)}, & t < t^* \\ S_t, & t > t^*. \end{cases}$$

- Let $\widetilde{\sigma}$ denote the volatility of $\widetilde{S}_t$ and assume $\widetilde{\sigma}$ to be constant rather than the volatility of S_t itself to be constant.

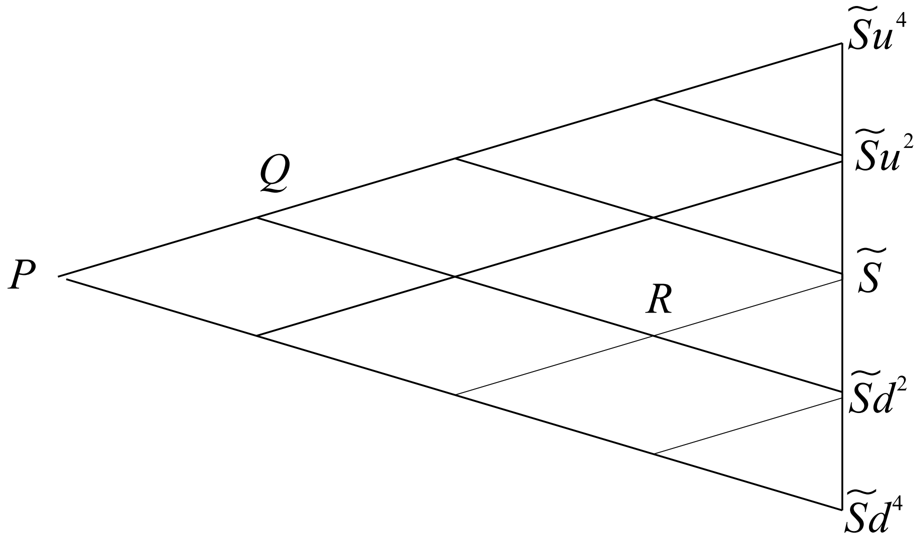
- Assume that a discrete dividend D is paid at time t^*, which lies between the k^{th} and $(k+1)^{\text{th}}$ time step.
- At the tip of the binomial tree, the risky component $\widetilde{S}$ is related to the asset price S by

$$\widetilde{S} = S - De^{-kr\Delta t}.$$

- The total value of asset price at the $(n,j)^{\text{th}}$ node, which corresponds to n time steps from the tip and j upward jumps, is given by

$$\widetilde{S}u^j d^{n-j} + De^{-(k-n)r\Delta t}\mathbf{1}_{\{n \le k\}},$$
$$n = 1, 2, \cdots, N \quad \text{and} \quad j = 0, 1, \cdots, n.$$

A reconnecting binomial tree with single discrete dividend D



Here, $N = 4$ and $k = 2$, and let $\widetilde{S}$ denote the risky component of the asset value at the tip of the binomial tree. The asset value at nodes P, Q and R are $\widetilde{S} + De^{-2r\Delta t}, \widetilde{S}u + De^{-r\Delta t}$ and $\widetilde{S}d$, respectively.

Example 3

Consider a 5-month American put option on a stock that is expected to pay a single dividend of \$2.06 during the life of the option. The initial stock price is \$52, the strike price is \$50, the risk-free interest rate is 10% per annum, the volatility is 40% per annum, and the ex-dividend date is in $3\frac{1}{2}$ months (which is 0.2917 years).

We construct a tree to model $\widetilde{S}$ (risky component of the asset price process), the stock price less the present value of future dividends during the life of the option. At time zero, the present value of the dividend is

$$2.06e^{-0.2917\times 0.1} = 2.00.$$

The initial value of $\widetilde{S}$ is therefore 50.00.

- Assuming that the 40% per annum volatility refers to $\widetilde{S}$, the earlier figure on P.30 provides a binomial tree for $\widetilde{S}$. Adding the present value of the dividend at each node leads to the figure on P.53, which is a binomial model for S.

- The probabilities at the nodes are 0.5073 for an up movement and 0.4927 for a down movement. Working back through the tree in the usual way gives the option price as 4.44.

Remark The exercise payoff is calculated using the actual asset price S, not the risky component $\widetilde{S}$.

Note that the lowest node at time 0.25 on P.53 has the put option value equals 14.22, which is not equal to the exercise payoff (see the binomial tree on P.30 for comparison). This result corresponds to the higher asset value, which is 37.41 on P.53 instead of 35.36 on P.30 for the same binomial node. The exercise payoff of the put option at $S = 37.41$ and $K = 50$ is 12.59, which is less than the continuation value of 14.22.

At each node:

Upper value = Underlying Asset Price

Lower value = Option Price

Shading indicates where option is exercised

Strike price = 50

Discount factor per step = $1/R = 0.9917$

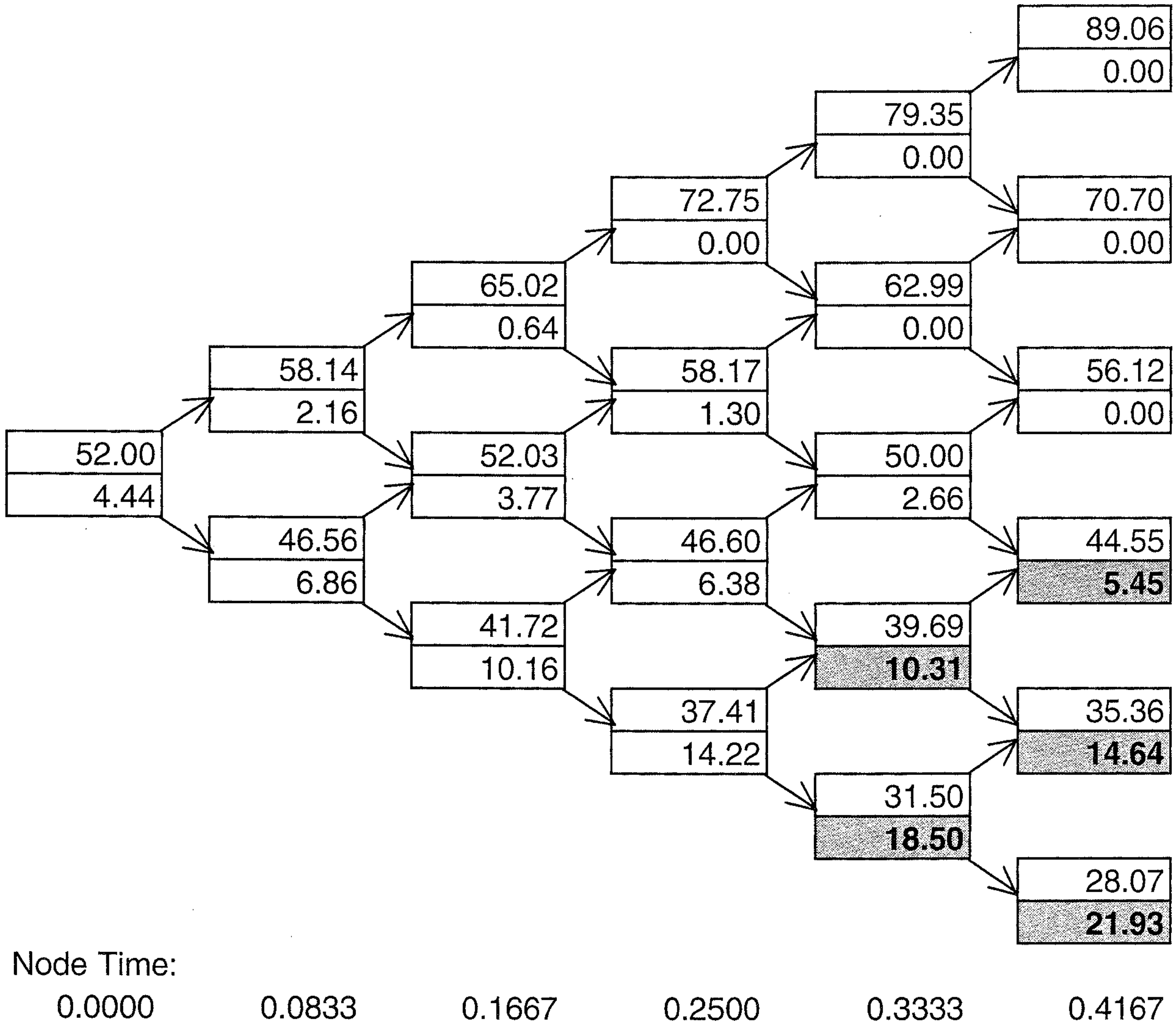
Time step, $\Delta t = 0.0833$ years, 30.42 days

Growth factor per step, $R = 1.0084$

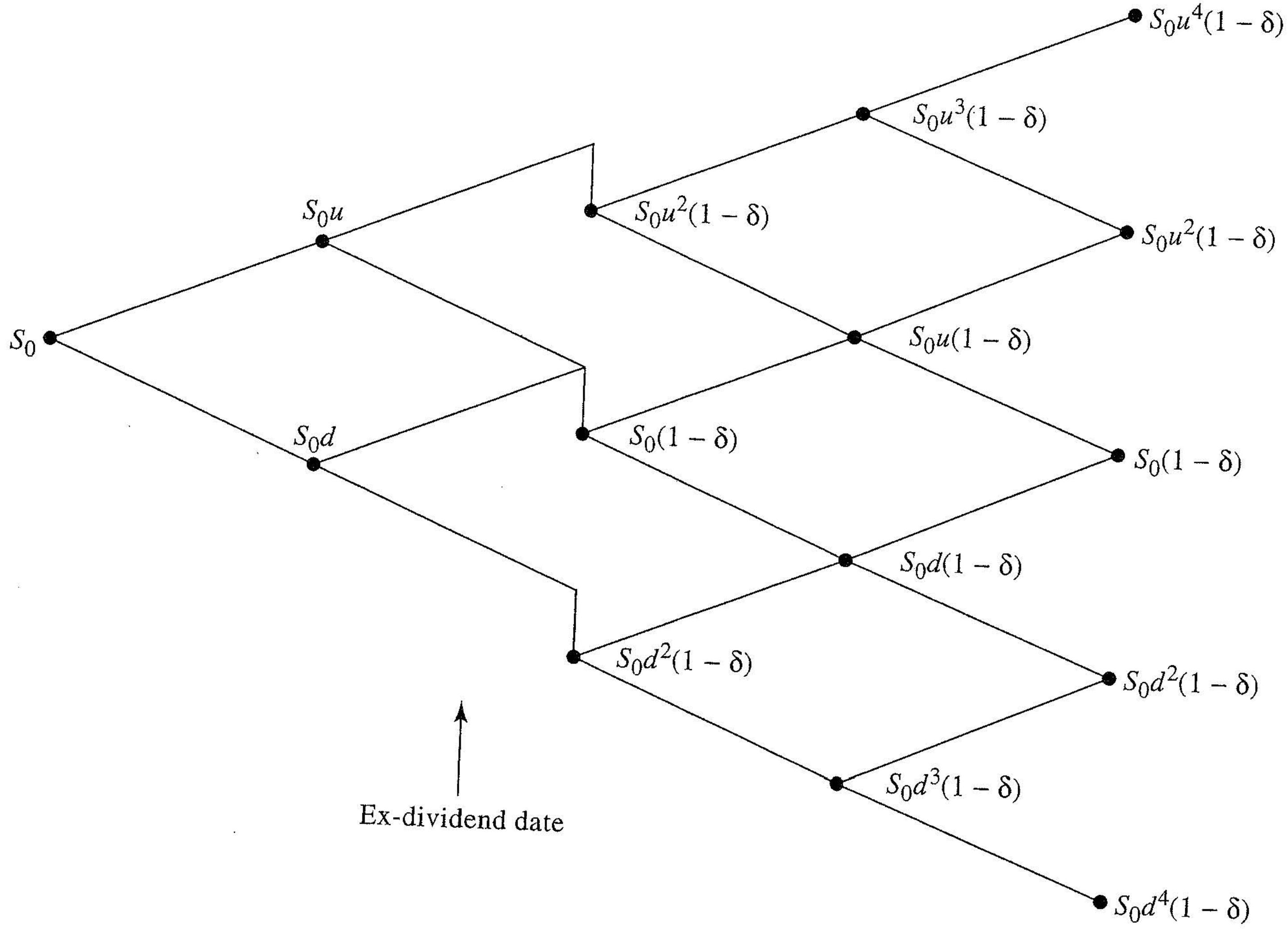
Risk neutral probability of up move, $p = 0.5073$

Proportional up jump factor, $u = 1.1224$

Proportional down jump factor, $d = 1/u = 0.8909$


89.06
0.00
79.35
0.00
72.75
0.00
70.70
0.00
65.02
0.64
62.99
0.00
58.14
2.16
58.17
1.30
56.12
0.00
52.00
4.44
52.03
3.77
50.00
2.66
46.56
6.86
46.60
6.38
44.55
5.45
41.72
10.16
39.69
10.31
37.41
14.22
35.36
14.64
31.50
18.50
28.07
21.93
Node Time:
0.0000
0.0833
0.1667
0.2500
0.3333
0.4167

Tree when stock pays a known dividend yield at one particular time. The dividend amount is equal to δ times the prevailing asset price. In this case, the interior nodes do recombine. Here, δ is the dividend yield.



Pricing of lookback options

A path-dependent derivative is a derivative where the payoff depends on the path followed by the price of the underlying asset, not just its final value. Two important properties:

1. The payoff from the derivative must depend on a single function, F, of the path followed by the underlying asset.

2. It must be possible to calculate the updated value of F at time $t + \Delta t$ from the known value of F at time t and the updated value of the underlying asset at time $t + \Delta t$.

For example, the realized maximum of a discrete asset price process over successive time steps is given by

$$S_{i+1}^{\text{max}} = \text{max}(S_i^{\text{max}}, S_{i+1}), \quad i = 1, 2, \ldots, n.$$

Research papers on numerical methods on lookback options

1. Hull, J. and A. White, "Efficient procedures for valuing European and American path-dependent options," *Journal of Derivatives* (Fall, 1993), p.21-31.

2. Cheuk, T.H.F. and T.C.F. Vorst, "Currency lookback options and observation frequency: a binomial approach," *Journal of International Money and Finance*, vol. 16(2) (1997), p.173-187.

3. Andreasen, J., "The pricing of discretely sampled Asian and lookback options: a change of numeraire approach," *Journal of Computational Finance* (Fall, 1998), p.5-30.

4. Babbs, S., "Binomial valuation of lookback options," *Journal of Economics and Dynamic Control*, vol. 24 (2000), p.1499-1525.

5. Kwok, Y.K. and K.W. Lau, "Accuracy and reliability considerations of option pricing algorithms," *Journal of Futures Markets*, vol. 21 (2001), p.875-903.

6. Yu, H, Y.K. Kwok and L.X. Wu, "Early exercise policies of American floating and fixed strike lookback options," *Nonlinear Science*, vol. 47 (2001), p.4591-4602.

7. Petrella, G. and S. Kou, "Numerical pricing of discrete barrier and lookback options via Laplace transforms," *Journal of Computational Finance* (Fall 2004), p.1-37.

American floating strike lookback put option on a non-dividend-paying stock (Hull-White, 1993)

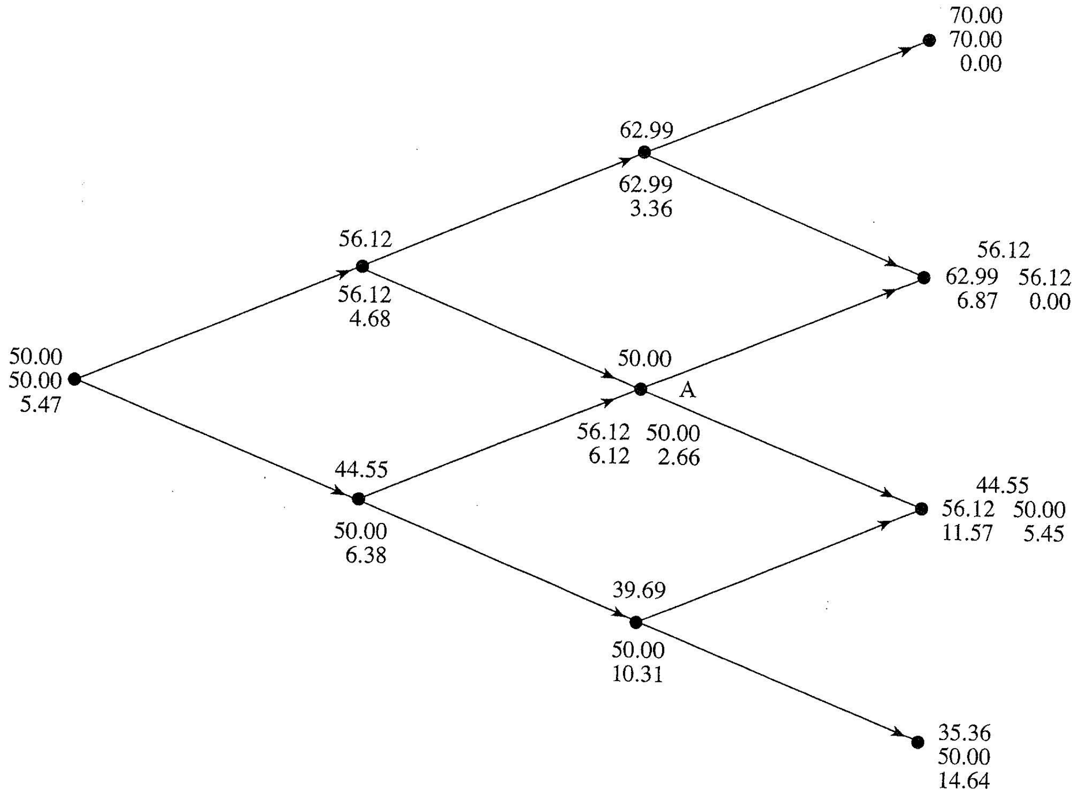
- If the American floating strike lookback put option is exercised at time τ, the exericse payoff is the amount by which the maximum stock price between time 0 and time τ exceeds the current stock price. That is,

$$\max_{t \in [0,\tau]} S_t - S_\tau.$$

  Note that the strike in the put payoff is reset to a new value when a new maximum asset value is realized.

