MAFS5250 – Computational Methods for Pricing Structured Products

Topic 2 – Implied binomial trees and calibration of interest rate trees

2.1 Implied binomial trees of fitting market data of option prices

- Arrow-Debreu prices and structures of the implied binomial trees
- Derman-Kani algorithm

2.2 Hull-White interest rate model and pricing of interest rate derivatives

- Analytic procedure of fitting the initial term structures of bond prices
- Calibration of interest rate trees against market discount curves

2.1 Implied binomial tree

The implied binomial tree method is a numerical procedure of computing a discrete approximation to the continuous risk neutral process for the underlying asset in a lattice tree that is consistent with observed market prices of options.

The implied binomial tree procedure should observe

- node transition probabilities fall between 0 and 1.

Suppose that options with any strike prices and maturities are available in the market and the implied binomial tree has been constructed to match the market prices of the options up to the n^{th} time step, how to devise a forward induction procedure to find stock prices and transition probabilities at the $(n+1)^{th}$ time step.

Derman-Kani binomial tree versus Cox-Ross-Rubinstein (CRR) binomial tree

In the CRR binomial tree, we assume σ to be constant. The upward jump ratio is $u = e^{\sigma\sqrt{\Delta t}}$. We obtain a symmetric recombining tree by setting $u = 1/d$. Let F be the price of the forward maturing one time step later. The martingale condition dictates the probability of up move, where

$$p = \frac{F - dS}{uS - dS} = \frac{e^{r\Delta t} - d}{u - d}, \text{ where } F = e^{r\Delta t}S.$$

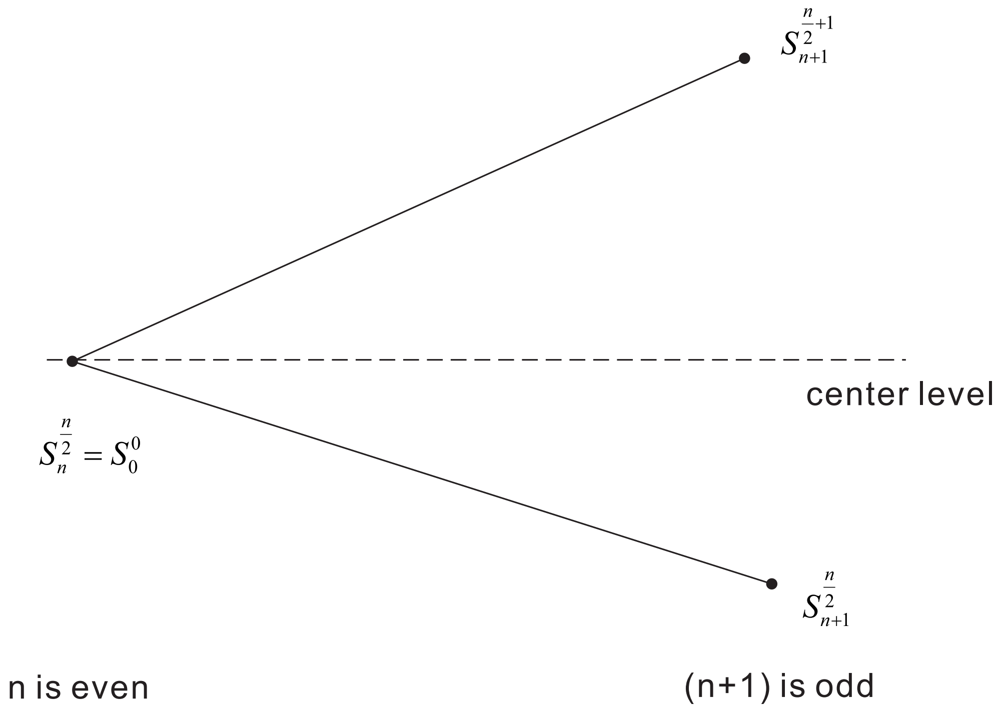
The martingale condition dictates the expected rate of return of the asset to be r.

In the derivation of the CRR tree, we equate mean and variance of the discrete and continuous asset price processes to determine p, u and d. Equating variance gives $u = e^{\sigma\sqrt{\Delta t}}$, equating mean is equivalent to setting the martingale condition. We are free to set $u = 1/d$ to generate a symmetric recombining tree.

In the Derman-Kani binomial tree, we do not prescribe σ. Instead, we enforce the amount of proportional jumps in asset price so that consistency with market observed call and put prices are observed. The jump ratio in the stock price tree reflects the level of volatility at the time level and stock price level, implicitly, $\sigma(S, t)$ (so called local volatility function).

At the n^{th} time level, the $n+1$ discrete asset prices are S_n^j, $j = 0, 1, \ldots, n$. Some structures of the implied binomial tree are imposed for the nodes along the center level, and the two nodes right above and below the center level.

1. Let the time step index n be even, say 4 time steps from the tip, the central node is set to lie at the center level with $S_n^{n/2} = S_0^0 = S_0$.



2. In the next time step, the corresponding index $n+1$ becomes odd, we set the two nodes just above and below the center level to have equal proportional jump in asset price from the central node in the last time step. That is,

$$\frac{S_{n+1}^{\frac{n}{2}+1}}{S_n^{\frac{n}{2}}} = \frac{S_n^{\frac{n}{2}}}{S_{n+1}^{\frac{n}{2}}}.$$

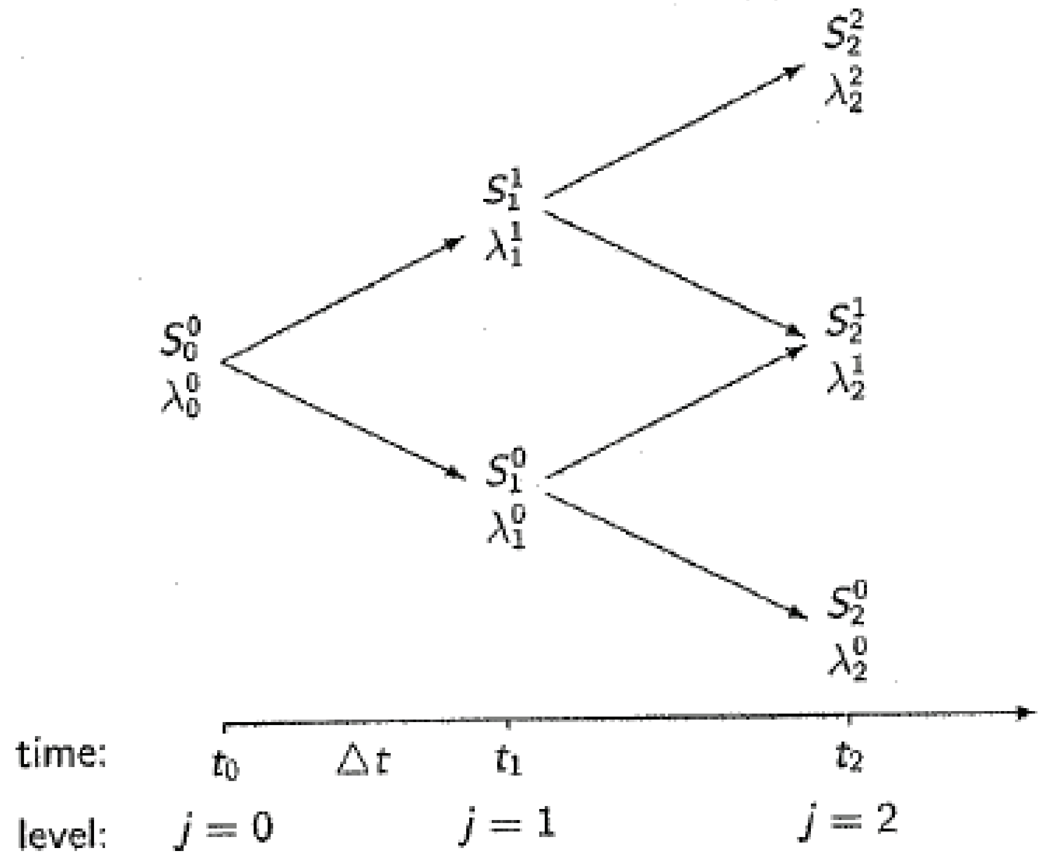
For example, when $n=8$, we have

$$\frac{S_9^5}{S_8^4} = \frac{S_8^4}{S_9^4}$$

Next, we determine the positions of the upper and lower nodes successively one node at a time by calibrating with known current market prices of call and put options, respectively.

Derman-Kani algorithm

Nodal stock prices, risk neutral transition probabilities and Arrow-Debreu prices (discounted risk neutral probabilities) in the implied binomial tree are calculated iteratively over successive time steps, starting at the level zero.



Construction of an implied binomial tree

- The forward price at level $n+1$ of S_n^i at level n is $F_n^i = e^{r\Delta t} S_n^i$.
- Conditional probability $P_{i+1}^n = P[S((n+1)\Delta t) = S_{n+1}^{i+1} | S(n\Delta t) = S_n^i]$ is the risk neutral transition probability of making an upward move from node (n,i) to $(n+1, i+1)$, $i = 0, 1, \ldots, n$.

Recall $F_t = E_Q[S_T | \mathcal{F}_t]$ or $F_n^i = E_Q[S((n+1)\Delta t) | S(n\Delta t) = S_n^i]$ based on martingale property of the asset price process, the risk neutral transition probability is given by

$$F_n^i = P_{i+1}^n S_{n+1}^{i+1} + (1 - P_{i+1}^n) S_{n+1}^i \tag{1}$$

so that

$$P_{i+1}^n = \frac{F_n^i - S_{n+1}^i}{S_{n+1}^{i+1} - S_{n+1}^i}.$$

Once the asset prices at the nodes of the implied binomial tree at the $(n+1)^{\text{th}}$ time step are known, the transition probabilities $P_i^n, i = 0, 1, \ldots, n$, can be determined based on the martingale property of the asset price process.

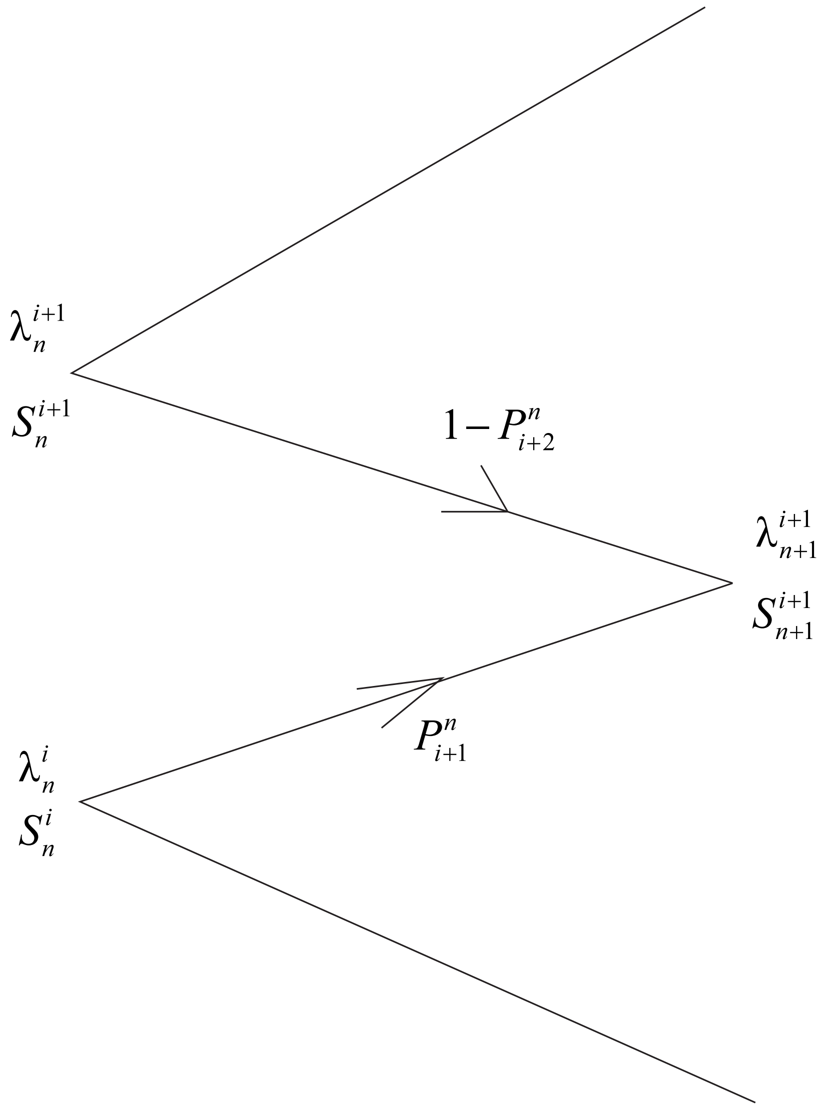
- The Arrow-Debreu price λ_n^i is the price of an option that pays \$1 if $S(n\Delta t)$ attains the value S_n^i, and 0 otherwise. Mathematically, it is given by the discounted probability that $S(n\Delta t)$ assumes S_n^i, where

$$\lambda_n^i = e^{-rn\Delta t} E[\mathbf{1}_{\{S(n\Delta t)=S_n^i\}} | S(0) = S_0].$$

Iterative scheme for computing λ_n^i: Starting with $\lambda_0^0 = 1$, based on law of total probability, we generate the successive iterates by

$$\begin{aligned}
\lambda_{n+1}^0 &= e^{-r\Delta t}[\lambda_n^0(1 - P_1^n)], \\
\lambda_{n+1}^{i+1} &= e^{-r\Delta t}[\lambda_n^i P_{i+1}^n + \lambda_n^{i+1}(1 - P_{i+2}^n)], \quad i = 0, 1, \ldots, n-1. \\
\lambda_{n+1}^{n+1} &= e^{-r\Delta t}\lambda_n^n P_{n+1}^n.
\end{aligned}$$

There is only one down-move branch that leads to the node λ_{n+1}^0 from the node λ_n^0. The corresponding probability of this downward move is $1 - P_1^n$.

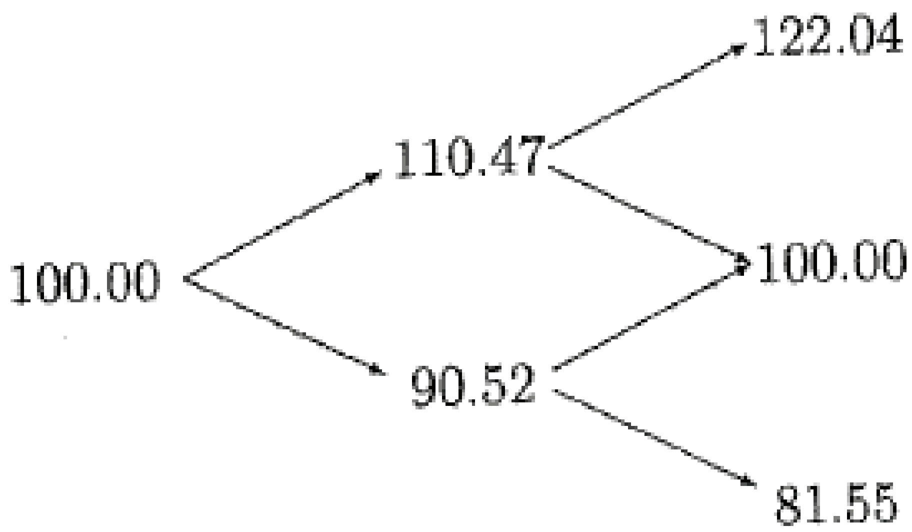


To reach S_{n+1}^{i+1}, we either move up from S_n^i with risk neutral probability P_{i+1}^n or move down from S_n^{i+1} with risk neutral probability $1 - P_{i+2}^n$.

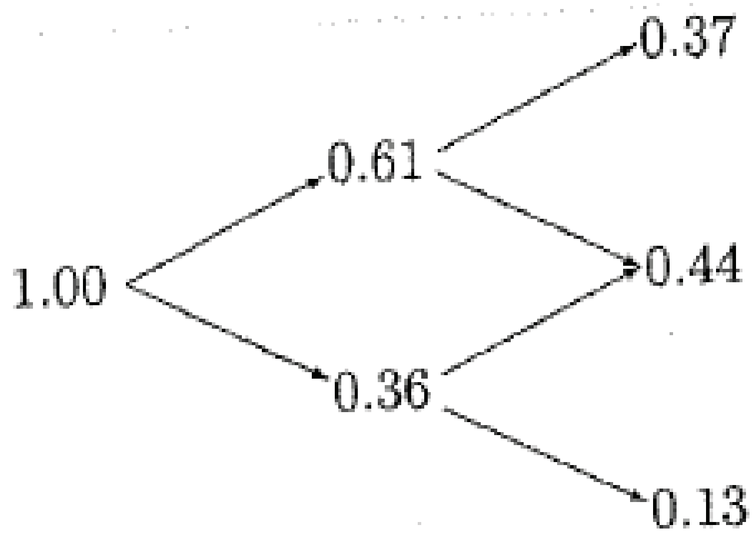
Arrow-Debreu price tree

The Arrow-Debreu price tree can be calculated from the asset price tree via the risk neutral transition probabilities.