- We suppose that the initial stock price is $50, the stock price volatility is 40% per annum, the risk-free interest rate is 10% per annum, the total life of the option is three months, and that stock price movements are represented by a three-step binomial tree. That is, $S_0 = 50, \sigma = 0.4, r = 0.10, \Delta t = 0.08333, u = 1.1224, d = 0.8909, R = 1.0084$, and $p = 0.5073$.

Binomial tree for valuing an American lookback put option



Rolling back through the tree gives the value of the American lookback put as \$5.47. In the first time level, we have

$$V_1^1 = \max((3.36 \times 0.5073 + 6.12 \times 0.4927)e^{-0.1 \times 0.0833}, 0) = 4.683$$
$$V_0^1 = \max((2.66 \times 0.5073 + 10.31 \times 0.4927)e^{-0.1 \times 0.0833}, 50 - 44.55) = 6.38.$$

- The top number at each node is the stock price. The next level of numbers at each node shows the possible maximum stock prices achievable on all paths leading to the node. The bottom level of numbers show the values of the derivative corresponding to each of the possible maximum stock prices.

- The values of the derivatives at the final nodes of the tree are calculated as the maximum stock price minus the actual stock price.

- To illustrate the rollback procedures, suppose that we are at node A, where the stock price is \$50. The maximum stock price achieved thus far is either 56.12 or 50 (depending on the path history of the asset price movement). Consider first where it is equal to 50. If there is an up movement, the maximum stock price becomes 56.12 and the value of the derivative is zero. If there is a down movement, the maximum stock price stays at 50 and the value of the terminal payoff is $5.45 = 50 - 44.55$.

- Assuming no early exercise, the value of the derivative at A when the maximum achieved so far is 50 is,

$$(0 \times 0.5073 + 5.45 \times 0.4927)e^{-0.1 \times 0.08333} = 2.66.$$

Clearly, it is not worth exercising at node A because the payoff from doing so is zero.

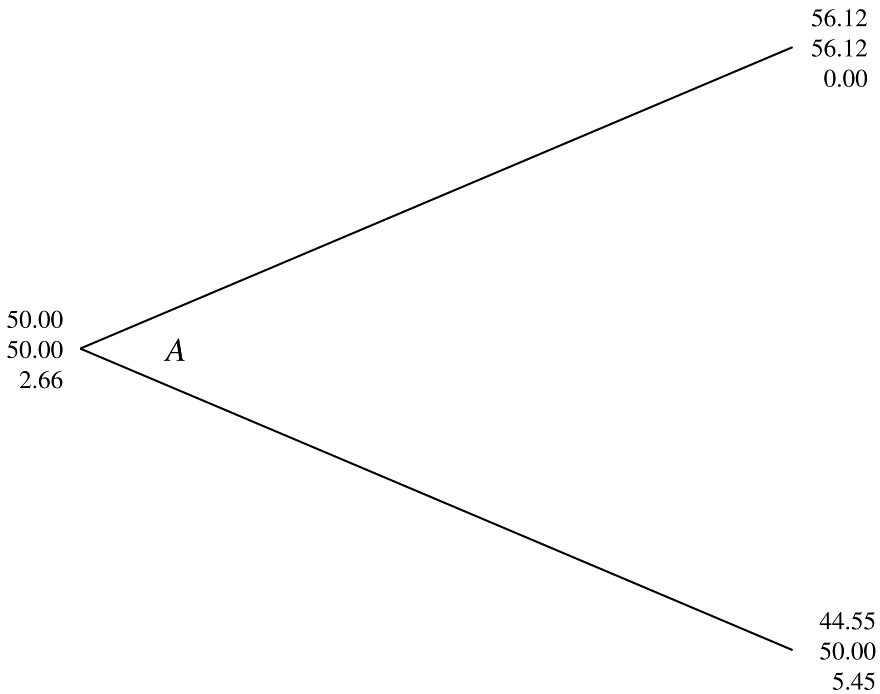
- A similar calculation for the situation where the maximum value at node A is 56.12 gives the value of the derivative at node A, without early exercise, to be

$$(0 \times 0.5073 + 11.57 \times 0.4927)e^{-0.1 \times 0.08333} = 5.65.$$

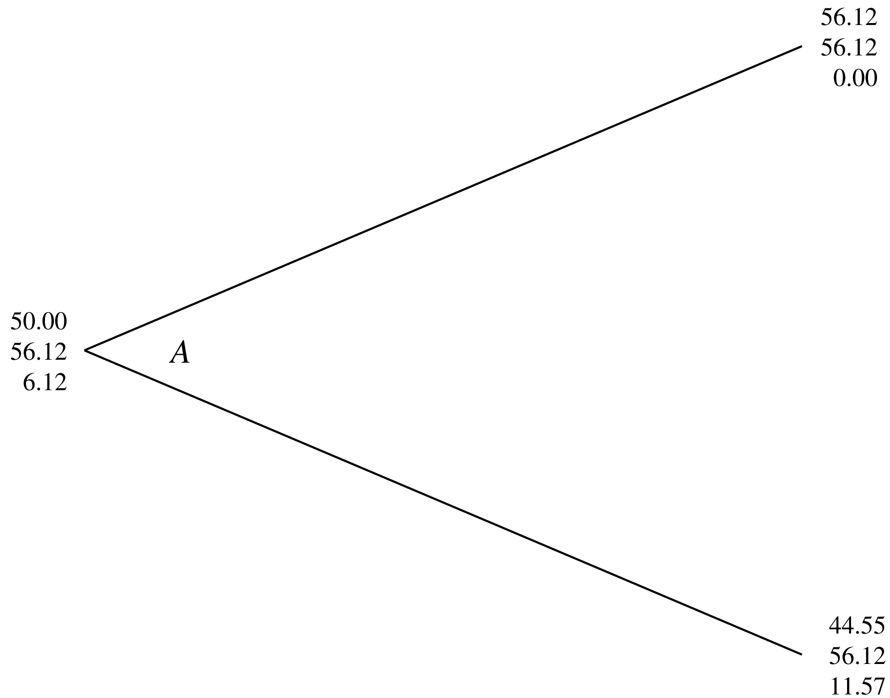
Early exercise gives a value of 6.12 and it is the optimal strategy.

- There may be multiple realized maximum asset values at each node. The different possible values of the path dependent function at a given node are linked to the corresponding path dependent function at the nodes that are one time step earlier.

There are 2 possible realized maximum at node A, one is 50.00 while the other is 56.12.



When the realized maximum at A is 50.00, the realized maximum becomes 56.12 when the asset price moves up while the realized maximum remains at 50.00 when the asset price moves down.



When the realized maximum at A is already 56.12, the realized maximum remains at 56.12 independent of whether the asset price moves up or down.

Alternative binomial algorithm (Cheuk-Vorst, 1997)

When the stock price S_t is used as the numeraire, the payoff of the floating strike lookback put takes the form:

$$\widetilde{V}_t = \frac{V_t}{S_t} = \frac{S_t^{\max}}{S_t} - 1, \quad \text{where } S_t^{\max} = \max_{u \in [0,t]} S_u.$$

We construct the truncated binomial tree for the process:

$$Y_t = \frac{S_t^{\max}}{S_t}, \quad Y_t \geq 1.$$

- At the tip of the binomial tree, $Y_0 = 1$.
- When $Y_t = 1$ where $S_t = S_t^{\max}$, then

$$Y_{t+\Delta t} = \begin{cases} u & \text{when } S_{t+\Delta t} = dS_t \\ 1 & \text{when } S_{t+\Delta t} = uS_t \end{cases}.$$

- When $Y_t = u^j$ for some $j \geq 1$, $S_t < S_t^{\max}$, then

$$Y_{t+\Delta t} = \begin{cases} u^{j+1} & \text{when } S_{t+\Delta t} = dS_t \\ u^{j-1} & \text{when } S_{t+\Delta t} = uS_t \end{cases}.$$

- Let $\widetilde{V}_j^n$ denote the numerical approximation to $\widetilde{V}_t = V_t/S_t$ at the $(n,j)^{\text{th}}$ node of the binomial tree for Y_t, where $t = n\Delta t, n \geq 0$ and $Y_t = u^j, j \geq 0$.

For $j \geq 1$, note that when the underlying S_t jumps up from state j to state $j+1$ with probability p, Y_t jumps down from state j to state $j-1$. In terms of $\tilde{V}_j^n$, the binomial scheme for pricing the lookback option is given by

$$\tilde{V}_j^n S(t_n) = e^{-r\Delta t}[p\tilde{V}_{j-1}^{n+1} uS(t_n) + (1-p)\tilde{V}_{j+1}^{n+1} dS(t_n)],$$

so that

$$\tilde{V}_j^n = e^{-r\Delta t}\left[p\tilde{V}_{j-1}^{n+1} u + (1-p)\tilde{V}_{j+1}^{n+1} d\right].$$

- The continuation value is then given by

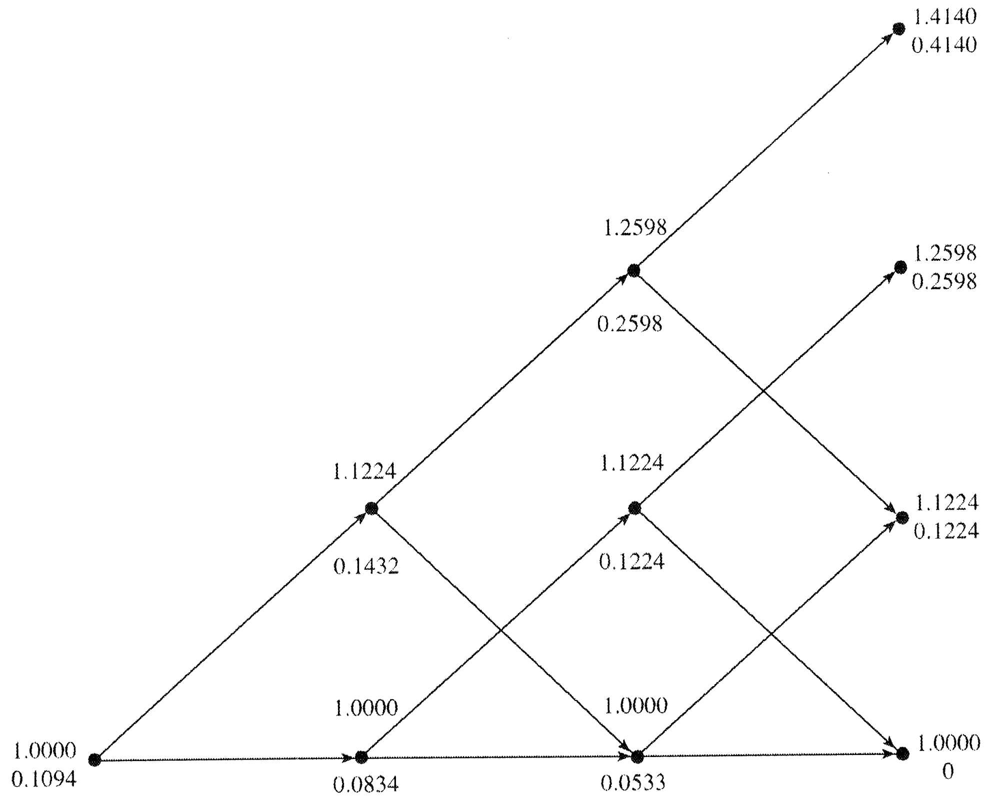
$$\begin{cases} e^{-r\Delta t}\left[(1-p)\widetilde{V}_{j+1}^{n+1}d + p\widetilde{V}_{j-1}^{n+1}u\right], & j \geq 1 \\ e^{-r\Delta t}\left[(1-p)\widetilde{V}_{j+1}^{n+1}d + p\widetilde{V}_{j}^{n+1}u\right], & j = 0 \end{cases}.$$

  Note that when $j = 0$, the upward jump of S_t keeps Y_t to stay at the same value $j = 0$.

Dynamic programming procedure for an American floating strike lookback option

$$\widetilde{V}_j^n = \begin{cases} \max\left\{Y_j - 1, e^{-r\Delta t}\left[(1-p)\widetilde{V}_{j+1}^{n+1}d + p\widetilde{V}_{j-1}^{n+1}u\right]\right\}, & j \geq 1 \\ \max\left\{Y_j - 1, e^{-r\Delta t}\left[(1-p)\widetilde{V}_{j+1}^{n+1}d + p\widetilde{V}_{j}^{n+1}u\right]\right\}, & j = 0 \end{cases}.$$

Dimension reduction is achieved by taking the stock price $S(t_n)$ as the numeraire. The exercise payoff of the American floating strike lookback option can be expressed solely in terms of $Y_j = \left(\frac{S_t^{\max}}{S_t}\right)_j$.



- The upper figures are values of Y_t while the lower figures are option values at the nodes.

Cheuk-Vorst's procedure for valuing an American-style floating strike lookback option. Though dimension reduction is achieved, the numerical scheme has very slow rate of convergence.

European floating strike currency lookback call ($S = 100$, $\tilde{r}_d = 0.04$, $\tilde{r}_f = 0.07$ and $T = 0.5$) with payoff: $S_T - \min_{[0,T]} S_\tau$. Here, N is the number of time steps in the binomial tree.

N	Option price		
	$\sigma = 0.1$	$\sigma = 0.2$	$\sigma = 0.3$
50	4.24	8.97	13.52
100	4.37	9.20	13.85
500	4.54	9.52	14.31
1000	4.58	9.60	14.42
5000	4.63	9.71	14.57
10000	4.65	9.73	14.60
Analytic	4.68	9.79	14.69

- The binomial results seem to converge very slowly to the analytical one (converge from below). The poor rate of convergence arises from the ineffective modeling of the recording of the newly realized maximum of the asset price (based on their intuitive derivation of the binomial formula at $j = 0$). See Qn4 of HW2 for an alternative derivation of the numerical boundary condition at $j = 0$.

- Cheuk-Vorst's algorithm suffers from an extremely slow rate of convergence when compared to other finite difference schemes for pricing continuously monitored lookback options. Finite difference schemes can incorporate boundary conditions effectively.

Comparison of the Numerical Accuracy of the Lookback Option Values Obtained from the Lax–Wendroff Scheme, Cheuk–Vorst Scheme, and Babbs Scheme

Volatility	*Numerical Schemes*	*Number of Time Steps* 50	100	500	1000	10,000	*Analytic Solution*
	LW scheme	4.6691	4.6745	4.6789	4.6794	4.6799	
$\sigma = 0.1$	CV scheme	4.24	4.37	4.54	4.58	4.65	4.6799
	Babbs scheme	4.6508	4.6653	4.6770	4.6784	4.6797	
	LW scheme	9.7415	9.7673	9.7870	9.7891	9.7912	
$\sigma = 0.2$	CV scheme	8.97	9.20	9.52	9.60	9.73	9.7915
	Babbs scheme	9.7362	9.7638	9.7859	9.7887	9.7912	
	LW scheme	14.5964	14.6419	14.6785	14.6826	14.6868	
$\sigma = 0.3$	CV scheme	13.52	13.85	14.31	14.42	14.60	14.6872
	Babbs scheme	14.6056	14.6464	14.6790	14.6831	14.6868	

The parameter values of the continuously monitored European floating strike lookback call option are: $S = m_{T_0}^t = 100$, $r = 0.04$, $q = 0.07$, and $\tau = 0.5$.

Binomial schemes for European fixed strike lookback call options

$$\text{Terminal payoff} = \max\left(\max_{0 \le i \le N} S(t_i) - K, 0\right),$$

where K is the fixed strike. Write $\overline{M}(t_j) = \max_{0 \le i \le j} S(t_i)$ as the realized maximum asset value up to time t_j, a known quantity at t_j. Note that

$$\max_{0 \le i \le N} S(t_i) = \max(\overline{M}(t_j), M(t_N; t_{j+1})),$$

where $M(t_N; t_{j+1}) = \max_{j+1 \le i \le N} S(t_i)$ is the random path dependent state variable for the future realized maximum asset value between time t_{j+1} and t_N.

The fixed strike lookback call value at time t_j and with known $\overline{M}(t_j)$ can be expressed as

$$\begin{aligned} & c_{fix}(S(t_j), \overline{M}(t_j), t_j) \\ = \; & e^{-r(t_N - t_j)} E_Q[\max(\max(\overline{M}(t_j), M(t_N; t_{j+1})) - K, 0)]. \end{aligned}$$

The terminal payoff can be decomposed into 2 terms:

$$
\begin{aligned}
&\max(\max(\overline{M}(t_j), M(t_N; t_{j+1})) - K, 0) \\
= &\begin{cases} \max(M(t_N; t_{j+1}) - K, 0) & \text{if } \overline{M}(t_j) \le K \\ \overline{M}(t_j) - K + \max(M(t_N; t_{j+1}) - \overline{M}(t_j), 0) & \text{if } \overline{M}(t_j) > K \end{cases} \\
= &\max(\overline{M}(t_j) - K, 0) + \max(M(t_N; t_{j+1}) - \max(\overline{M}(t_j), K), 0).
\end{aligned}
$$

The decomposition reveals

- $\overline{M}(t_j) \le K$
  $\overline{M}(t_j)$ has no effect on the final option payoff.
- $\overline{M}(t_j) > K$
  Guaranteed to receive at least $\overline{M}(t_j) - K$ at maturity, plus higher payoff if a higher realized maximum value is achieved at later time instants.

How to achieve dimension reduction?

Define the adjusted exercise price $K'(t_j)$, where

$$K'(t_j) = \max(\overline{M}(t_j), K).$$

- Since $\overline{M}(t_j)$ records the maximum among discrete nodal values of the stock prices in the binomial tree, so $\overline{M}(t_j)$ is equal to $S_0 u^n$ for some integer n. One may set the original strike price K be equal to $S_0 u^m$ for some integer m (at least as a numerical approximation). In fact, it would not be quite a restriction if the number of time steps is sufficiently large. Taking $M = \max(n, m)$, then $K'(t_j) = S_0 u^M$.

This paves the adoption of dimension reduction in the discrete pricing model by normalizing the price function with respect to asset price.

We relate the adjusted strike $K'(t_j)$ with $S(t_j)$ in terms of an index k (as power of u) via

$$k = \ln \frac{S(t_j)}{K'} / \ln u \Leftrightarrow K' = S(t_j) u^{-k},$$

then k is always non-positive. This is because $K' \geq \overline{M}(t_j) \geq S(t_j)$.

Once we have set $K' = S_0 u^{\ell}$ for some integer ℓ and a similar form for $S(t_j)$, it is seen that the fixed strike lookback call value $c_X(S(t_j), K', t_j)$ is homogeneous in $S(t_j)$, since the ratio of the payoff to the prevailing asset price can be expressed as a power function in u. Homogeneity in $S(t_j)$ helps achieve dimension reduction. To this goal, we consider

$$X(k, t_j) = \frac{c_X(S(t_j), K', t_j)}{S(t_j)}.$$

The next move in the asset price may or may not result in the updating of K'. If there is no updating of K', then k is increased (decreased) by one for an up-move (down-move) of the asset price.

- $k \leq -1$ [$S(t_j) \leq K'/u$ or $K' \geq S(t_j)u$ so that it is not possible to have an updated K' in the next time step.]

  We adopt the usual backward induction procedure for the call value normalized by $S(t_j)$:

  $$X(k, t_j) = [(1-p)X(k-1, t_{j+1})d + pX(k+1, t_{j+1})u]e^{-r\Delta t}.$$

  When the asset price moves up (down) while K' does not change, the index k increases (decreases) to $k+1$ ($k-1$) since $k = \ln \dfrac{S(t_j)}{K'} / \ln u$.

- $k = 0$ [$S(t_j) = K' = \max(\overline{M}(t_j), K)$]
  A downward move of the underlying asset price makes k to become -1.
  For an upward move, the underlying asset price exceeds the adjusted strike K'. The option holder is entitled to receive an extra payoff equal to $S(t_j)(u-1)$ at maturity. Also, the value of k remains to be zero with the updating value of $K' = S(t_{j+1})$ at t_{j+1}.

Conditional on an upward move, the present value of this extra payment is

$$S(t_j)(u-1)e^{-(N-j)r\Delta t}.$$

The guaranteed payoff arising from updating of realized maximum of asset price comes from the accumulation of these payments. The corresponding binomial scheme at $k=0$ is modified as

$$\begin{aligned} X(0,t_j) = & \ [(1-p)X(-1,t_{j+1})d + pX(0,t_{j+1})u]e^{-r\Delta t} \\ & + p(u-1)e^{-(N-j)r\Delta t}. \end{aligned}$$

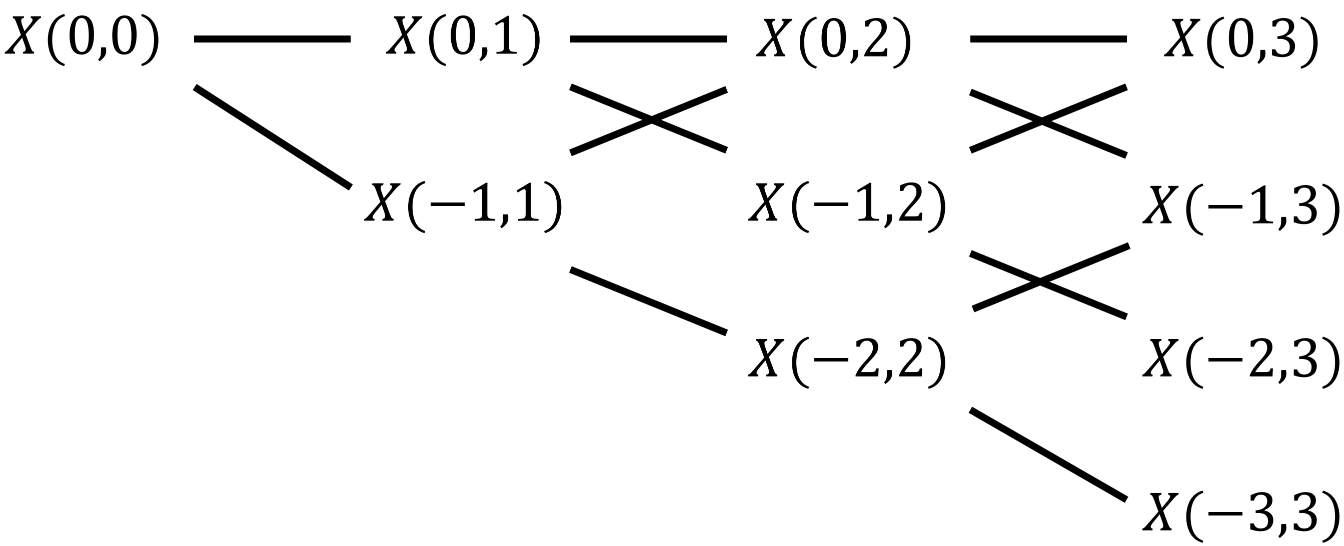
Terminal condition

At maturity, the option value is zero since $\overline{M}(t_N) \leq K' = \max(\overline{M}(t_N), K)$. We then have $X(k, t_N) = 0$ for all values of k.

Though the terminal option value is set to be zero under the present framework of adjusted strike, we have been accumulating the sum of the present value of extra payments whenever a new updated K' is recorded.

Truncated tree representation

- When $S(t_0) \geq K$, the fixed strike lookback call option is sure to be in-the-money. The truncated binomial tree starts at $(0, 0)$.



- When $S(t_0) < K$, we start the binomial tree at $(k, 0)$ with $k < 0$.

European fixed strike currency lookback call ($S = 100$, $K = 100$, $\tilde{r}_d = 0.04$, $\tilde{r}_f = 0.07$ *and* $T = 0.5$). Here, N is the number of time steps in the binomial tree.

N	Option price		
	$\sigma = 0.1$	$\sigma = 0.2$	$\sigma = 0.3$
50	6.20	11.51	17.05
100	6.34	11.80	17.50
500	6.53	12.20	18.14
1000	6.57	12.30	18.30
5000	6.64	12.44	18.51
10000	6.65	12.47	18.56
Analytic	6.78	12.70	18.92

- The rate of convergence is very slow. Even with $10,000$ time steps, the numerical results barely achieve accuracy within 2% error.

1.2 Trinomial schemes

In a trinomial model, the asset price S is assumed to jump to either uS, mS or dS after one time period $\triangle t$, where $u > m > d$. We consider a trinomial formula of option valuation of the form

$$V = \frac{p_1 V_u^{\Delta t} + p_2 V_m^{\Delta t} + p_3 V_d^{\Delta t}}{R}, \qquad R = e^{r\triangle t}.$$

This is deduced from the risk neutral valuation principle: the current option value is the discounted expectation of the terminal option value under the risk neutral pricing measure.

There are 6 unknowns: p_1, p_2, p_3, u, m and d. We take $m = 1, u = 1/d$. We obtain 3 equations by

(i) equating mean, (ii) equating variance,

(iii) setting sum of probabilities $= 1$. We are left with one free parameter.

Discounted expectation approach

Under the assumption of the Geometric Brownian motion followed by the continuous asset price process, we write

$$\ln S_{t+\triangle t} = \ln S_t + \zeta,$$

where ζ is a normal random variable with mean $\left(r - \frac{\sigma^2}{2}\right)\triangle t$ and variance $\sigma^2 \triangle t$. We approximate ζ by an approximate discrete random variable ζ^a with the following distribution

$$\zeta^a = \begin{cases} v & \text{with probability } p_1 \\ 0 & \text{with probability } p_2 \\ -v & \text{with probability } p_3 \end{cases}$$

where $v = \lambda\sigma\sqrt{\triangle t}$ and $\lambda \geq 1$. The corresponding values for u, m and d in the trinomial scheme are: $u = e^v, m = 1$ and $d = e^{-v}$. This is because when $\ln \frac{S_{t+\Delta t}}{S_t}$ assumes the value v, then $\frac{S_{t+\Delta t}}{S_t}$ assumes the value e^v. Since we allow zero displacement with probability $p_2 > 0$, we expect stronger discrete move to the right or left (giving $\lambda \geq 1$) when compared to the binomial random walk. Also, we expect $\lambda = 1$ when $p_2 = 0$.

To find the probability values p_1, p_2 and p_3, the mean and variance of the approximating discrete trinomial random walk variable ζ^a are chosen to be equal to those of ζ. These lead to

$$E[\zeta^a] = v(p_1 - p_3) = \left(r - \frac{\sigma^2}{2}\right)\triangle t$$

$$\text{var}(\zeta^a) = v^2(p_1 + p_3) - v^2(p_1 - p_3)^2 = \sigma^2 \triangle t.$$

We see that $v^2(p_1 - p_3)^2 = O(\Delta t^2)$. We may drop this term so that

$$v^2(p_1 + p_3) = \sigma^2 \triangle t,$$

while still maintaining $O(\Delta t)$ accuracy.

By considering the approximation of $\ln \frac{S_{t+\Delta t}}{S_t}$ instead of $\frac{S_{t+\Delta t}}{S_t}$, the algebraic equations for solving p_1, p_2 and p_3 involve only linear functions of Δt rather than exponential functions of Δt.

Lastly, the probabilities must be summed to one so that

$$p_1 + p_2 + p_3 = 1.$$

We then solve together to obtain

$$\begin{aligned} p_1 &= \frac{1}{2\lambda^2} + \frac{(r - \frac{\sigma^2}{2})\sqrt{\triangle t}}{2\lambda\sigma} \\ p_2 &= 1 - \frac{1}{\lambda^2} \\ p_3 &= \frac{1}{2\lambda^2} - \frac{(r - \frac{\sigma^2}{2})\sqrt{\triangle t}}{2\lambda\sigma}, \end{aligned}$$

here λ is a free parameter.

- In order that $p_2 \geq 0$, we must choose $\lambda \geq 1$. Numerical experiments indicate that the optimal choice of λ is $\sqrt{3}$ so that $p_2 = 2/3$. Indeed, Kwok and Lau (2001) show mathematically that the truncation error is smallest when $\lambda = \sqrt{3}$.

- Trinomial schemes are first order accurate, where $V_{\text{num}} - V_{\text{exact}} = K\Delta t$. The different choices of λ amount to different values of K.

- Note that $p_2 = 0$ when $\lambda = 1$, which reduces to the Cox-Ross-Rubinstein binomial scheme. This illustrates an effective mean of deriving the binomial/trinomial parameters using the discrete approximation of the logarithm of the price ratio at successive time steps.

- When $\lambda = 1$, we have $p_1 = \dfrac{1}{2} + \dfrac{\left(r - \frac{\sigma^2}{2}\right)\sqrt{\Delta t}}{2\sigma}$. This would agree with the Taylor expansion of $p = \dfrac{R-d}{u-d}, u = 1/d = e^{\sigma\sqrt{\Delta t}}$ up to $O(\Delta t)$.

Multistate extension – Kamrad-Ritchken's approach

- We assume the joint density of the prices of the two underlying assets S_1 and S_2 to be bivariate lognormal.
- Let σ_i be the volatility of asset price S_i, $i = 1, 2$ and ρ be the correlation coefficient between the two lognormal diffusion processes.
- Let S_i and $S_i^{\triangle t}$ denote, respectively, the price of asset i at the current time and one period $\triangle t$ later.
- Under the risk neutral measure, we have

$$\ln \frac{S_i^{\triangle t}}{S_i} = \zeta_i, \qquad i = 1, 2,$$

where ζ_i is a normal random variable with mean $\left(r - \frac{\sigma_i^2}{2}\right)\triangle t$ and variance $\sigma_i^2 \triangle t$.

The instantaneous correlation coefficient between ζ_1 and ζ_2 is ρ. The joint bivariate normal processes $\{\zeta_1, \zeta_2\}$ is approximated by a pair of joint discrete random variables $\{\zeta_1^a, \zeta_2^a\}$ with the following discrete distribution

ζ_1^a	ζ_2^a	probability
v_1	v_2	p_1
v_1	$-v_2$	p_2
$-v_1$	$-v_2$	p_3
$-v_1$	v_2	p_4
0	0	p_5

where $v_i = \lambda_i \sigma_i \sqrt{\triangle t}$, $i = 1, 2$. We first assume 2 free parameters λ_1 and λ_2. Later, we argue that we must choose $\lambda_1 = \lambda_2$ for consistency.

The above form of the discrete distribution can be shown to be sufficient to serve as the discrete approximation of the correlated diffusion processes with drifts. It is redundant to include scenarios, like $\zeta_1^a = v_1$ and $\zeta_2^a = 0$, $\zeta_1^a = 0$ and $\zeta_2^a = v_2$, etc.

Equating the corresponding means gives

$$E[\zeta_1^a] = v_1(p_1 + p_2 - p_3 - p_4) = \left(r - \frac{\sigma_1^2}{2}\right)\triangle t \qquad (i)$$

$$E[\zeta_2^a] = v_2(p_1 - p_2 - p_3 + p_4) = \left(r - \frac{\sigma_2^2}{2}\right)\triangle t. \qquad (ii)$$

By equating the variances and covariance to $O(\triangle t)$ accuracy, we have

$$\text{var}(\zeta_1^a) = v_1^2(p_1 + p_2 + p_3 + p_4) = \sigma_1^2\triangle t \qquad (iii)$$

$$\text{var}(\zeta_2^a) = v_2^2(p_1 + p_2 + p_3 + p_4) = \sigma_2^2\triangle t \qquad (iv)$$

$$E[\zeta_1^a\zeta_2^a] = v_1v_2(p_1 - p_2 + p_3 - p_4) = \sigma_1\sigma_2\rho\triangle t. \qquad (v)$$

In order that Eqs. (iii) and (iv) are consistent, we must set $\lambda_1 = \lambda_2$.