CRR binomial tree for Arrow-Debreu prices with $T = 2$ years, $\Delta t = 1$, $\sigma = 0.1$ and $r = 0.03$.



asset price tree



Arrow-Debreu price tree

We start with

$$F_0^0 = S_0^0 e^{0.03} = 103.05,$$

so that the risk neutral transition probability is obtained as follows:

$$P_1^0 = \frac{F_0^0 - S_1^0}{S_1^1 - S_1^0} = \frac{103.05 - 90.52}{110.47 - 90.52} = \ 0.628.$$

In a similar manner, we can compute P_1^1 and P_2^1 from the information given in the asset price tree. The Arrow-Debreu prices are found to be (see the Arrow-Debreu price tree)

$$\begin{aligned}
\lambda_1^0 &= e^{-r\Delta t}\lambda_0^0(1 - P_1^0) = 0.36 \\
\lambda_1^1 &= e^{-r\Delta t}\lambda_0^0 P_1^0 = e^{-0.03} \times 0.628 = 0.61 \\
\lambda_2^0 &= e^{-r\Delta t}\lambda_1^0(1 - P_1^1) = 0.13 \\
\lambda_2^1 &= e^{-r\Delta t}[\lambda_1^0 P_1^1 + \lambda_1^1(1 - P_2^1)] = 0.44 \\
\lambda_2^2 &= e^{-r\Delta t}\lambda_1^1 P_2^1 = 0.37.
\end{aligned}$$

Option prices and Arrow-Debreu prices

Based on the discounted expectation valuation principle under a risk neutral measure, option prices maturing on $(n+1)\Delta t$ are related to the Arrow-Debreu prices:

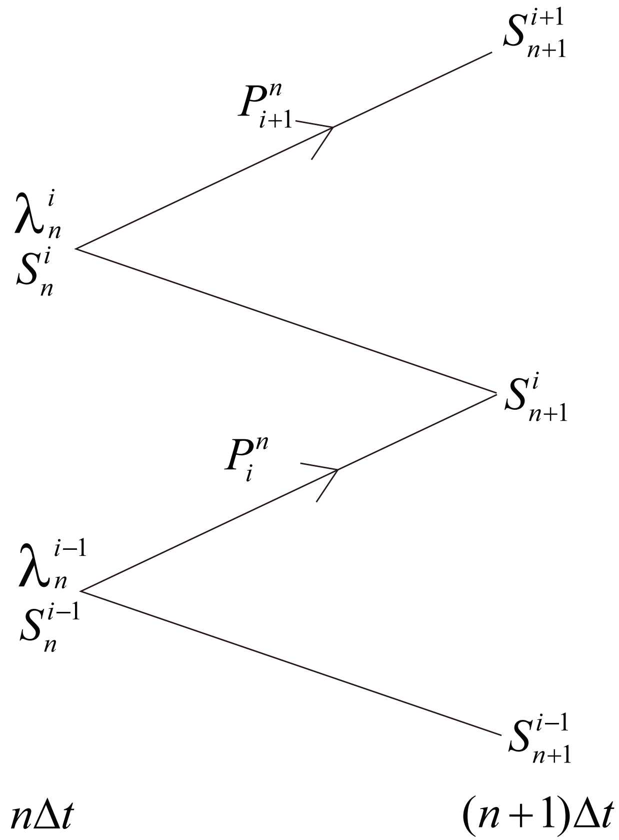
$$C((n+1)\Delta t; K) = \sum_{i=0}^{n+1} \lambda_{n+1}^i \max(S_{n+1}^i - K, 0) \quad (2)$$

$$P((n+1)\Delta t; K) = \sum_{i=0}^{n+1} \lambda_{n+1}^i \max(K - S_{n+1}^i, 0). \quad (3)$$

The call option price formula represents the sum of the contribution to the option value from the payoff $\max(S_{n+1}^i - K, 0)$ when $S((n+1)\Delta t) = S_{n+1}^i$, $i = 0, 1, \ldots, n+1$. The call option is equivalent to a portfolio of Arrow-Debreu securities with number of units $\max(S_{n+1}^i - K, 0)$ corresponding to the state S_{n+1}^i.

The forward price formula [eq.(1)] and the call and put option price formulas [eqs.(2) and (3)] are used to compute the tree parameters in the implied binomial tree. The implied binomial tree is built from the center level up and down.

$\lambda_n^i, S_n^i, i = 0, 1, \ldots, n$ are assumed to be known at the n^{th} time level.



$S_{n+1}^i, i = 0, 1, \ldots, n+1$

$P_i^n, i = 1, \ldots, n+1$

We determine S_{n+1}^i, $i = 0, 1, \ldots, n+1$, sequentially from the center level up and down using market option prices at the $(n+1)^{\text{th}}$ time level and known S_n^j, $j = 0, 1, \ldots, n$. The risk neutral transition probabilities P_i^n, $i = 1, \ldots, n+1$, are determined subsequently.

Determination of the asset prices at the upper nodes

The *upper* part of the implied binomial tree grows from the central node up one by one by using market *call* prices.

Applying the call option price formula at discrete times and using the relation of the Arrow-Debreu prices at successive time steps, we obtain

$$\begin{aligned} & e^{r\Delta t}C((n+1)\Delta t; K) \\ = \;& \lambda_n^0(1-P_1^n)\max(S_{n+1}^0 - K, 0) + \lambda_n^n P_{n+1}^n \max(S_{n+1}^{n+1} - K, 0) \\ & + \sum_{j=0}^{n-1}\{\lambda_n^j P_{j+1}^n + \lambda_n^{j+1}(1-P_{j+2}^n)\}\max(S_{n+1}^{j+1} - K, 0). \end{aligned}$$

Next, we set K to be S_n^i so that the call option is in-the-money at $t = (n+1)\Delta t$ when the stock price at the $(n+1)^{\text{th}}$ time level equals S_{n+1}^j, $j = i+1, i+2, \ldots, n+1$. Only those terms in the summation for $j = i, i+1, \ldots, n-1$ survive. We then have

$$
\begin{aligned}
& e^{r\Delta t}C((n+1)\Delta t; S_n^i) \\
= \ & \{\lambda_n^i P_{i+1}^n + \lambda_n^{i+1}(1 - P_{i+2}^n)\}(S_{n+1}^{i+1} - S_n^i) + \lambda_n^n P_{n+1}^n (S_{n+1}^{n+1} - S_n^i) \\
& + \sum_{j=i+1}^{n-1} \{\lambda_n^j P_{j+1}^n + \lambda_n^{j+1}(1 - P_{j+2}^n)\}(S_{n+1}^{j+1} - S_n^i).
\end{aligned}
$$

Note that we deliberately isolate the term that corresponds to $j = i$. We group the terms with common λ_n^j by changing the summation index and obtain

$$
\begin{aligned}
& e^{r\Delta t}C((n+1)\Delta t; S_n^i) \\
= \ & \lambda_n^i P_{i+1}^n (S_{n+1}^{i+1} - S_n^i) \\
& + \sum_{j=i+1}^{n-1} \lambda_n^j P_{j+1}^n (S_{n+1}^{j+1} - S_n^i) + \lambda_n^n P_{n+1}^n (S_{n+1}^{n+1} - S_n^i) \\
& + \lambda_n^{i+1}(1 - P_{i+2}^n)(S_{n+1}^{i+1} - S_n^i) + \sum_{j=i+2}^{n} \lambda_n^j (1 - P_{j+1}^n)(S_{n+1}^j - S_n^i) \\
= \ & \lambda_n^i P_{i+1}^n (S_{n+1}^{i+1} - S_n^i) \\
& + \sum_{j=i+1}^{n} \lambda_n^j [(1 - P_{j+1}^n)(S_{n+1}^j - S_n^i) + P_{j+1}^n (S_{n+1}^{j+1} - S_n^i)].
\end{aligned}
$$

Recall $F_n^j = P_{j+1}^n S_{n+1}^{j+1} + (1 - P_{j+1}^n) S_{n+1}^j$ and the terms involving S_n^i reduce to $-\lambda_n^j S_n^i$. Therefore, the time-0 price of the call option maturing at $(n+1)\Delta t$ and with strike S_n^i is given by

$$C((n+1)\Delta t; S_n^i) = \left[\lambda_n^i P_{i+1}^n (S_{n+1}^{i+1} - S_n^i) + \sum_{j=i+1}^{n} \lambda_n^j (F_n^j - S_n^i)\right] e^{-r\Delta t}.$$

Lastly, we may eliminate P_{i+1}^n in the above equation using

$$P_{i+1}^n = \frac{F_n^i - S_{n+1}^i}{S_{n+1}^{i+1} - S_{n+1}^i}.$$

This gives an equation that expresses S_{n+1}^{i+1} in terms of S_{n+1}^i, $C(S_n^i, (n+1)\Delta t)$ and other known quantities at the n^{th} time level. The solution for S_{n+1}^{i+1} is given in eq.(4) on P.21.