Writing $\lambda = \lambda_1 = \lambda_2$, we have the following four independent equations for the five probability values

$$p_1 + p_2 - p_3 - p_4 = \frac{(r - \frac{\sigma_1^2}{2})\sqrt{\triangle t}}{\lambda \sigma_1}$$

$$p_1 - p_2 - p_3 + p_4 = \frac{(r - \frac{\sigma_2^2}{2})\sqrt{\triangle t}}{\lambda \sigma_2}$$

$$p_1 + p_2 + p_3 + p_4 = \frac{1}{\lambda^2}$$

$$p_1 - p_2 + p_3 - p_4 = \frac{\rho}{\lambda^2}.$$

Since the probabilities must be summed to one, this gives the remaining condition as

$$p_1 + p_2 + p_3 + p_4 + p_5 = 1.$$

The solution of the above linear algebraic system of equations gives

$$p_1 = \frac{1}{4}\left[\frac{1}{\lambda^2} + \frac{\sqrt{\triangle t}}{\lambda}\left(\frac{r - \frac{\sigma_1^2}{2}}{\sigma_1} + \frac{r - \frac{\sigma_2^2}{2}}{\sigma_2}\right) + \frac{\rho}{\lambda^2}\right]$$

$$p_2 = \frac{1}{4}\left[\frac{1}{\lambda^2} + \frac{\sqrt{\triangle t}}{\lambda}\left(\frac{r - \frac{\sigma_1^2}{2}}{\sigma_1} - \frac{r - \frac{\sigma_2^2}{2}}{\sigma_2}\right) - \frac{\rho}{\lambda^2}\right]$$

$$p_3 = \frac{1}{4}\left[\frac{1}{\lambda^2} + \frac{\sqrt{\triangle t}}{\lambda}\left(-\frac{r - \frac{\sigma_1^2}{2}}{\sigma_1} - \frac{r - \frac{\sigma_2^2}{2}}{\sigma_2}\right) + \frac{\rho}{\lambda^2}\right]$$

$$p_4 = \frac{1}{4}\left[\frac{1}{\lambda^2} + \frac{\sqrt{\triangle t}}{\lambda}\left(-\frac{r - \frac{\sigma_1^2}{2}}{\sigma_1} + \frac{r - \frac{\sigma_2^2}{2}}{\sigma_2}\right) - \frac{\rho}{\lambda^2}\right]$$

$$p_5 = 1 - \frac{1}{\lambda^2}, \qquad \lambda \geq 1 \text{ is a free parameter.}$$

Two-state trinomial model

- For convenience, we write $u_i = e^{v_i}$, $d_i = e^{-v_i}$, $i = 1, 2$.
- Let $V_{u_1 u_2}^{\Delta t}$ denote the option price at one time period later with asset prices $u_1 S_1$ and $u_2 S_2$, and similar meaning for $V_{u_1 d_2}^{\Delta t}, V_{d_1 u_2}^{\Delta t}$ and $V_{d_1 d_2}^{\Delta t}$.
- We let $V_{0,0}^{\Delta t}$ denote the option price one period later with no jumps in asset prices.
- The corresponding 5-point formula for the two-state trinomial model based on the risk neutral valuation approach can be expressed as

$$V = (p_1 V_{u_1 u_2}^{\triangle t} + p_2 V_{u_1 d_2}^{\triangle t} + p_3 V_{d_1 d_2}^{\triangle t} + p_4 V_{d_1 u_2}^{\triangle t} + p_5 V_{0,0}^{\triangle t})/R.$$

- When $\lambda = 1$, we have $p_5 = 0$ and the above 5-point formula reduces to the 4-point formula.

1.3 Forward shooting grid methods (strongly path dependent options)

- For path dependent options, the option value also depends on the path function $F_t = F(S, t)$ defined specifically for the given nature of path dependence, say, the minimum asset price realized along a specific asset price path.
- Since option value depends also on F_t, we find the value of the path dependent option at each node in the lattice tree for all alternative values of F_t that can occur.
- The approach of appending an auxiliary state vector at each node in the lattice tree to model the correlated evolution of F_t with S_t is commonly called the *forward shooting grid* (FSG) *method*.

- Consider a trinomial tree whose probabilities of upward, zero and downward jump of the asset price are denoted by p_u, p_0 and p_d, respectively.
- Let $V^n_{j,k}$ denote the numerical option value of the exotic path dependent option at the n^{th}-time level (n time steps from the tip of the tree). Also, j denotes the j upward jumps from the initial asset value and k denotes the numbering index for the various possible values of the augmented state variable F_t at the $(n, j)^{\text{th}}$ node.
- Let G denote the function that describes the correlated evolution of F_t with S_t over the time interval Δt, that is,

$$F_{t+\Delta t} = G(F_t, t, S_{t+\Delta t}).$$

- Let $g(k, n, j)$ denote the grid function which is considered as the discrete analog of the evolution function G. Here, k is the index for F_t, n is the index for t and j is the index for $S_{t+\Delta t}$.

- The trinomial version of the FSG scheme can be represented as follows

$$V_{j,k}^{n} = \left[p_u V_{j+1,g(k,n,j+1)}^{n+1} + p_0 V_{j,g(k,n,j)}^{n+1} + p_d V_{j-1,g(k,n,j-1)}^{n+1}\right] e^{-r\Delta t},$$

  where $e^{-r\Delta t}$ is the discount factor over time interval Δt.

- To price a specific path dependent option, the design of the FSG algorithm requires the specification of the grid function $g(k, n, j)$.

  For notational convenience, if the grid function has no dependence on t, we simply write it as $g(k, j)$.

Cumulative Parisian feature of knock-out

- Let M denote the prespecified number of cumulative breaching occurrences that is required to activate knock-out, and let k be the integer variable that counts the cumulative number of breaching occurrences so far.

- Let B denote the down barrier associated with the knock-out feature. Let x_j denote the value of $x = \ln S$ that corresponds to j upward moves in the trinomial tree. That is, $x_j = \ln S_0 + j\Delta x$, where S_0 is the initial asset price and Δx is the stepwidth of the state variable x.

- When $n\Delta t$ happens to be a monitoring instant, the index k increases its value by 1 if the asset price S falls on or below the barrier B, that is, $x_j \leq \ln B$.

Counting the number of time steps that x_j falls below or at $\ln B$

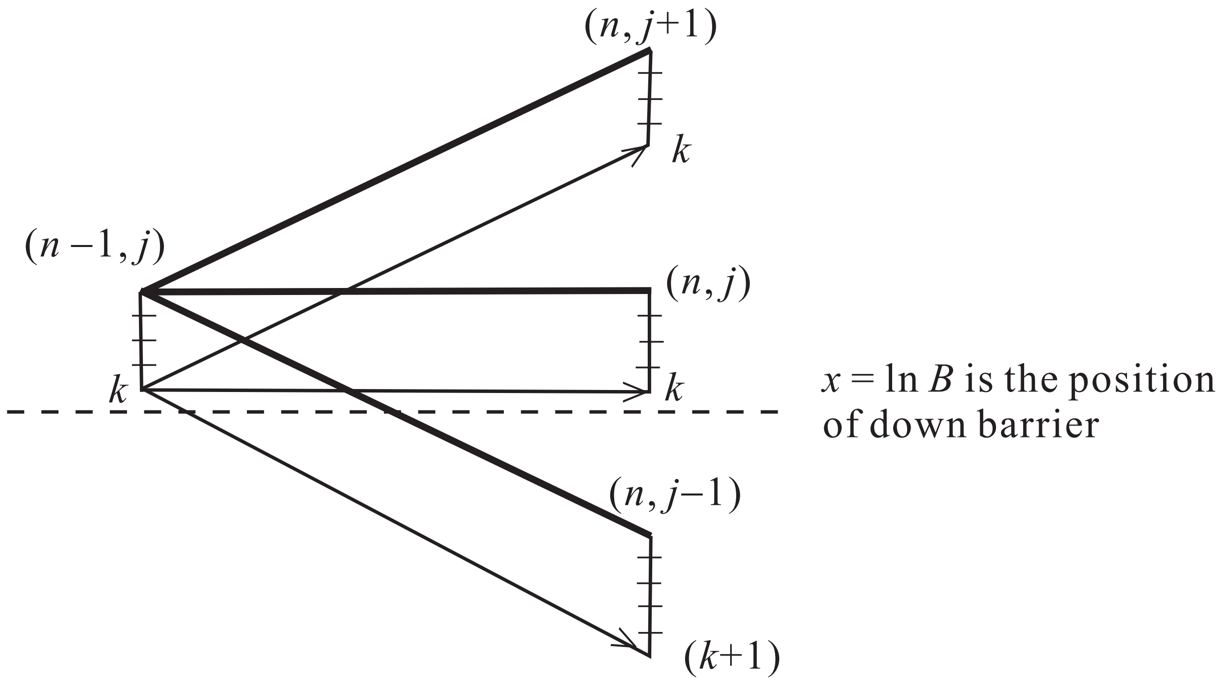
To incorporate the cumulative Parisian feature, the appropriate choice of the grid function $g_{cum}(k,j)$ is defined by

$$g_{cum}(k,j) = k + \mathbf{1}_{\{x_j \le \ln B\}}.$$

The forward shooting grid algorithm is exemplified by

$$V_{j,k}^{n-1} = \begin{cases} [p_u V_{j+1,k}^n + p_0 V_{j,k}^n + p_d V_{j-1,k}^n] e^{-r\Delta t} \\ \quad \text{if } n\Delta t \text{ is not a monitoring instant} \\ [p_u V_{j+1,g_{cum}(k,j+1)}^n + p_0 V_{j,g_{cum}(k,j)}^n + p_d V_{j-1,g_{cum}(k,j-1)}^n] e^{-r\Delta t} \\ \quad \text{if } n\Delta t \text{ is a monitoring instant} \end{cases}.$$

The number of breaching occurrences k is updated to $g_{cum}(k,j+1)$ when the updated asset price at the n^{th} time level is S_{j+1}^n [up move from S_j^{n-1} at the $(n-1)^{\text{th}}$ time level]. The knock-out condition is defined by $V_{j,M}^n = 0$.



Schematic diagram that illustrates the construction of the grid function $g_{cum}(k,j)$ that models the cumulative Parisian feature. The down barrier $\ln B$ is placed mid-way between two horizontal rows of trinomial nodes. Here, the n^{th}-time level is a monitoring instant. In this example, since $x_{j-1} < \ln B$, the forward shooting grid algorithm is

$$V_{j,k}^{n-1} = \left[p_u V_{j+1,k}^n + p_0 V_{j,k}^n + p_d V_{j-1,k+1}^n\right] e^{-r\Delta t}, \quad k = 1, 2, \cdots.$$

1. The pricing of options with the continuously monitored cumulative Parisian feature is obtained by setting all time steps to be monitoring instants.

2. The computational time required for pricing an option with the cumulative Parisian feature requiring M breaching occurrences to knock out is about M times that of an one-touch knock-out barrier option.

3. The size of the augmented state vector appended at each node grows from zero at the tip of the trinomial tree to the maximum size of M as we proceed the time marching in the trinomial calculations. At maturity, we set

$$\begin{cases} V_{j,k}^N = V_T(S_j^N), & 0 \leq k < M, \\ V_{j,M}^N = 0, & k = M. \end{cases}$$

4. Applications of the cumulative counting feature can also be found in structured products, say, the coupons (as in reverse convertibles) are accrued contingent on the underlying stock price lying within certain range of values.

5. The *consecutive Parisian feature* counts the number of consecutive breaching occurrences that the asset price stays in the knock-out region. The count is reset to zero once the asset price moves out from the knock-out region. Assuming B to be the down barrier, the appropriate grid function $g_{con}(k, j)$ in the FSG algorithm is given by

$$g_{con}(k, j) = (k + 1)\mathbf{1}_{\{x_j \leq \ln B\}}.$$

6. The consecutive counting feature can be found in the soft call provision in a convertible bond. In most convertible bond contracts, the issuer is allowed to issue the notice of redemption conditional on the underlying stock price staying above the preset hurdle price for a prespecified number of trading days.

Call options with the strike reset feature

- Consider a call option with the strike reset feature where the option's strike price is reset to the prevailing asset price on a preset reset date if the option is out-of-money on that date.
- Let $t_i, i = 1, 2, \cdots, M$, denote the i^{th} reset date and X_i denote the strike price specified on t_i based on the above reset rule. Write X_0 as the strike price set at initiation, then X_i is given by

$$X_i = \min(X_{i-1}, S_{t_i}), \quad i = 1, 2, \cdots, M,$$

  where S_{t_i} is the prevailing asset price on the reset date t_i.
- Why does it become superfluous to set

$$X_i = \min(X_{i-1}, S_{t_i}, X_0), \quad i = 1, 2, \cdots, M?$$

  Since $X_1 = \min(X_0, S_{t_1})$, the information of the initial strike price X_0 has been embedded in the strike reset procedure. Suppose the discrete realized minimum asset price has not reached as low as X_0 after i time steps, then $X_i = X_0$.

- The strike price at expiry of this call option is not fixed since its value depends on the realization of the asset price on the reset dates.

- When we apply the backward induction procedure in the trinomial calculations, we encounter the difficulty in defining the terminal payoff since the strike price can assume many possible values due to the reset mechanism.

- These difficulties can be resolved easily using the FSG approach by tracking the evolution of the asset price and the strike reset through an appropriate choice of the grid function. The terminal payoff in the FSG lattice tree is computed with respect to all possible values of k that can be realized at maturity.

Remark If we do not impose the initial strike X_0, then this strike reset call option resembles the discretely monitored floating strike lookback call option with terminal payoff: $\max(S_T - S_{min}, 0)$.

- Suppose the original strike price X_0 corresponds to the index k_0, this would mean $X_0 = S_0 u^{k_0}$. For convenience, we may choose the proportional jump parameter u such that k_0 is an integer. In terms of these indexes, the grid function that models the correlated evolution between the reset strike price and asset price is given by

$$g_{reset}(k, j) = \min(k, j),$$

where k denotes the index that corresponds to the strike price reset in the last reset date and j is the index that corresponds to the prevailing asset price at the reset date.

- Since the strike price is reset only on a reset date, we perform the usual trinomial calculations for those time levels that do not correspond to a reset date while the augmented state vector of strike prices are adjusted according to the grid function $g_{reset}(k, j)$ for those time levels that correspond to a reset date.

- The FSG algorithm for pricing the reset call option is given by

$$V_{j,k}^{n-1} = \begin{cases} \left[p_u V_{j+1,k}^n + p_0 V_{j,k}^n + p_d V_{j-1,k}^n\right] e^{-r\Delta t} \\ \qquad \text{if } n\Delta t \neq t_i \text{ for some } i \\ \left[p_u V_{j+1,g_{reset}(k,j+1)}^n + p_0 V_{j,g_{reset}(k,j)}^n + p_d V_{j-1,g_{reset}(k,j-1)}^n\right] e^{-r\Delta t}, \\ \qquad \text{if } n\Delta t = t_i \text{ for some } i \end{cases}.$$

- The payoff values along the terminal nodes at the N^{th} time level in the trinomial tree are given by

$$V_{j,k}^N = \max(S_0 u^j - S_0 u^k, 0), \quad j = -N, -N+1, \cdots, N,$$

and k assumes values that lie between k_0 and the index corresponding to the lowest asset price on the last reset date (since there is no reset on maturity date). It is necessary to list all possible nodal values that can be assumed by the reset strike.

Floating strike arithmetic averaging call

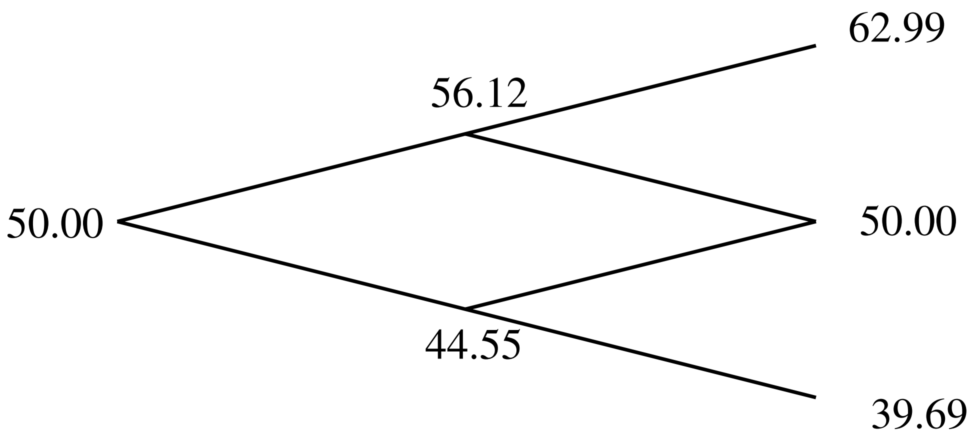
- To price an Asian option, we find the option value at each node for all possible values of the path function $F(S,t)$ that can occur at that node.