Financial interpretation of the call price formula

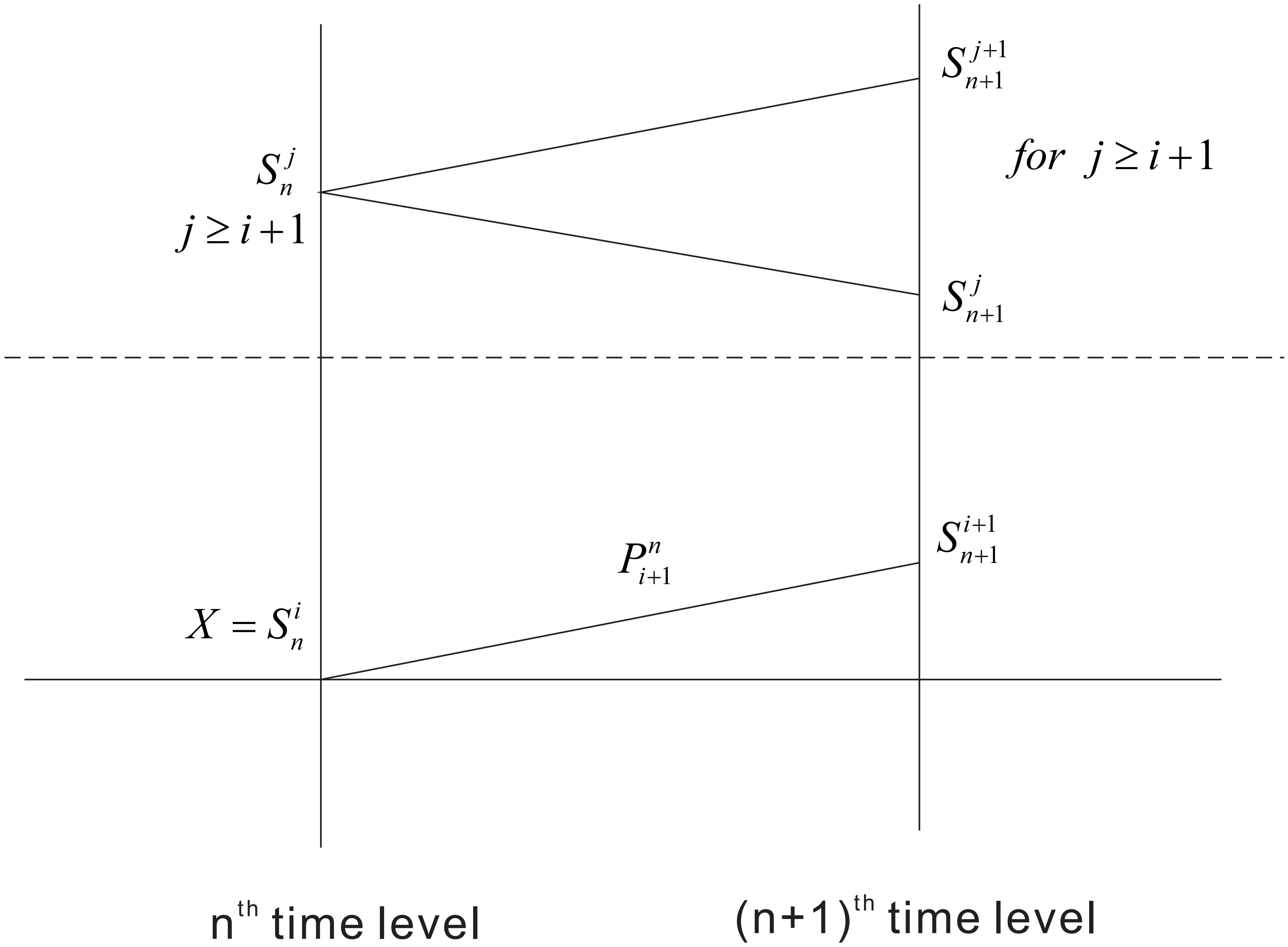
The call with strike $X = S_n^i$ expires in-the-money at $(n+1)\Delta t$ when

(i) at the n^{th} time level, $S(n\Delta t) = S_n^i$ and moves up to S_{n+1}^{i+1} with conditional probability P_{i+1}^n;

(ii) $S(n\Delta t) = S_n^j$, $j \geq i+1$.

By nested expectation, the time-0 price of the call maturing at $t = (n+1)\Delta t$ is given by the discounted expectation of reaching the state S_n^j (which is simply given by λ_n^j) followed by taking the discounted conditional expectation of the terminal payoff of the call based on reaching the state S_n^j. We write $C((n+1)\Delta t; S_n^i)$ as the terminal payoff of the call with strike S_n^i, then the discounted conditional expectation is given by (see a similar proof on P.57 for interest rate tree)

$$
\begin{aligned}
& e^{-r\Delta t} E[C((n+1)\Delta t; S_n^i) | S(n\Delta t) = S_n^j] \\
= & \begin{cases} e^{-r\Delta t} P_{i+1}^n (S_{n+1}^{i+1} - S_n^i) & j = i \\ e^{-r\Delta t} (F_n^j - S_n^i) & j = i+1, i+2, \ldots, n. \end{cases}
\end{aligned}
$$



With K being set to be S_n^i, the call expires in-the-money at S_{n+1}^j, $j \geq i+1$.

To avoid arbitrage, upward move probabilities must lie between 0 and 1.

- Suppose $P^n_{i+1} > 1$, then $F^i_n > S^{i+1}_{n+1}$. The forward price cannot be higher than the stock price even when the stock price at the next move is in the upstate. We demand $F^i_n < S^{i+1}_{n+1}$.
- Suppose $P^n_{i+2} < 0$, then $F^{i+1}_n < S^{i+1}_{n+1}$. The forward price cannot be lower than the stock price even when the stock price at the next move is in the down-state. We demand $S^{i+1}_{n+1} < F^{i+1}_n$.

Combining the results together, we require $F^i_n < S^{i+1}_{n+1} < F^{i+1}_n$. If the asset price S^{i+1}_{n+1} obtained from the above procedure violates this inequality, we override the option price that produces it. Instead, we choose an asset price that keeps the logarithmic spacing between this node and its adjacent node the same as that between corresponding nodes at the previous time level. That is,

$$\frac{S^{i+1}_{n+1}}{S^i_{n+1}} = \frac{S^{i+1}_n}{S^i_n}.$$

Implicitly, volatility is assumed to stay at the same value in the next time level and similar asset price level.

Key procedures in the Derman-Kani algorithm

1. For the nodes above the center level, we are able to obtain S_{n+1}^{i+1} in terms of S_{n+1}^{i}, $C((n+1)\Delta t; S_n^i)$, F_n^i, and other known quantities at the n^{th} time level. We obtain

$$S_{n+1}^{i+1} = \frac{S_{n+1}^{i}[C((n+1)\Delta t; S_n^i)e^{r\Delta t} - \rho_i^u] - \lambda_n^i S_n^i (F_n^i - S_{n+1}^i)}{C((n+1)\Delta t; S_n^i)e^{r\Delta t} - \rho_i^u - \lambda_n^i (F_n^i - S_{n+1}^i)}, \quad (4)$$

where ρ_i^u denotes the following summation term:

$$\rho_i^u = \sum_{j=i+1}^{n} \lambda_n^j (F_n^j - S_n^i).$$

The above formula is used to find S_{n+1}^{i+1} knowing S_{n+1}^{i}, starting from the central nodes in the tree and going upwards.