- Unfortunately, the number of possible values for the averaging value F at a binomial node for the arithmetic averaging option grows exponentially at 2^n, where n is the number of time steps from the tip of the binomial tree. (Why 2^n? Since there are 2^n possible realized asset paths after n time steps and each path gives a unique arithmetic averaging value.)

- Therefore, the binomial schemes that place no constraint on the number of possible F values at the binomial nodes would become computationally infeasible.

Illustration

Consider the following tree



There are $4 = 2^2$ possible arithmetic averaging values after 2 time steps, namely,

$$A_{uu} = \frac{50.00 + 56.12 + 62.99}{3}, \quad A_{ud} = \frac{50.00 + 56.12 + 50.00}{3},$$
$$A_{du} = \frac{50.00 + 44.55 + 50.00}{3}, \quad A_{dd} = \frac{50.00 + 44.55 + 39.69}{3}.$$

Note that these arithmetic averaging values do not coincide with the asset prices at the nodes at the 2nd time level. Extending to a 3-step binomial tree, there are $8 = 2^3$ possible arithmetic averaging values, namely, $A_{uuu}, A_{uud}, A_{udu}, \cdots, A_{ddd}$.

Geometric averaging values

- Two-step binomial tree

$$
\begin{aligned}
G_{uu} &= \sqrt[3]{(S_0)(S_0u)(S_0u^2)} = S_0u, \\
G_{dd} &= \sqrt[3]{S_0(S_0u^{-1})(S_0u^{-2})} = S_0u^{-1}, \\
G_{ud} &= \sqrt[3]{(S_0)(S_0u)(S_0)} = S_0u^{1/3}, \\
G_{du} &= \sqrt[3]{(S_0)(S_0u^{-1})(S_0)} = S_0u^{-1/3}.
\end{aligned}
$$

- Three-step binomial tree

$$
\begin{aligned}
G_{uuu} &= \sqrt[4]{(S_0)(S_0u)(S_0u^2)(S_0u^3)} = S_0u^{1.5}, \\
G_{ddd} &= \sqrt[4]{(S_0)(S_0u^{-1})(S_0u^{-2})(S_0u^{-3})} = S_0u^{-1.5}, \\
G_{uud} &= \sqrt[4]{(S_0)(S_0u)(S_0u^2)(S_0u)} = S_0u, \\
G_{udu} &= S_0u^{0.5}, \quad G_{duu} = S_0, \\
G_{udd} &= \sqrt[4]{(S_0)(S_0u)(S_0)(S_0u^{-1})} = S_0, \\
G_{dud} &= S_0u^{-0.5}, G_{ddu} = S_0u^{-1}.
\end{aligned}
$$

There are 7 possible geometric averaging values after 3 time steps.

- A possible remedy is to restrict the possible values for F to a certain set of predetermined values. The option value $V(S, F, t)$ for other values of F is obtained from the known values of V at predetermined F values by an interpolation between the nodal values.

- The methods of interpolation include the nearest node interpolation, linear (between 2 neighboring nodes) and quadratic interpolation (between 3 neighboring nodes).

- How to cope with the exponentially large number of possible values assumed by taking the arithmetic averaging of the realized asset price path? We limit the number of averaging values to some multiple of the number of values assumed by the asset price (here, the multiple is $1/\rho$).

For a given time step Δt, we fix the respective step width for the logarithm of asset price and average to be

$$\Delta W = \sigma\sqrt{\Delta t} \quad \text{and} \quad \Delta Y = \rho \Delta W, \quad \rho < 1,$$

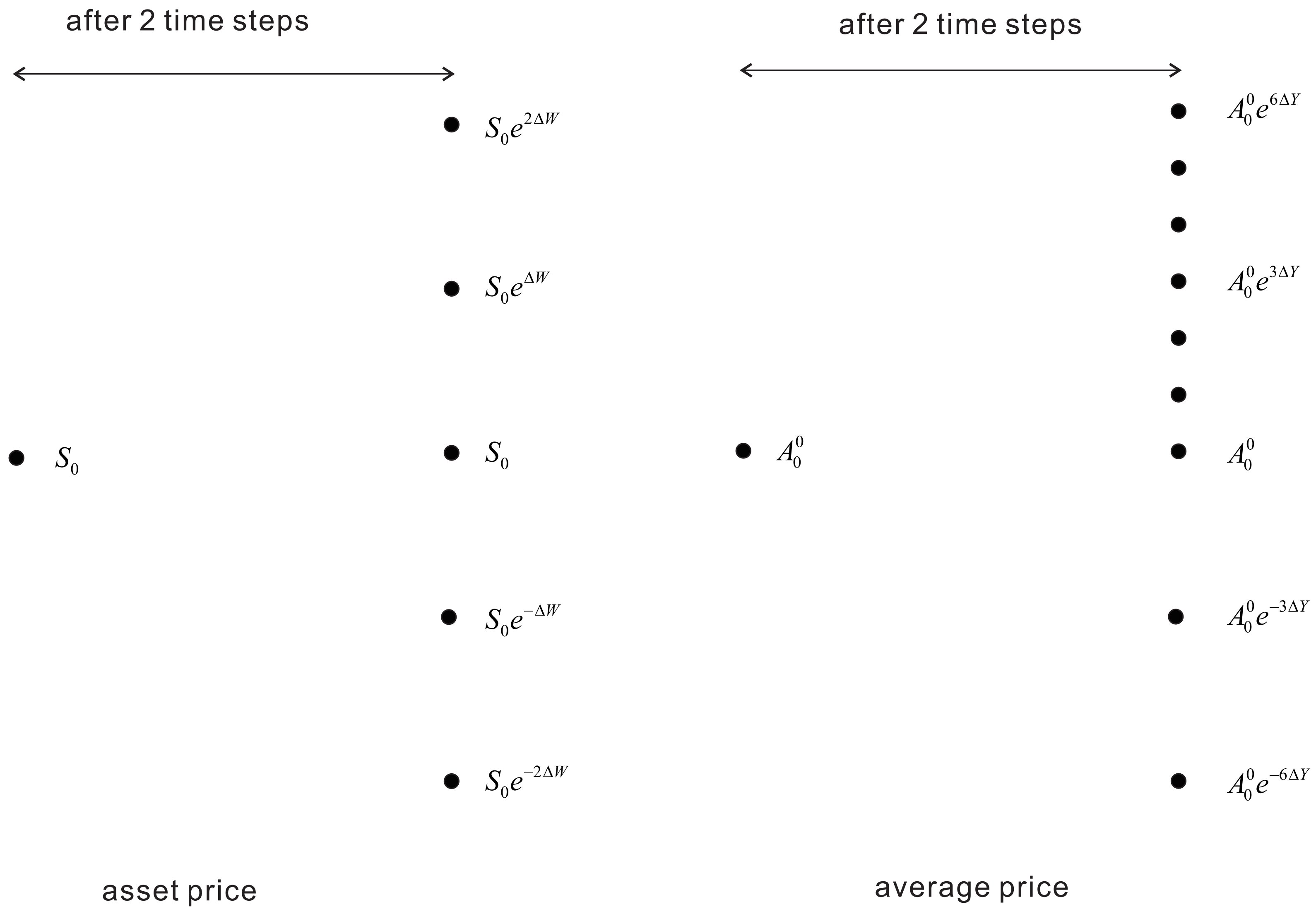
and define the possible values for S_t and A_t at the n^{th} time step by

$$S_j^n = S_0 e^{j\Delta W} \quad \text{and} \quad A_k^n = S_0 e^{k\Delta Y},$$

where j and k are integers, and S_0 is the asset price at the tip of the binomial tree.

- We take $1/\rho$ to be an integer. The larger integer value chosen for $1/\rho$, the finer the quantification of the arithmetic averaging asset value.

Quantification of arithmetic averaging asset value $\left(\text{Here, } \dfrac{1}{\rho} = 3 \text{ is taken.} \right)$



The continuous version of the arithmetic averaging state variable is defined by

$$A_t = \frac{1}{t} \int_0^t S_u \, du.$$

- The terminal payoff of the floating strike Asian call option is given by $\max(S_T - A_T, 0)$, where A_T is the arithmetic average of S_t over the time period $[0, T]$.

- Similarly, the terminal payoffs of other related Asian options are

  (i) Floating strike Asian put option: $\max(A_T - S_T, 0)$;

  (ii) Fixed strike Asian call option: $\max(A_T - X, 0)$, where X is the fixed strike price.

Updating rule of A_t over successive discrete time points

Consider the following relation between A_t and S_t in differential form:

$$d(tA_t) = S_t\,dt \text{ or } dA_t = \frac{1}{t}(S_t - A_t)\,dt,$$

we approximate the above differential at time $t + \Delta t$ by adopting

$$(t + \Delta t)[A_{t+\Delta t} - A_t] = (S_{t+\Delta t} - A_{t+\Delta t})\Delta t,$$

so that

$$A_{t+\Delta t} = \frac{(t + \Delta t)A_t + \Delta t\ S_{t+\Delta t}}{t + 2\Delta t} \equiv G(t, A_t, S_{t+\Delta t}).$$

This is the updating rule of $A_{t+\Delta t}\,\Delta t$ at the new time level $t + \Delta t$ based on the old value A_t at the previous time level t and the updated asset value $S_{t+\Delta t}$ at the new time level $t + \Delta t$.

Suppose $t = n\Delta t$, then

$$A_{n+1} = \frac{(n + 1)A_n + S_{n+1}}{n + 2}.$$

Note that there are $n + 1$ asset prices recorded between 0 and $n\Delta t$.

Consider the binomial procedure at the $(n, j)^{\text{th}}$ node, suppose we have an upward move in the asset price from S_j^n to S_{j+1}^{n+1} and let $A_{k^+(j)}^{n+1}$ be the corresponding updated value of A_t changing from A_k^n when the asset price moves up from S_j^n to S_{j+1}^{n+1}. Setting $A_0^0 = S_0$ and taking $t = n\Delta t$, the equivalence of the above discrete updating rule is given by

$$A_{k^+(j)}^{n+1} = \frac{(n+1)A_k^n + S_{j+1}^{n+1}}{n+2}. \qquad (a)$$

For a downward move in the asset price from S_j^n to S_{j-1}^{n+1}, A_k^n changes to $A_{k^-(j)}^{n+1}$ where

$$A_{k^-(j)}^{n+1} = \frac{(n+1)A_k^n + S_{j-1}^{n+1}}{n+2}. \qquad (b)$$

Note that $A_{k^\pm(j)}^{n+1}$ in general do not coincide with $A_{k'}^{n+1} = S_0 e^{k'\Delta Y}$, for some integer k'.

Recall $A_{k^\pm(j)}^{n+1} = S_0 e^{k^\pm(j)\Delta Y}$ and $S_{j\pm1}^{n+1} = S_0 e^{(j\pm1)\Delta W}$.

In terms of ΔW and ΔY, after canceling the common factor S_0, eqs. (a) and (b) can be expressed as

$$e^{k^{\pm}(j)\Delta Y} = \frac{(n+1)e^{k\Delta Y} + e^{(j\pm 1)\Delta W}}{n+2}.$$

Accordingly, we compute the indexes $k^{\pm}(j)$ by

$$g(n,k,j\pm 1) = k^{\pm}(j) = \frac{\ln \frac{(n+1)e^{k\Delta Y}+e^{(j\pm 1)\Delta W}}{n+2}}{\Delta Y}. \qquad (1)$$

- We define the integers $k^{\pm}_{floor}$ such that $A^{n+1}_{k^{\pm}_{floor}}$ are the largest possible $A^{n+1}_{k'}$ values less than or equal to $A^{n+1}_{k^{\pm}(j)}$. We then set $k^{+}_{floor} =$ floor$(k^{+}(j))$ and $k^{-}_{floor} =$ floor$(k^{-}(j))$, where floor(x) denotes the largest integer less than or equal to x. Equation (1) corresponds to the evolution of A^n_k to $A^{n+1}_{k^{\pm}(j)}$ depending on the updated value of $S^{n+1}_{j\pm 1}$ [in terms of the indexes k and $k^{\pm}(j)$].

Restricting the size of the augmented state vector representing possible averaging values

- What would be the possible range of k at the n^{th} time step? We observe that the arithmetic averaging state variable A_t must lie between the maximum asset value S_n^n and the minimum asset value S_{-n}^n, so k must lie between $-\frac{n}{\rho} \leq k \leq \frac{n}{\rho}$. Unless ρ assumes a very small value, the number of predetermined values for A_t is in general manageable.

- Consider A_ℓ^n, where ℓ is in general a real number. We write $\ell_{floor} =$ floor(ℓ) and let $\ell_{ceil} = \ell_{floor} + 1$, then A_ℓ^n lies between $A_{\ell_{floor}}^n$ and $A_{\ell_{ceil}}^n$. Though the number of possible values of ℓ grows exponentially with the number of time steps in the binomial tree, both ℓ_{floor} and ℓ_{ceil} at the n^{th} time level assume an integer value lying between $-\dfrac{n}{\rho}$ and $\dfrac{n}{\rho}$.

Linear interpolation

- Let $c_{j,\ell}^n$ denote the numerical approximation to the Asian call value at the $(n,j)^{\text{th}}$ node with the averaging state variable assuming the value A_ℓ^n, and similar notations for $c_{j,\ell_{floor}}^n$ and $c_{j,\ell_{ceil}}^n$.
- For non-integer value $\ell, c_{j,\ell}^n$ is approximated through linear interpolation using the call values $c_{j,\ell_{floor}}^n$ and $c_{j,\ell_{ceil}}^n$ at the neighboring nodes.

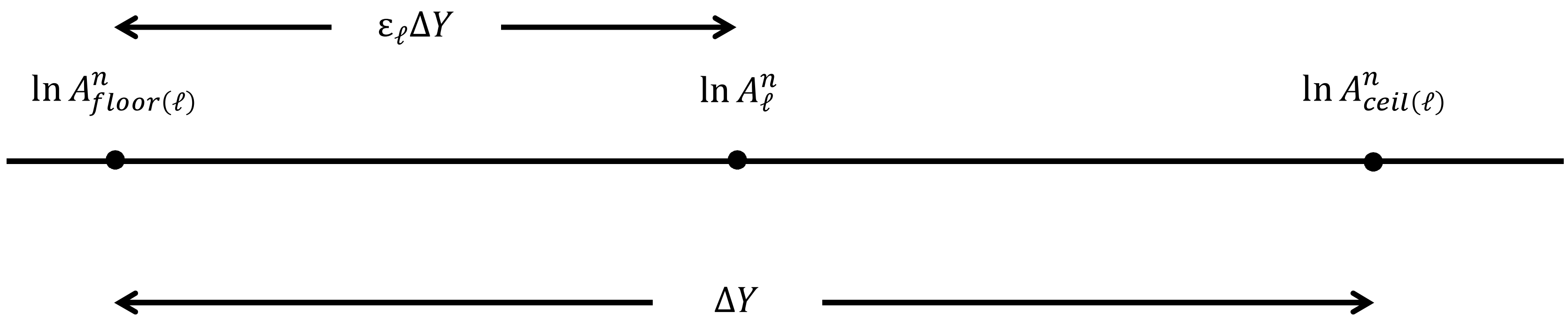
$$\begin{aligned} c_{j,\ell}^n &= c_{j,\ell_{floor}}^n + \epsilon_\ell \left(c_{j,\ell_{ceil}}^n - c_{j,\ell_{floor}}^n \right) \\ &= \epsilon_\ell c_{j,\ell_{ceil}}^n + (1-\epsilon_\ell) c_{j,\ell_{floor}}^n, \end{aligned}$$

where

$$\epsilon_\ell = \frac{\ln A_\ell^n - \ln A_{\ell_{floor}}^n}{\Delta Y}.$$

Here, ϵ_ℓ is the fraction of one step width ΔY between $\ln A_{ceil(\ell)}^n$ and $\ln A_{floor(\ell)}^n$, where

$$A_\ell^n = A_{\ell_{floor}}^n e^{\epsilon_\ell \Delta Y}.$$



- Here, ℓ is a real number lying between two consecutive integers, floor(ℓ) and ceil(ℓ), where ceil(ℓ) = floor(ℓ) + 1.

- Numerical option values are available only at $A^n_{floor(\ell)}$ and $A^n_{ceil(\ell)}$, where the index k in A^n_k assumes an integer value [like floor(ℓ) or ceil(ℓ)].

- We use the log distance between $A^n_{\text{floor}(\ell)}$, A^n_ℓ and $A^n_{\text{ceil}(\ell)}$ as reference for linear interpolation. For ℓ to be non-integer, we approximate $c^n_{j,\ell}$ by linear interpolation between $c^n_{j,floor(\ell)}$ and $c^n_{j,ceil(\ell)}$.