(a) In the initiation step for the first upward node at $j = \frac{n}{2} + 1$ when $n+1$ is odd, we do not know $S_{n+1}^{\frac{n}{2}}$. By applying

$$S_{n+1}^{\frac{n}{2}+1} = \frac{\left(S_n^{\frac{n}{2}}\right)^2}{S_{n+1}^{\frac{n}{2}}}$$

and substituting into eq.(4) with the elimination of $S_{n+1}^{\frac{n}{2}}$, we obtain

$$S_{n+1}^{\frac{n}{2}+1} = \frac{S_n^{\frac{n}{2}}\left[C((n+1)\Delta t; S_n^{\frac{n}{2}})e^{r\Delta t} + \lambda_n^{\frac{n}{2}} S_n^{\frac{n}{2}} - \rho_{\frac{n}{2}}^u\right]}{\lambda_n^{\frac{n}{2}} F_n^{\frac{n}{2}} - C((n+1)\Delta t; S_n^{\frac{n}{2}})e^{r\Delta t} + \rho_{\frac{n}{2}}^u}.$$

Recall $S_n^{\frac{n}{2}} = S_0$. Once $S_{n+1}^{\frac{n}{2}+1}$ has been determined, we apply eq.(4) to determine S_{n+1}^j, $j = \frac{n}{2}+2, \frac{n}{2}+3, \ldots, n+1$.

(b) When $n+1$ is even, we set $S_{n+1}^{\frac{n+1}{2}} = S_0^0$. Again, we apply eq.(4) to determine S_{n+1}^j, $j = \frac{n+3}{2}, \frac{n+5}{2}, \ldots, n+1$, successively.

2. We calculate the parameters in the lower nodes using known market put prices $P(S_n^i, (n+1)\Delta t)$. In a similar manner, we obtain

$$S_{n+1}^i = \frac{S_{n+1}^{i+1}[e^{r\Delta t}P((n+1)\Delta t; S_n^i) - \rho_i^\ell] + \lambda_n^i S_n^i (F_n^i - S_{n+1}^{i+1})}{e^{r\Delta t}P((n+1)\Delta t; S_n^i) - \rho_i^\ell + \lambda_n^i (F_n^i - S_{n+1}^{i+1})},$$

where ρ_i^ℓ denotes the sum over all nodes below the one with price S_n^i:

$$\rho_i^\ell = \sum_{j=0}^{i-1} \lambda_n^j (S_n^i - F_n^j).$$

Once S_{n+1}^i, $i = 0, 1, \ldots, n+1$, are obtained, the transition probabilities and Arrow-Debreu prices can be calculated accordingly.

Remark

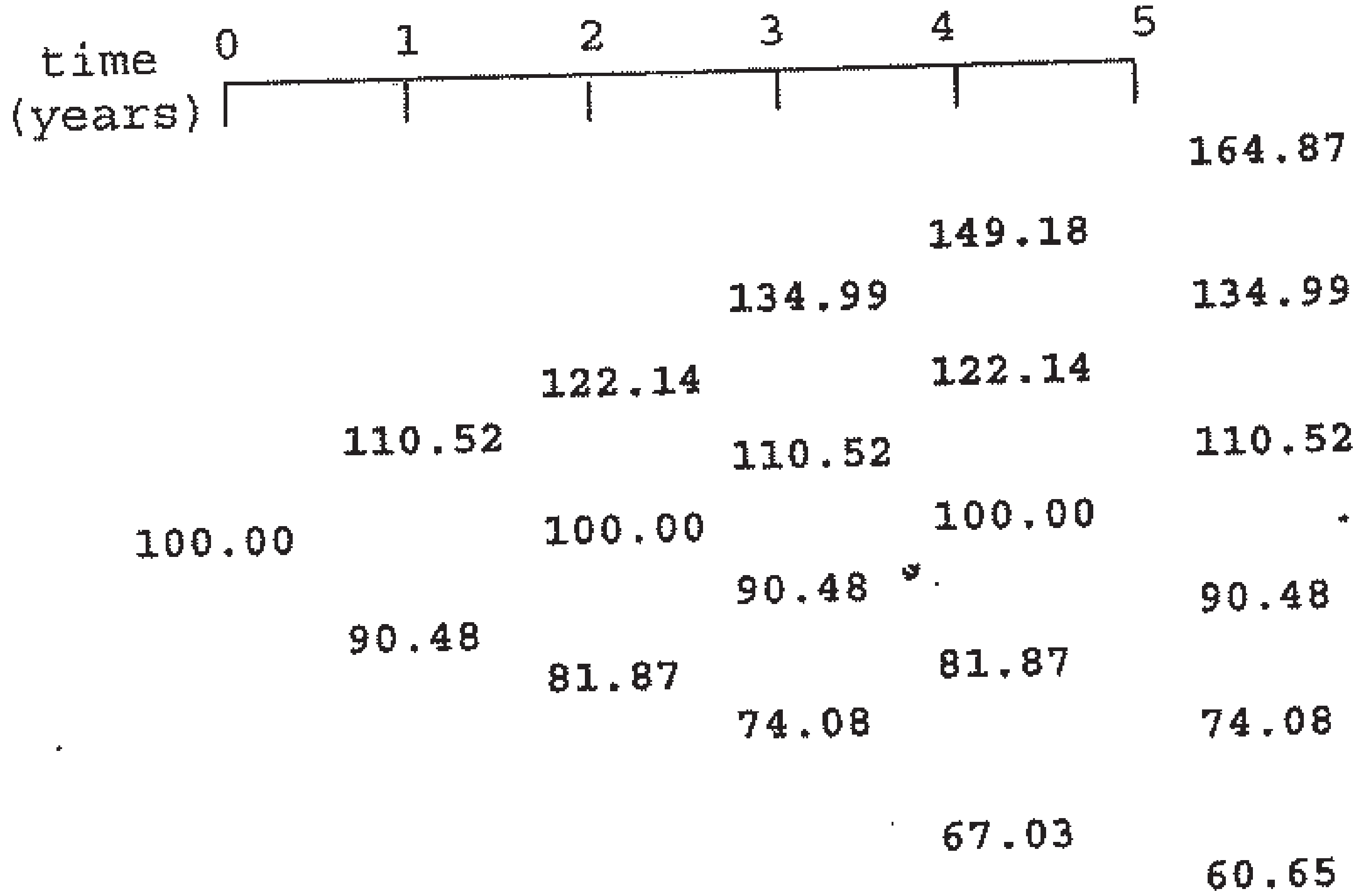
Market option prices may not be available at the required strikes and maturity dates. Interpolation is commonly used to estimate the required market option prices in the algorithm from limited data set of observed market option prices.

Numerical example ("Volatility Smile and its Implied Tree," E.Derman and I.kani, 1994)