- By applying the above linear interpolation formula [taking ℓ to be $k^+(j)$ and $k^-(j)$ successively], the FSG algorithm with linear interpolation for pricing the floating strike arithmetic averaging call option is given by

$$\begin{aligned} c_{j,k}^n &= e^{-r\Delta t}\left[pc_{j+1,k^+(j)}^{n+1} + (1-p)c_{j-1,k^-(j)}^{n+1}\right] \\ &\approx e^{-r\Delta t}\left\{p\left[\epsilon_{k^+(j)}c_{j+1,k_{ceil}^+}^{n+1} + (1-\epsilon_{k^+(j)})c_{j+1,k_{floor}^+}^{n+1}\right]\right. \\ &\quad \left. + (1-p)\left[\epsilon_{k^-(j)}c_{j-1,k_{ceil}^-}^{n+1} + (1-\epsilon_{k^-(j)})c_{j-1,k_{floor}^-}^{n+1}\right]\right\}, \end{aligned} \tag{2}$$

$n = N-1, \cdots, 0, j = -n, -n+2, \cdots, n, k$ is an integer between $-\frac{n}{\rho}$ and $\frac{n}{\rho}$, $k^\pm(j)$ are given by Eq. (i) while

$$\epsilon_{k^\pm(j)} = \frac{\ln A_{k^\pm(j)}^{n+1} - \ln A_{k_{floor}^\pm}^{n+1}}{\Delta Y}. \tag{3}$$

Terminal payoff of a floating strike Asian call option

The final condition is

$$
\begin{aligned}
c_{j,k}^N &= \max(S_j^N - A_k^N, 0) \\
&= \max(S_0 e^{j\Delta W} - S_0 e^{k\Delta Y}, 0), \quad j = -N, -N+2, \cdots, N.
\end{aligned}
$$

The upper (lower) bound of arithmetic averaging values can be deduced by assuming upward (downward) moves of the stock price at all time steps. We can then deduce the range of values that can be assumed by k.

In summary, we compute the updated arithmetic average values based on n, A_k^n and $S_{j\pm 1}^{n+1}$.

$$A_k^n \longrightarrow A_{k^+(j)}^{n+1} \quad \text{when} \quad S_j^n \longrightarrow S_{j+1}^{n+1}$$
$$A_k^n \longrightarrow A_{k^-(j)}^{n+1} \quad \text{when} \quad S_j^n \longrightarrow S_{j-1}^{n+1}$$

Note that k is an integer while $k^+(j)$ and $k^-(j)$ are in general non-integers. Since the numerical call option values at the $(n+1)^{th}$ time step are known at integer value of the index k' for $A_{k'}^{n+1}$, we use the interpolation scheme to estimate $c_{j,k^\pm(j)}^{n+1}$ as follows:

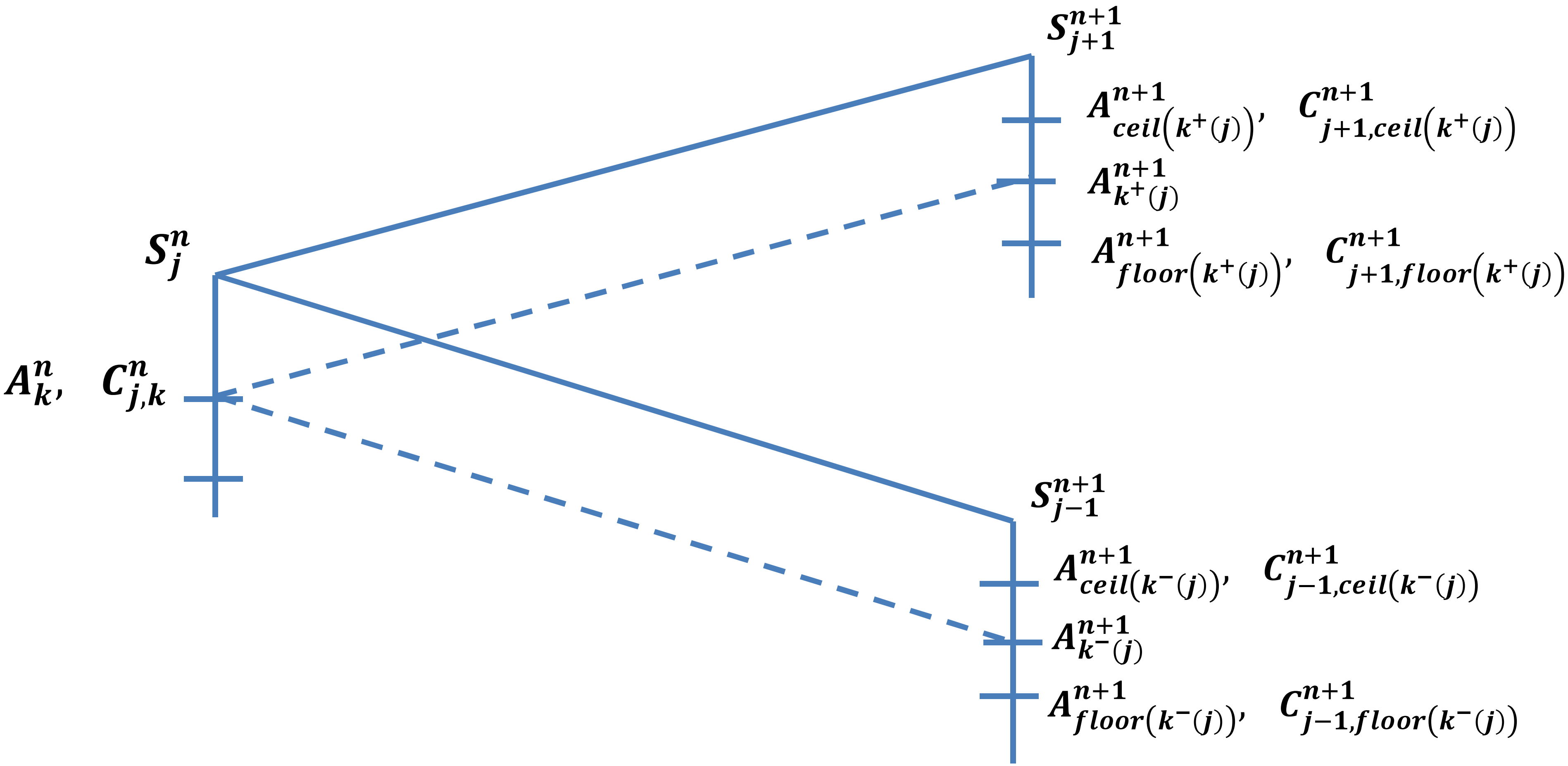
$$c_{j,k^\pm(j)}^{n+1} = \epsilon_{k^\pm(j)} c_{j,ceil(k^\pm(j))}^{n+1} + (1 - \epsilon_{k^\pm(j)}) c_{j,floor(k^\pm(j))}^{n+1},$$

where

$$\epsilon_{k^\pm(j)} = \frac{\ln A_{k^\pm(j)}^{n+1} - \ln A_{floor(k^\pm(j))}^{n+1}}{\Delta Y}.$$

Using the discounted expectation approach, we have

$$c_{j,k}^n = \left[p c_{j+1,k^+(j)}^{n+1} + (1-p) c_{j-1,k^-(j)}^{n+1} \right] e^{-r\Delta t}.$$



- Recall that $S_j^n = S_0 u^j = S_0 e^{j\Delta W}$ and $A_k^n = S_0 e^{k\Delta Y} = S_0 e^{k\rho\Delta W}$. Also, A_k^n becomes $A_{k^+(j)}^{n+1}$ when $S_j^n \to S_{j+1}^{n+1}$ and $A_{k^-(j)}^{n+1}$ when $S_j^n \to S_{j-1}^{n+1}$.
- At each time step, we compute the numerical option values at all possible integer values of k.

Remarks

1. When dealing with the discretely monitored Asian options, we only update the values of k at a time step that corresponds to a monitoring instant. At a time step that does not correspond to a monitoring instant, we compute $c_{j,k}^n$ using the binomial formula:

$$c_{j,k}^n = [pc_{j+1,k}^{n+1} + (1-p)c_{j-1,k}^{n+1}]e^{-r\Delta t}.$$

2. The range of averaging value of the asset price at a given time step can be deduced by finding the largest possible averaging value (upward move at every time step) and the smallest value (downward move at every time step). Given the step width ΔY for the averaging value, we can determine $k_{\text{max}}^{(n)}$ and $k_{\text{min}}^{(n)}$ that correspond to the upper bound and lower bound on the averaging value at the n^{th} time step.

Geometric averaging

Extension to Asian options on geometrical averaging of asset values

$$\ln G_n = \frac{1}{n+1}(\ln S_0 + ... + \ln S_n)$$

$$\ln G_{n+1} = \frac{1}{n+2}(\ln S_0 + ... + \ln S_n + \ln S_{n+1}),$$

so the evolution of G_{n+1} in terms of n, G_n and S_{n+1} is

$$(n+2)\ln G_{n+1} - (n+1)\ln G_n = \ln S_{n+1}$$

$$G_{n+1} = (G_n)^{\frac{n+1}{n+2}}(S_{n+1})^{\frac{1}{n+2}}.$$

It is necessary to convert this correlated evolution function between $G_{t+\Delta t}$, G_t and $S_{t+\Delta t}$ in terms of the indexes that correspond to the discrete values of the three state variables.

Suppose we write

$$G_k^n = G_0^0 e^{k\Delta Y} = S_0 e^{k\rho\Delta W}$$

$$S_{j\pm 1}^{n+1} = S_0 e^{(j\pm 1)\Delta W},$$

we deduce that

$$e^{k^\pm(j)\rho\Delta W} = (e^{k\rho\Delta W})^{\frac{n+1}{n+2}}(e^{(j\pm 1)\Delta W})^{\frac{1}{n+2}}.$$

This gives the grid function

$$k^\pm(j) = g(n,k,j\pm 1) = k\frac{n+1}{n+2} + \frac{j\pm 1}{\rho}\frac{1}{n+2}.$$

In general, $k^\pm(j)$ would not assume integer values. The option value at $k^\pm(j)$ at the $(n+1)^{\text{th}}$ time step is obtained by linear interpolation at $floor(k^\pm(j))$ and $ceil(k^\pm(j))$. We have

$$V_{j\pm 1,k^\pm(j)}^{n+1} = V_{j\pm 1,floor(k^\pm(j))}^{n+1} + \epsilon_{k^\pm(j)}\left[V_{j\pm 1,ceil(k^\pm(j))}^{n+1} - V_{j\pm 1,floor(k^\pm(j))}^{n+1}\right]$$

where

$$\epsilon_{k^\pm(j)} = k^\pm(j) - floor(k^\pm(j)).$$

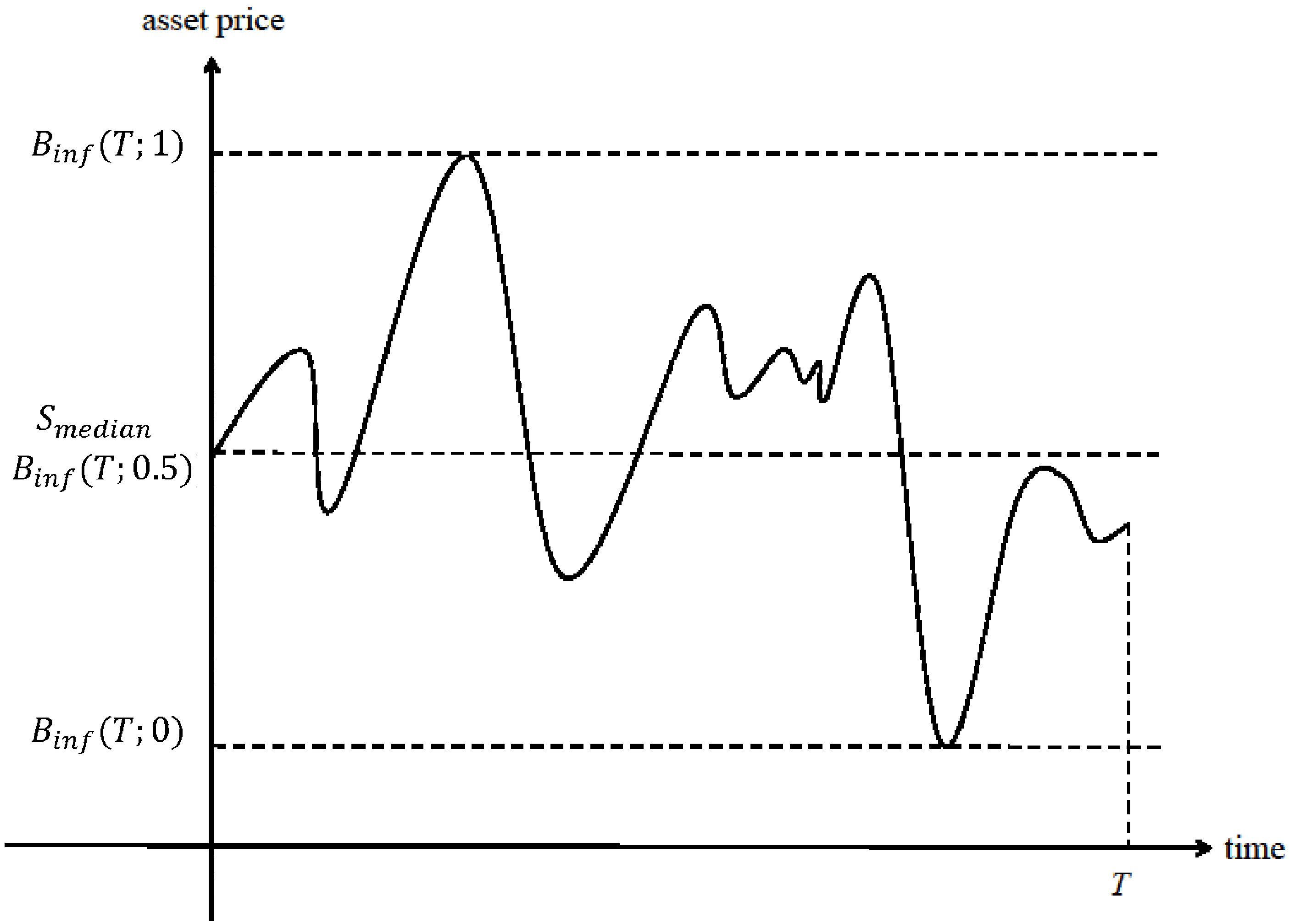
Alpha quantile option

The α-quantile option takes the barrier level to be a stochastic state variable that defines the terminal payoff.

For a given percentile α, $0 \leq \alpha \leq 1$, the α-quantile of $\{S_t\}_{t\in[0,T]}$ is defined as

$$B_{\text{inf}}(T;\alpha) = \text{inf}\left\{B : \frac{1}{T}\int_0^T \mathbf{1}_{\{S_t \leq B\}}\,dt \geq \alpha\right\}. \tag{A}$$

We gradually lower the barrier B and eventually the percentage of time that S_t stays at or below B just hits at α. In other words, $B_{\text{inf}}(T;\alpha)$ is the barrier level such that the asset price S_t is at or below $B_{\text{inf}}(T;\alpha)$ exactly α of the monitoring period. When $\alpha = 0.5$, $B_{\text{inf}}(T;0.5)$ is the median S_{median} of the asset price process over the time period $[0,T]$.



- The asset price is below S_{median} exactly half of the time period $[0, T]$.
- $B_{\text{inf}}(T; 1)$ is the realized maximum asset price over $[0, T]$ since the asset price is below this barrier level (infimum among all barrier levels) 100% of the time period.

For a European α-quantile call option, the terminal payoff is gven by

$$V_\alpha(S,T) = \max(B_{\text{inf}}(T;\alpha) - X, 0),$$

where X is the strike price.

- In the discrete trinomial tree model with N time steps, we write S_j^N as the discrete terminal asset price at maturity, $j = -N, -N+1, ..., N$. We assume that the possible values taken by the stochastic variable B_{inf} are limited to S_j, $j = -N, ..., N-1, N$; $S_j = S_0 u^j$, where u is the up-jump parameter. One may adopt a finer resolution of the discrete values that can be taken by B_{inf} for better accuracy (say, allowing the jump parameter of B_{inf} to be ρu, where $\rho < 1$).
- The numerical approximate value of the continuously monitored European α-quantile call option is given by

$$V_\alpha(S,0) = e^{-rT} \sum_{j=-N}^{N} P[B_{\text{inf}} = S_j] \max(S_j - X, 0), \;\; S_j = S_0 u^j.$$

  This is the summation of the state price of the event $\{B_{\text{inf}} = S_j\}$, $j = -N, \ldots, N$, multiplied by the corresponding terminal payoff at $B_{\text{inf}} = S_j$.