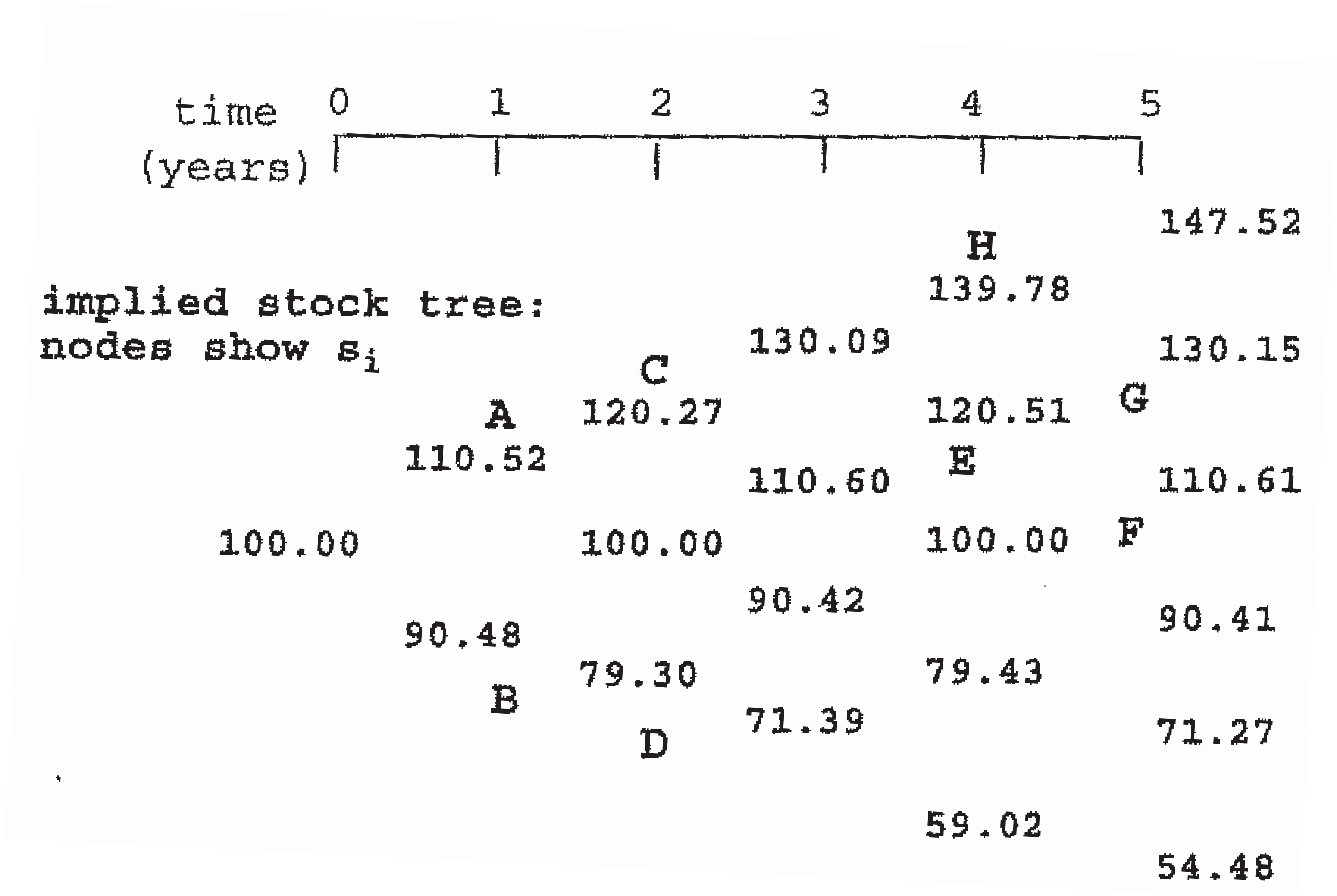
We assume that the current value of the index is 100, its dividend yield is zero, and that the annually compounded riskless interest rate is 3% per year for all maturities.

We assume that the annual implied volatility of an at-the-money European call is 10% for all expirations, and that implied volatility increases (decreases) linearly by 0.5 percentage points with every 10 point drop (rise) in the strike. This defines the smile in this numerical example.

We show the standard (not implied) CRR binomial stock tree for a local volatility of 10% everywhere. This tree produces no smile. It is the discrete binomial analog of the continuous-time Black-Scholes equation. We use the binomial tree for a given σ set at the implied volatility to convert implied volatilities into quoted option prices. Its up and down moves are generated by factors exp($\pm\sigma\Delta t$). The transition probability at every node is 0.625.



Binomial stock tree with constant 10% **stock volatility**



Implied stock tree obtained in the numerical example

We determine the transition probabilities and Arrow-Debreu prices sequentially once the stock prices have been determined in the implied stock prices one time step at a time.

```
transition probability tree:                 0.796
nodes show p_i                       0.700

                             0.682             0.692
                     0.682               0.678
                             0.624             0.623
           0.625
                                         0.666
                     0.671
                             0.541             0.551
                                         0.711

                                               0.376
```

Arrow-Debreu price tree:
nodes show λ_t

					0.140
				0.181	
			0.266		0.257
		0.402		0.329	
	0.607		0.381		0.255
1.000		0.425		0.259	
	0.364		0.216		0.151
		0.116		0.106	
			0.052		0.051
				0.015	
					0.009

The assumed 3% interest rate means that the forward price one year later for any node is $1.03 = 1 + 0.03$ times that node's stock price.

Today's stock price at the first node on the implied tree is 100, and the corresponding initial Arrow-Debreu price $\lambda_0 = 1.000$. Let us find the node A stock price in level 2. For even levels, we set $S_{i+1} = S_A$, $S = 100$, $e^{r\Delta t} = 1.03$ and $\lambda_1 = 1.000$, then

$$S_A = \frac{100[1.03 \times C(100, 1) + 1.000 \times 100 - \Sigma]}{1.000 \times 103 - 1.03 \times C(100, 1) + \Sigma},$$

where $C(100, 1)$ is the value today of a one-year call with strike 100.

Note that Σ must be set to zero because there are no higher nodes than the one with strike above 100 at level 0.

According to the smile, we must value the call $C(100, 1)$ at an implied volatility of 10%. In the simplified binomial world, $C(100, 1) = 6.38$ when valued on the CRR tree. Inserting these values into the above equation yields $S_A = 110.52$. The price corresponding to the lower node B in the implied tree is given by our chosen centering condition $S_B = S^2/S_A = 90.48$. The transition probability at the node in year 0 is

$$P = \frac{103 - 90.48}{110.52 - 90.48} = 0.625$$

Using forward induction, the Arrow-Debreu price at node A is given by $\lambda_A = \lambda_0 P/1.03 = (1.00 \times 0.625)/1.03 = 0.607$, as shown on the bottom tree. In this way, the smile has implied the second level of the tree.

We choose the central node to lie at 100. The next highest node C is determined by the one-year forward value $F_A = 113.84$ of the stock price $S_A = 110.52$ at node A, and by the two-year call $C(S_A, 2)$ struck at S_A.

Since there are no nodes with higher stock values than that of node A in year 1, the $\sum$ term is again zero, we obtain

$$S_C = \frac{100[1.03 \times C(S_A, 2)] - 0.607 \times S_A \times (F_A - 100)}{1.03 \times C(S_A, 2) - 0.607 \times (F_A - 100)}.$$

The value of $C(S_A, 2)$ at the implied volatility of 9.47% $=$ 10% $-$ 0.05 $\times$ (110.52 $-$ 100) corresponding to a strike of 110.52 is 3.92 in our binomial world.

Substituting the values into the above equation yields $S_C = 120.27$. The transition probability is given by

$$P_A = \frac{113.84 - 100}{120.27 - 100} = 0.682.$$

We can similarly find the new Arrow-Debreu price λ_C. We can also show that the stock price at node D must be 79.30 to make the put price $P(S_B, 2)$ have an implied volatility of 10.47% consistent with the smile.

Suppose that we have already constructed the implied tree up to year 4, and also found the value of S_F at node F to be 110.61. The stock price S_G at node G is given by

$$S_G = \frac{S_F[1.03 \times C(S_E, 5) - \Sigma] - \lambda_E \times S_E \times (F_E - 110.61)}{[1.03 \times C(S_E, 5) - \Sigma] - \lambda_E \times (F_E - 110.61)},$$

where $S_E = 120.51$ and $F_E = 120.51 \times 1.03 = 124.13$ and $\lambda_E = 0.329$.

The smile's interpolated implied volatility at a strike of 120.51 is 8.86%, corresponding to a call value $C(120.51, 5) = 6.24$. The value of the Σ term in the above equation is given by the contribution to this call from the node H above node E in year 4. We obtain

$$\begin{aligned} \Sigma &= \lambda_H(F_H - S_E) \\ &= 0.181 \times (1.03 \times 139.78 - 120.51) \\ &= 4.247 \end{aligned}$$

Substituting these values gives $S_G = 130.15$.

2.2. Hull-White interest rate model and pricing of interest rate derivatives

Analytic procedure of fitting the initial term structures of bond prices

In the Hull-White short rate model, $\phi(t)$ in the drift term is the only time dependent parameter function in the model. Under the risk neutral measure Q, the instantaneous short rate r_t is assumed to follow

$$dr_t = [\phi(t) - \alpha r_t] \; dt + \sigma \, dZ_t,$$

where α and σ are constant parameters. The model includes the mean reversion property. When $r_t > \phi(t)/\alpha$, the drift becomes negative and pulls r_t back to the mean reversion level of $\phi(t)/\alpha$.

We assume that the two constant parameters α and σ can be estimated by some other means. We illustrate the analytic procedure for the calibration of $\phi(t)$ using the information of the current term structure of bond prices.

The discount bond price $B(r,t;T)$ is given by $E_Q^t[e^{-\int_t^T r_u\ du}]$. By virtue of the Feynman-Kac representation theorem, the governing partial differential equation for the discount bond price $B(r,t;T)$ is given by

$$\frac{\partial B}{\partial t} + \frac{\sigma^2}{2}\frac{\partial^2 B}{\partial r^2} + [\phi(t) - \alpha r]\frac{\partial B}{\partial r} - rB = 0, \quad B(r,T;T) = 1.$$

We assume that the bond price function to be the affine form [linear in r for $\ln B(t,T)$]

$$B(t,T) = e^{a(t,T)-b(t,T)r}.$$

By substituting the assumed affine solution into the partial differential equation and collecting like terms with and without r, the governing ordinary differential equations for $a(t,T)$ and $b(t,T)$ are found to be

$$\frac{db}{dt} - \alpha b + 1 = 0,\ t < T;\ b(T,T) = 0;$$
$$\frac{da}{dt} + \frac{\sigma^2}{2}b^2 - \phi(t)b = 0,\ t < T;\ a(T,T) = 0.$$

Solving the pair of ordinary differential equations for $a(t,T)$ and $b(t,T)$, we obtain

$$\begin{aligned} b(t,T) &= \frac{1}{\alpha}\left[1 - e^{-\alpha(T-t)}\right], \\ a(t,T) &= \frac{\sigma^2}{2}\int_t^T b^2(u,T)\ du - \int_t^T \phi(u)b(u,T)\ du. \end{aligned}$$

It is easy to check that $b(T,T) = a(T,T) = 0$ so that $B(T,T) = 1$. Our goal is to determine $\phi(T)$ in terms of the current term structure of bond prices $B(r,t;T)$.

Applying the relation:

$$\ln B(r,t;T) + rb(t,T) = a(t,T),$$

we have

$$\int_t^T \phi(u)b(u,T)\ du = \frac{\sigma^2}{2}\int_t^T b^2(u,T)\ du - \ln B(r,t;T) - rb(t,T). \qquad (1)$$

To solve for $\phi(u)$ in the above integral equation, the first step is to obtain an explicit expression for $\int_t^T \phi(u)\ du$. Given that $b(t,T)$ only involves a constant and an exponential function, this can be achieved by differentiating $\int_t^T \phi(u) b(u,T)\ du$ with respect to T and subtracting the terms involving $\int_t^T \phi(u) e^{-\alpha(T-t)}\ du$.

The differentiation of the left hand side of Eq. (1) with respect to T gives

$$\begin{aligned} \frac{\partial}{\partial T}\int_t^T \phi(u) b(u,T)\ du &= \phi(u) b(u,T)\Big|_{u=T} + \int_t^T \phi(u) \frac{\partial}{\partial T} b(u,T)\ du \\ &= \int_t^T \phi(u) e^{-\alpha(T-u)}\ du. \end{aligned}$$

We equate the derivatives on both sides to obtain

$$\int_t^T \phi(u) e^{-\alpha(T-u)}\ du = \frac{\sigma^2}{\alpha}\int_t^T [1-e^{-\alpha(T-u)}]e^{-\alpha(T-u)}\ du - \frac{\partial}{\partial T}\ln B(r,t;T) - re^{-\alpha(T-t)}. \tag{2}$$

We multiply Eq. (1) by α and obtain

$$\int_t^T \phi(u)\left[1-e^{-\alpha(T-u)}\right]du = \frac{\sigma^2}{2}\int_t^T \frac{1}{\alpha}\left[1-2e^{-\alpha(T-u)}+e^{-2\alpha(T-u)}\right]du - \alpha\ln B(r,t;T) - r\left[1-e^{-\alpha(T-t)}\right]. \tag{3}$$

Adding Eq.(2) and Eq.(3) together, we have

$$\int_t^T \phi(u)\ du = \frac{\sigma^2}{2\alpha}\int_t^T [1-e^{-2\alpha(T-u)}]\ du - r - \frac{\partial}{\partial T}\ln B(r,t;T) - \alpha\ln B(r,t;T).$$

Recall that $\ln B(r,t;T)$ can be observed directly from the current term structure of the discount bond prices.

By differentiating the above equation with respect to T again and noting that r is independent of T, we obtain $\phi(T)$ in terms of the current term structure of bond prices $B(r,t;T)$ as follows:

$$\begin{aligned}\phi(T) &= \frac{\sigma^2}{2\alpha}[1 - e^{-2\alpha(T-t)}] - \frac{\partial^2}{\partial T^2}\ln B(r,t;T) \\ &\quad - \alpha\frac{\partial}{\partial T}\ln B(r,t;T).\end{aligned}$$

Alternatively, one may express $\phi(T)$ in terms of the current term structure of the instantaneous forward rates $F(t,T)$, where

$$B(r,t;T) = \exp\left(-\int_t^T F(t,u)\ du\right).$$

Note that $-\dfrac{\partial}{\partial T}\ln B(r,t;T) = F(t,T)$ so that we may rewrite $\phi(T)$ in the form

$$\phi(T) = \frac{\sigma^2}{2\alpha}[1 - e^{-2\alpha(T-t)}] + \frac{\partial}{\partial T}F(t,T) + \alpha F(t,T).$$

The merit of using $F(t,T)$ is analytic simplicity where the second derivative term is avoided.

Remarks on various versions of interest rates

1. Instantaneous short rate r_t

   This is the instantaneous interest rate known at time t and being applied over $(t, t + dt)$. For $u > t$, r_u is not known at time t. In terms of r_t, the discount bond price is given by

$$B(r, t; T) = E_Q^t \left[e^{-\int_t^T r_u \; du} \right].$$

2. Instantaneous forward rate $F(t, u)$

   This is the instantaneous interest rate known at time t and being applied over $(u, u + du)$, where $u > t$. Obviously, $F(t, t) = r_t$. In terms of $F(t, u)$, the discount bond price is given by

$$B(r, t; T) = e^{-\int_t^T F(t,u) \; du}.$$

   In the reverse sense, this relation dictates the determination of $F(t, u)$ in terms of observed term structure of discount bond price, where

$$F(t, T) = -\frac{\partial}{\partial T} \ln B(r, t; T).$$

Calibration of interest rate trees against market discount curves

In the discrete world, the interest rates on the Hull-White tree are interpreted as the Δ-period rates over a finite period Δ, not the same as the instantaneous short rate r. Let $R(t)$ denote the Δt-period rate at time t applied over the finite time interval $(t, t+\Delta t)$. We can equate the discount bond price $B(r,t;t+\Delta t)$ and the discount factor over $(t, t+\Delta t)$ based on known Δt-period rate $R(t)$ to give

$$B(r,t;t+\Delta t) = e^{-R(t)\Delta t} = e^{a(t,t+\Delta t)} e^{-b(t,t+\Delta t)r(t)}.$$

The two rates $r(t)$ and $R(t)$ are related by

$$r(t) = \frac{R(t)\Delta t + a(t,t+\Delta t)}{b(t,t+\Delta t)}.$$

- We assume that the Δt-rate, R, follows a similar mean reversion process as the instantaneous short rate r:

$$dR = [\alpha(t) - aR]\,dt + \sigma\,dZ.$$

Tree construction procedures

Unlike the usual trinomial trees used in equity pricing, the calibrated interest rate trees are distorted. The size of the displacement is the same for all nodes at a particular time t.

- The first stage in building a tree for this model is to construct a tree for a variable R^* that is initially zero and follow the process

$$dR^* = -aR^*\,dt + \sigma\,dZ, \quad a > 0.$$

  We build a symmetrical tree similar to Figure 2 for R^*.

- In the second stage, we build the tree for R that calibrates to the initial term structures of discount bond prices. This is done by determining $\alpha_m = \alpha(m\Delta t)$, $m = 0, 1, 2, \ldots$, recursively.

- Though there is a mean reversion term $-aR^*$, the mean reversion level is zero. Hence, the stochastic process $R^*(t)$ is symmetrical about $R^* = 0$. The mean and variance of the increment of $R^*(t)$ over Δt are

$$\begin{aligned} E\left[R^*(t+\Delta t) - R^*(t)\right] &= -aR^*(t)\Delta t \\ \text{var}\left(R^*(t+\Delta t) - R^*(t)\right) &= \sigma^2 \Delta t \\ &= E\left[\left[R^*(t+\Delta t) - R^*(t)\right]^2\right] - a^2 R^{*2}(t)\Delta t^2. \end{aligned}$$

- We define ΔR as the spacing between interest rates on the tree and set $\Delta R = \sigma\sqrt{3\Delta t}$. This proves to be a good choice of ΔR from the viewpoint of minimization of discretization errors. When σ is constant, ΔR has the same value at all nodes.