Binary cumulative options

Let $V^{bin}_{cum}(\alpha, B)$ denote the value of a binary option that pays $\$1$ at maturity T if the cumulative time staying at or below the down-barrier B is less than α of the total life of the option, $0 \leq \alpha \leq 1$; otherwise the terminal payoff of the option is zero. This option value is equivalent to the state price of the following event:

$$\frac{1}{T}\int_0^T \mathbf{1}_{\{S_t \leq B\}}\, dt < \alpha.$$

For a fixed value of α, the payoff of this binary option is \$1 (corresponding to the occurrence of the above event) only if the specified down-barrier B is below the realized value of $B_{\text{inf}}(T;\alpha)$. If otherwise, suppose $B \geq B_{\text{inf}}(T;\alpha)$, according to eq.(A), then $\frac{1}{T}\int_0^T \mathbf{1}_{\{S_t \leq B\}}\, dt \geq \alpha$, a contradiction to the fact that the above event occurs. The fair value of this binary cumulative barrier option is $e^{-rT}P[B_{\text{inf}}(T;\alpha) > B]$. In other words, if B is set to be too low such that $B_{\text{inf}}(T;\alpha) > B$, this is equivalent to expire in-the-money for the binary option $V^{bin}_{cum}(\alpha, B)$.

In the discrete world of the trinomial tree, we choose $B = S_j$ for some j. We then have

$$V_{cum}^{bin}(\alpha, S_j) = e^{-rT} P[B_{\text{inf}} > S_j]$$

so that

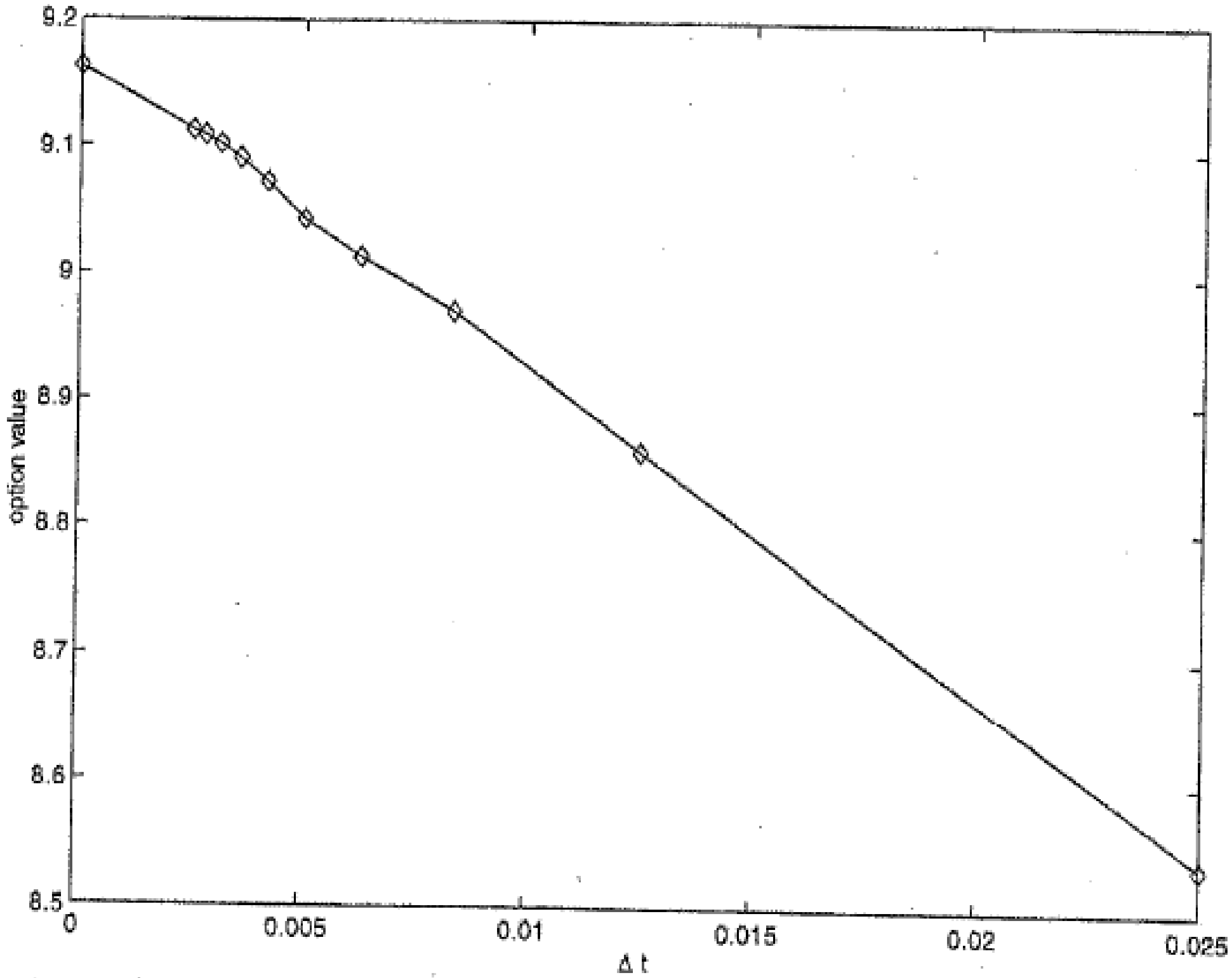
$$\begin{aligned} e^{-rT} P[B_{\text{inf}} = S_j] &= e^{-rT} \{P[B_{\text{inf}} > S_{j-1}] - P[B_{\text{inf}} > S_j]\} \\ &= V_{cum}^{bin}(\alpha, S_{j-1}) - V_{cum}^{bin}(\alpha, S_j). \end{aligned}$$

The terminal payoff of $V_{cum}^{bin}(\alpha, B)$ is given by

$$V_{j,k}^N = \begin{cases} 1 & \text{if } 0 \le k < \alpha N \\ 0 & \text{if } k \ge \alpha N \end{cases},$$

where k counts the number of time steps in the total number of N time steps that $S_j^n \le B$ and N is the total number of time steps in the lattice tree calculations. Note that the terminal payoff is independent of j (the index for the asset price) since the payoff of the binary cumulative barrier option is independent of asset price.

Numerical Option Values of Continuously Monitored Alpha Quantile Call Option



Parameter values: $\alpha = 0.8$, $S = 100$, $X = 95$, $r = 0.05$, $q = 0$, $\sigma = 0.2$ and $T = 0.25$.

Accumulators

- Entails the investor entering into a commitment to purchase a fixed number of shares per day at a pre-agreed price (the "Accumulator Price"). This Price is set (typically 10-20%) below the market price of the shares at initiation. This is portrayed as the "discount" to the market price of the shares.

Example

Citic Pacific entered into an Australian dollar accumulator as hedges "with a view to minimizing the currency exposure of the company's iron ore mining project in Australia". The company benefits from a strengthening in the A$ above A$1 = US$0.87.

Citic Pacific's（中信泰富）*bitter story*

- Citic Pacific signed an accumulator that not only set the highest gains but failed to include a floor for losses. The Australian dollar's value was rising when the contract was signed.
- After July, 2008, the AUD's value against the USD declined, sliding as low as 1 to 0.65. The firm also said its highest, marked-to-market loss could reach HK\$14.7 billion. Some analysts say if the AUD falls to 1 to 0.50 USD, the mark-to-market loss would rise to HK\$26 billion.
- Citic Pacific shares fell 80% on the Hong Kong exchange to HK\$5.06 a share on October 24, compared with HK\$28.20 a share on July 2.
- The company was driven by a "mixture of greed and a gambling mentality" to use the accumulator. Why not simply buy the less risky currency futures to hedge the iron ore mining project?

Cap on upside gain

If the market price of the shares rises above a pre-specified level ("Knock-Out price") then the obligation to purchase shares ceases. This Price is set (typically 2% to 5%) above the market price of the shares at initiation.

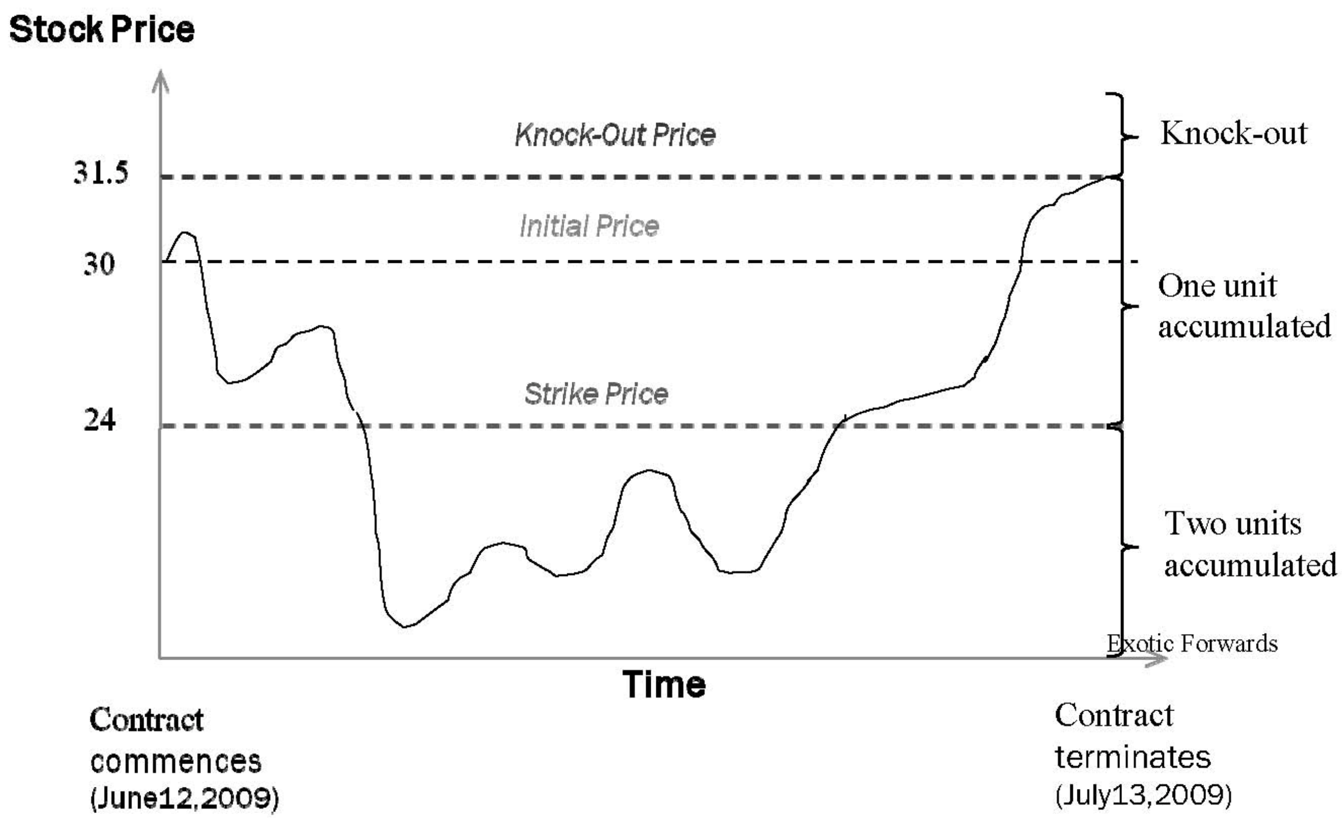
Intensifying downside losses ("I will kill you later")

If the market price falls below the Accumulator Price (10-20% below the market price at initiation), then the investor would be obligated to purchase more shares. This is called the Step-Up feature. The Step-Up factor can be 2 or up to 5.

- Margin is required to minimize counterparty risks. The investor generally benefits where the share prices remain relatively stable, preferably between the Knock Out Price and the Accumulator Price.

Example of an accumulator on China Life Insurance Company

- *Stock Price Movement of China Life Insurance Company Limited (June 12, 2009 - July 13, 2009)*



SGD-Equity Accumulator Structure

Underlying Shares:	SEMBCORP INDUSTRIES LTD
Start Date:	05 November 2007
Final accumulation Date:	03 November 2008
Maturity Date:	06 November 2008 (subject to adjustment if a Knock-Out Event has occurred)
Strike Price:	$4.7824

Knock-Out Price: $6.1425

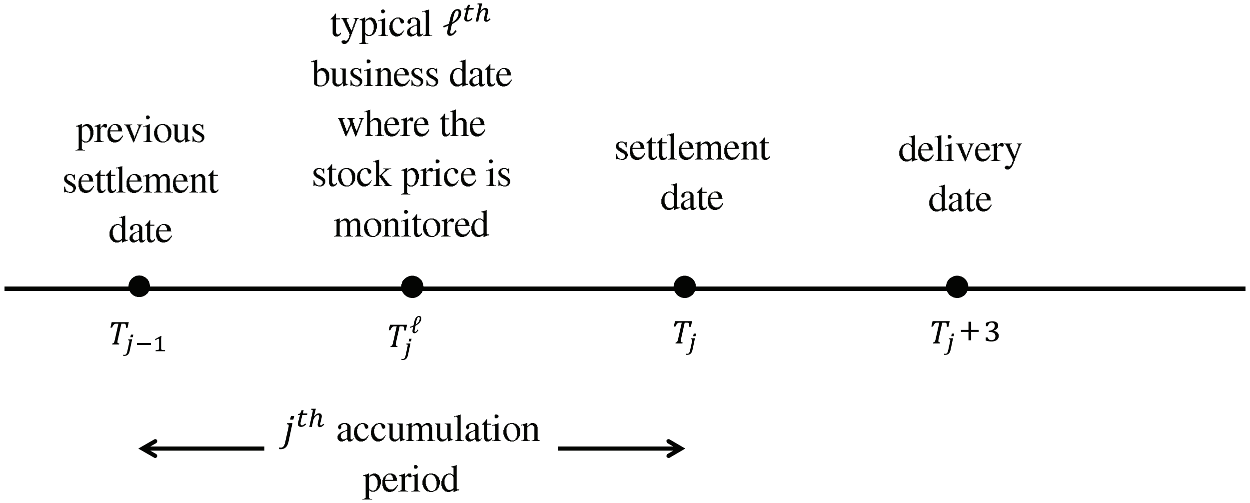
Knock-Out Event: A Knock-Out Event occurs if the official closing price of the Underlying Share on any Scheduled Trading Day is greater than or equal to the Knock-Out Price. Under such event, there will be no further daily accumulation of Shares from that day onward. The aggregate number of shares accumulated will be settled on the Early Termination Date, which is the third business day following the occurrence of Early Termination Event.

Shares Accumulation:	On each Scheduled Trading Day prior to the occurrence of Early Termination Event, the number of shares accumulated will be 1,000 when Official Closing Price for the day is higher than or equal to the Strike Price 2,000 when Official Closing Price for the day is lower than the Strike Price
Monthly Settlement Date:	The Shares accumulated for each Accumulation Period will be delivered to the investor on the third business day following the end of each monthly Accumulation Period
Total Number of Shares:	Up to the maximum of 500,000 shares (the worst scenario is $2,000$ shares purchased for 250 trading days)

Accumulation Period and Delivery Schedule

12 accumulation periods in total

Accumulation Period	Number of days	Delivery Date
05 Nov 07 to 03 Dec 07	20	06 Dec 07
04 Dec 07 to 02 Jan 08	19	07 Jan 08
03 Jan 08 to 04 Feb 08	23	11 Feb 08
05 Feb 08 to 03 Mar 08	18	06 Mar 08
04 Mar 08 to 02 Apr 08	21	07 Apr 08
03 Apr 08 to 02 May 08	21	06 May 08
05 May 08 to 02 Jun 08	20	05 Jun 07
03 Jun 08 to 02 Jul 08	22	07 Jul 08
03 Jul 08 to 04 Aug 08	23	07 Aug 08
05 Aug 08 to 02 Sep 08	21	05 Sep 08
03 Sep 08 to 02 Oct 08	21	07 Oct 08
03 Oct 08 to 03 Nov 08	21	06 Nov 08



Let $V(j,\ell,i,k)$ denote the value of the accumulator at the j^{th} accumulation period, ℓ^{th} business date (ℓ^{th} time step if the time step is taken to be one business day), i^{th} stock price level, and k units of shares accumulated in the j^{th} period up to the ℓ^{th} day.

- 04 Dec 07 corresponds to the first day in the second accumulation period, so $\ell = 1, j = 2$; 02 Oct 08 corresponds to the 21st day in the 11^{th} accumulation period, so $\ell = 21, j = 11$.

- The last day of the j^{th} accumulation period can be considered as the 0^{th} day of the $(j+1)^{\text{th}}$ accumulation period.

An accumulator as a portfolio of occupation time derivatives

- Without the "intensifying loss" feature, the product is like a portfolio of forward contracts with the knock-out feature. Purchases are conditional on survival until the date of accumulation of shares.

- The "intensifying loss" feature can be considered as a portfolio of forward contracts with the "excursion time" feature. The accumulated amount of shares depends on the total excursion time of the stock price staying below the strike price, again conditional on survival until the date of accumulation of share.

In other words, one has to count the number of days that the stock price stays below the strike price, conditional on "no knock-out". The knock-out feature limits the upside gain of the accumulator investor.

1. Use the forward shooting grid technique to keep track of the total number of shares to be purchased. The grid function is defined by

$$G(k,i) = k + 2,000\mathbf{1}_{\{i \leq I_{strike}\}} + 1,000\mathbf{1}_{\{i > I_{strike}\}}.$$

We set I_{strike} such that $S_0 u^{I_{strike}+\frac{1}{2}}$ is the actual strike price to avoid the ambiguity of determining whether 1,000 or 2,000 stocks to be bought when the stock price is exactly equal to the strike price.

Suppose we take m time steps per each business day, $m \geq 1$; and let ℓ denote the number of time steps lapsed from the last settlement date. For those time steps that do not correspond to the time of stock accumulation, we have the usual binomial scheme:

$$V(j,\ell,i,k) = e^{-r\Delta t}[pV(j,\ell+1,i+1,k) + (1-p)V(j,\ell+1,i-1,k)];$$

while at a time step that corresponds to stock accumulation, we have

$$\begin{aligned} V(j,\ell,i,k) &= e^{-r\Delta t}[pV(j,\ell+1,i+1,G(k,i+1)) \\ &+ (1-p)V(j,\ell+1,i-1,G(k,i-1)]. \end{aligned}$$

2. Jump conditions are applied across each settlement date (ending date of an accumulation period).

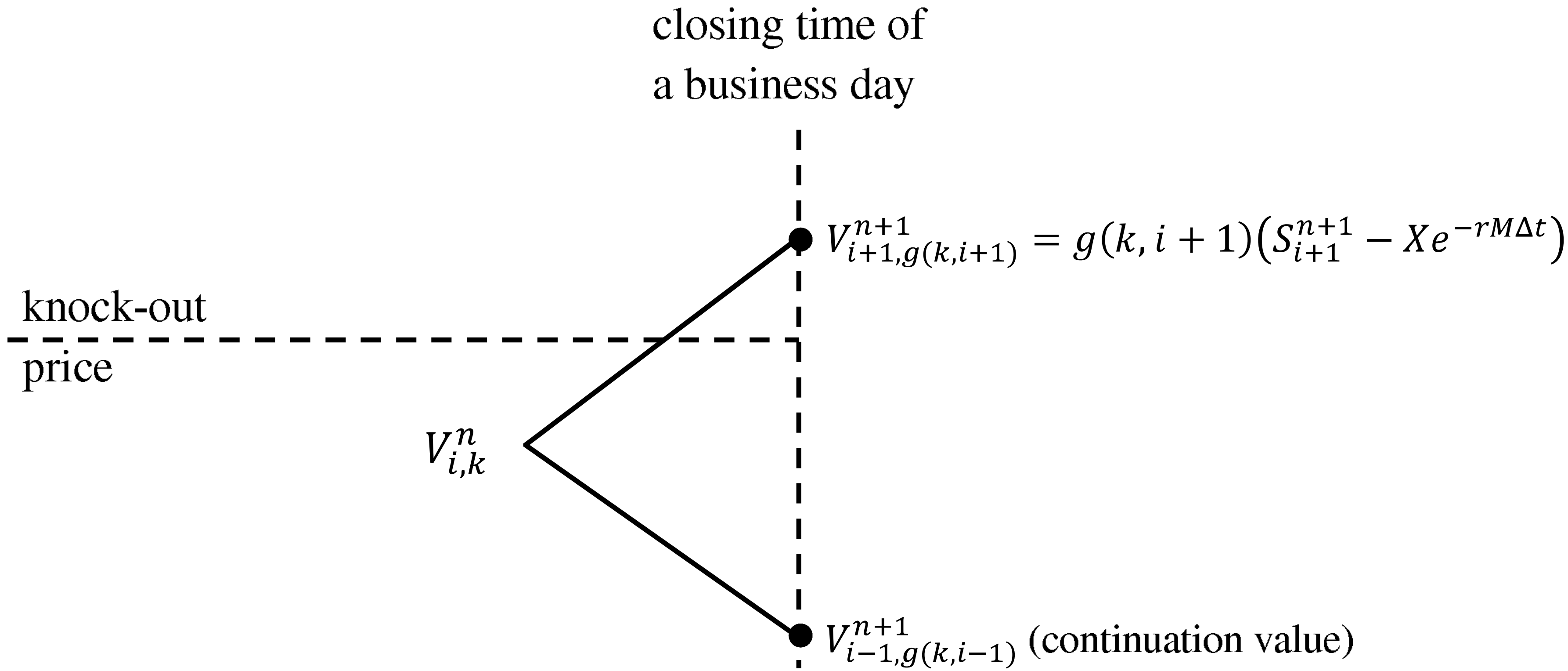
   - Right after the settlement of the accumulated shares, k is reset to zero. Right before delivery, compute the value function for all possible values of k. If there are $\ell_{\text{max}(j)}$ days in the j^{th} period, then k assumes values from $1,000 \times \ell_{\text{max}(j)}, 1,000 \times [\ell_{\text{max}(j)} + 1], \cdots, 2,000 \times \ell_{\text{max}(j)}$.

   - The jump in the accumulator value across each settlement date is the value of the accumulated units of stock on the settlement date. Moving from the j^{th} period to the $(j+1)^{\text{th}}$ period, the value of the accumulator is split into the continuation value of the accumulator with k being reset to zero and the value of the stocks transacted. The delivery of the accumulated stocks is done M time steps after the end of the accumulation period, so

$$V(j, m\ell_{\text{max}(j)}, i, k) = V(j+1, 0, i, 0) + k(S_i - Xe^{-rM\Delta t}).$$

     Here, M is the number of time steps between the settlement date and delivery date.

Rebate upon knock-out at the upstream knock-out price



When S_{i+1}^{n+1} is above the knock-out price, knock-out event occurs and the value of the accumulator becomes $g(k, i+1)\left(S_{i+1}^{n+1} - Xe^{-rM\Delta t}\right)$. Suppose the stocks are delivered 3 days after the knock-out date, then $M = 3m$, where m is the number of time steps for each business date.

Decomposition of an accumulator under immediate settlement

Under the assumption of monitoring of the upper knock-out barrier on each business day and immediate settlement of the accumulated stock, one can decompose an accumulator into a portfolio of up-and-out barrier call and put options. The accumulator survives up to t_i if $\max\limits_{0\le\tau\le t_{i-1}} S_\tau < H$.

The payoff on the observation date t_i is given by

$$\begin{cases} 0 & \text{if } \max_{0\le\tau\le t_{i-1}} S_\tau \ge H \\ S_{t_i} - K & \text{if } \max_{0\le\tau\le t_{i-1}} S_\tau < H \text{ and } S_{t_i} \ge K \\ 2(S_{t_i} - K) & \text{if } \max_{0\le\tau\le t_{i-1}} S_i < H \text{ and } S_{t_i} < K, \end{cases}$$

where $K =$ strike price and $H =$ upper knock-out level. Here, the realized maxima of S_τ, $0 \le \tau \le t_{i-1}$ is sampled on each business day.

- The delivery of one or two stocks is determined by $S_{t_i} \ge K$ or otherwise, independent of whether knock-out occurs or not on t_i.

$n =$ total number of observation dates
$c_{uo} =$ up-and-out barrier call option
$p_{uo} =$ up-and-out barrier put option

Fair value of an accumulator $= \sum_{i=1}^{n}[c_{uo}(t_i; K, H) - 2p_{uo}(t_i; K, H)]$.

- When $S_{t_i} \geq K$, the t_i-maturity put option is out-of-the-money and the t_i-maturity call option has the payoff $S_{t_i} - K$.
- When $S_{t_i} < K$, the call option is out-of-the-money and the put option becomes in-the-money with payoff $K - S_{t_i}$. When the two put options are in short position, the payoff is $-2(K - S_{t_i}) = 2(S_{t_i} - K)$.

The up-and-out barrier put and call price formulas are available under continuous monitoring of the barrier. Approximation formulas of barrier options under discrete monitoring of the barrier can be obtained by appropriate adjustment to the continuous monitoring counterpart.

Cautious note

In an actual contract, when the accumulator is not knocked out in an accumulation period, the accumulated shares within the accumulation period will be delivered 3 trading dates after the end of the accumulation period. The accumulator may be knocked out prior to the end of the accumulation period. When the accumulator is knocked out, the delivery of the accumulated shares will be 3 trading days after the knock-out date. This is unlike the above simplified assumption that the stocks are delivered immediately on each business date. The discount factor in the strike depends on the delivery date, which is uncertain due to the uncertainty of the knock-out event.

Due to uncertainty of the delivery date of the accumulated shares, it is necessary to use the forward shooting grid method that accounts for all possible cases of delivery of the accumulated shares, either 3 days after knocked out or 3 days after the end of the previous accumulation period.

Numerical studies on risk characteristics [taken from "Accumulator Pricing" by K. Lam et al.
(2009)]

One-year tenor, 21 trading days in each month, $n = 252, H = \$105$. The initial stock price S_0 is \$100, quantity bought on each day is either 1 or 2 depending on the stock price staying above the down-region or otherwise.

- Since the accumulator parameters (H and K) are designed so that it has a near zero-cost structure, the fair price for the sample accumulator is typically small.

FAIR VALUES OF ACCUMULATOR CONTRACTS

Volatility (σ)	Discounted Purhcase Price K				Zero-cost discounted price
	78	84	90	96	
10%	2639.5	1821.5	978.4	24.2	96.14
15%	1785.8	1108.4	369.8	-499.5	92.70
20%	1217.4	604.0	-82.2	-883.4	89.32
25%	790.0	211.6	-437.1	-1180.8	86.04
30%	445.2	-109.3	-727.2	-1423.3	82.86
35%	155.2	-380.6	-972.4	-1629.2	79.80
40%	-95.2	-615.9	-1185.4	-1809.6	76.84

The parameter values are: $S_0 = 100, H = 105, r = 0.03, q = 0.00$. For a zero-cost accumulator with monthly settlement at $\sigma = 20\%$, the fair discounted purchase price is shown to be 89.32 (interpolated between $K = 90$ and $K = 84$ along the row of $\sigma = 20\%$). At moderate level of $\sigma = 20\%$ and $S_0 = \$100$, the accumulator contract should set $K = 89.32$ (slightly more than 10% discount) in order that the initial value is zero.

Implied Volatility

For given values of H and K, we find the volatility such that the value of the accumulator is zero.

IMPLIED VOLATILITIES OF ACCUMULATOR CONTRACTS

Barrier Level, H	Discounted Strike Price K				
	80	84	88	92	96
107	**36.06**%	**29.55**%	23.30%	17.26%	11.31%
105	**34.63**%	**28.16**%	21.97%	16.00%	10.16%
103	**33.10**%	**26.68**%	20.54%	14.63%	8.91%

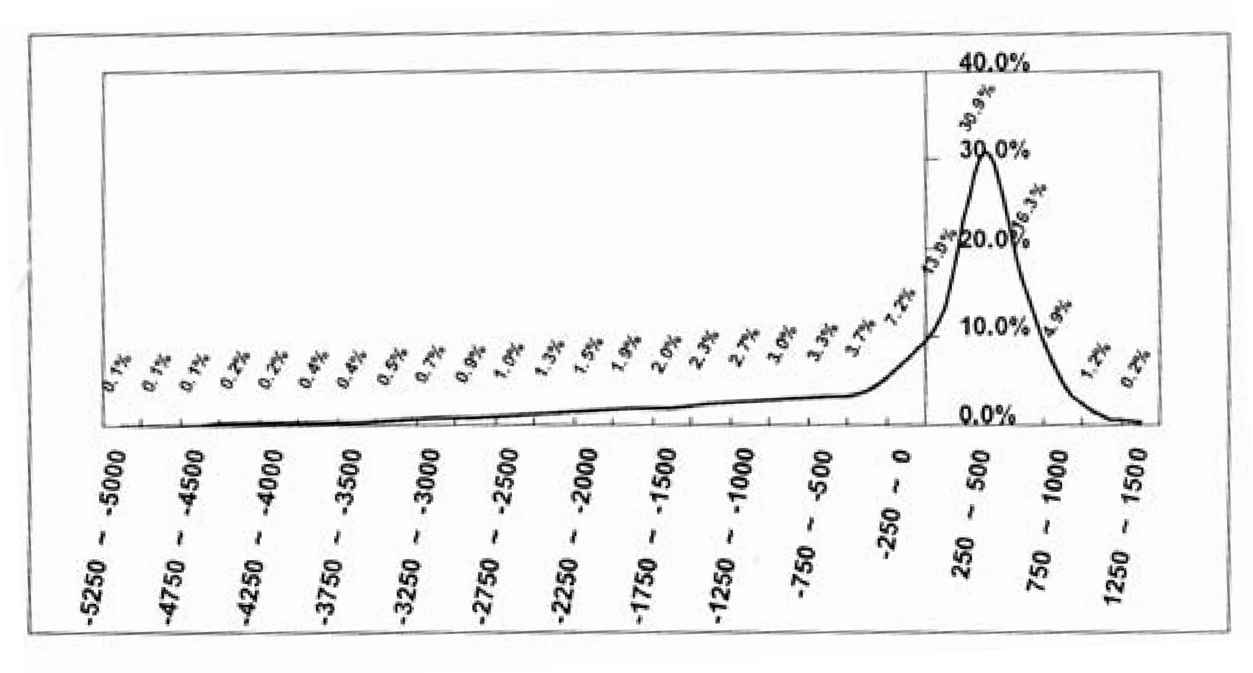
For a fixed value of K, the implied volatility value is higher for a higher barrier level H. For a fixed barrier level, the implied volatility is a decreasing function of the strike price K.

- Suppose an investor anticipates a volatility of 25% in the next one year. This investor will find the barrier-strike combination in the upper left corner (bold area in Table) favorable because the implied volatilities in those cells have implied volatility larger than 25%.

- The investor should be compensated with a higher barrier level and/or lower purchase price at a higher volatility level since she is shorting two puts and long only one call with maturity on each business date.

Value at risk analysis

- Profit/loss distribution is highly asymmetric.



Probability distribution of profit and loss of the sample accumulator

- It has a long left tail meaning that an extreme loss for the investor is possible.

- An extreme profit for the investor is unlikely as the distribution has a short right tail. This is because the contract will be knocked out once the stock price breaches the upper barrier H.

- For the sample accumulator contract analyzed, the lower 5-percentile is $-\$2424.50$. This means that at the maturity of the contract, there is a 5% chance to run a loss more than \$2424.50.

- For the seller of the contract, we can estimate his/her corresponding loss using the same confidence level 0.95. Computation result shows that the value at risk at maturity is \$841.01 with 95% confidence. Based on these two values at risk, we can conclude that the seller runs a much smaller risk than the buyer.

Greek values calculations ($S_0 = 100$, $H = 105$, $K = 90$, $r = 0.03$, $q = 0$, $\sigma = 0.2$, $n = 252$)

GREEKS OF ACCUMULATOR CONTRACTS

Spot price S		88	92	96	100	104
Immediate settlement of stocks	Delta	290.12	211.95	137.19	65.98	-3.54
	Gamma	-18.49	-19.40	-18.13	-17.55	-16.23
	Vega	-12139	-12507	-11182	-8072	-2966
Delay settlement (3 days)	Delta	288.05	209.63	134.88	63.48	-6.47
	Gamma	-18.67	-19.35	-18.16	-17.62	-16.37
	Vega	-12201	-12554	-11220	-8100	-2978

Gamma is the sensitivity of delta to stock price. Vega is the sensitivity of contract value to volatility. Delta, gamma, and vega are all sizable because an accumulator contract is composed of many option contracts with varying expiration dates.

- There is an asymmetry in the delta and vega values. When the spot price is low (say $S = 88$), the magnitude of delta and vega values are much larger than those when the spot price is high (say $S = 104$).

- Delta values are decreasing function of S because gamma values remain at a negative level. Delta has a magnitude of 288.05 (delay settlement) when $S = 88$, but its magnitude drops to -6.47 when $S = 104$. This means that losing buyers will be more vulnerable to price changes than winning buyers.

- Vega has a magnitude of 12201 when $S = 88$, but drops to a magnitude of 2978 when $S = 104$ meaning that compared to winning buyers, losing buyers are more vulnerable to volatility changes as well. This may be one reason why some buyers of the contract become very desperate when the market turns south.

- This asymmetry in risk exposure between the two parties is consistent with the finding that the value at risk of the buyer is several times that of the seller.