- In the first stage of this procedure, we build a tree similar to that shown in Figure 2 for R^*. Due to the mean reversion feature of R^*, we must resolve which of the three branching methods shown in Figure 1 will apply at each node. This will determine the overall geometry of the tree. Once this is done, the branching probabilities must also be calculated.

Which of the three branching methods shown in Figure 1 will apply at each node?

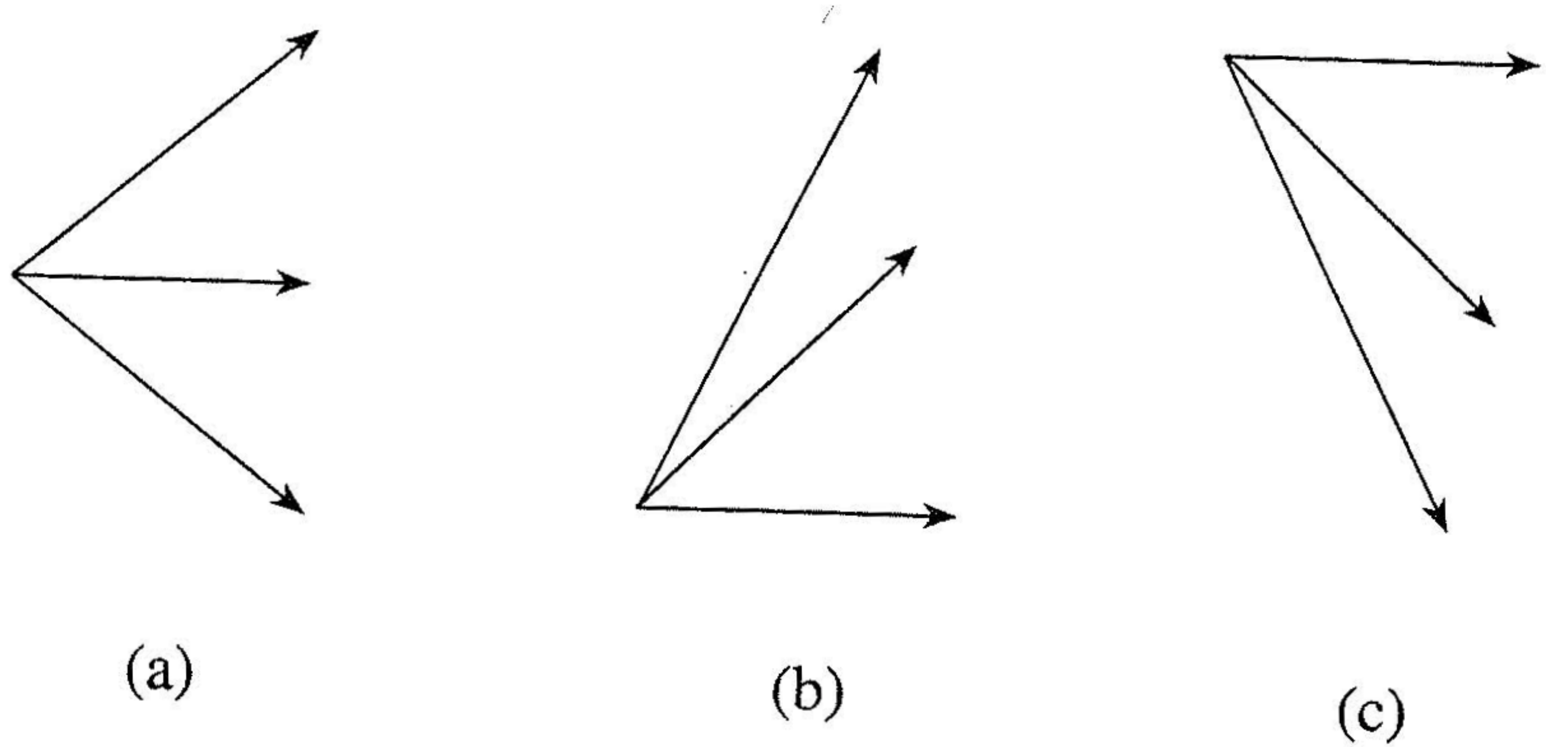


Figure 1. Alternative branching methods in a trinomial tree.

Mean reversion feature

The branching method used at a node must lead to the probabilities on all three branches being positive. For those nodes that are not close to the edge of the trinomial tree, the normal branching shown in Figure 1(a) is appropriate.

It is necessary to switch from the normal branching in Figure 1(a) to the downward branching in Figure 1(c) for a sufficiently large j. The mean reversion drift becomes stronger at a high value of j, so further upward moves of R^* should be prohibited. Similarly, it is necessary to switch from the normal branching in Figure 1(a) to the upward branching in Figure 1(b) when j is sufficiently negative.

Define (i, j) as the node where $t = i\Delta t$ and $R^* = j\Delta R$. The variable i is a positive integer and j is a positive or negative integer. Define p_u, p_m, and p_d as the probabilities of the highest, middle, and lowest branches emanating from the node. The probabilities of discrete moves of R^* are chosen to match with the expected change and variance of the change in R^* under the continuous model over the next time interval Δt. The probabilities must also sum to unity. This leads to three equations in the three probabilities.

Normal branching

If the branching has the form shown in Figure 1(a), the mean and variance of the discrete moves of R^* over Δt is $p_u\Delta R - p_d\Delta R$ and $p_u\Delta R^2 + p_d\Delta R^2 - (p_u\Delta R - p_d\Delta t)^2$, respectively. The nodal transition probabilities p_u, p_m, and p_d at node (i, j) must satisfy the following three equations:

$$
\begin{aligned}
&p_u\Delta R - p_d\Delta R = -aj\Delta R\Delta t \\
&p_u\Delta R^2 + p_d\Delta R^2 = \sigma^2\Delta t + a^2 j^2\Delta R^2\Delta t^2 \\
&p_u + p_m + p_d = 1.
\end{aligned}
$$

Using $\Delta R = \sigma\sqrt{3\Delta t}$, the solution to these equations is

$$\begin{aligned} p_u &= \frac{1}{6} + \frac{1}{2}(a^2 j^2 \Delta t^2 - aj\Delta t) \\ p_m &= \frac{2}{3} - a^2 j^2 \Delta t^2 \\ p_d &= \frac{1}{6} + \frac{1}{2}(a^2 j^2 \Delta t^2 + aj\Delta t). \end{aligned}$$

The nodal transition probabilities have dependence on j, where p_d becomes larger when j is larger. This is consistent with the mean reversion feature.

We observe that p_m remains positive when $|aj\Delta t| < \sqrt{2/3}$. This imposes a constraint on the time step Δt, which becomes more restrictive when j is large. It is desirable not to allow j to be too large. One can show that $x^2 - x + \frac{1}{3} > 0$ and $x^2 + x + \frac{1}{3} > 0$ for all values of x, so p_u and p_d are always positive.

Upward branching

If the branching has the form shown in Figure 1(b), the corresponding equations are

$$\begin{aligned}
&2p_u\Delta R + p_m\Delta R = -aj\Delta R\Delta t\\
&4p_u\Delta R^2 + p_m\Delta R^2 = \sigma^2\Delta t + a^2j^2\Delta R^2\Delta t^2\\
&p_u + p_m + p_d = 1.
\end{aligned}$$

The solution to these equations give the following probabilities

$$\begin{aligned}
p_u &= \frac{1}{6} + \frac{1}{2}(a^2j^2\Delta t^2 + aj\Delta t)\\
p_m &= -\frac{1}{3} - a^2j^2\Delta t^2 - 2aj\Delta t\\
p_d &= \frac{7}{6} + \frac{1}{2}(a^2j^2\Delta t^2 + 3aj\Delta t).
\end{aligned}$$

Note that the roots of $x^2 + 2x + \frac{1}{3} = 0$ are $-1 \pm \sqrt{2/3}$, so p_m remains positive when $-1 - \sqrt{2/3} < aj\Delta t < -1 + \sqrt{2/3}$. When this condition is satisfied, both p_u and p_d remain positive as well. Recall that upward branching is chosen when j is sufficiently negative in value.

Downward branching

If the branching has the form shown in Figure 1(c), the probabilities are

$$\begin{aligned} p_u &= \frac{7}{6} + \frac{1}{2}(a^2 j^2 \Delta t^2 - 3aj\Delta t) \\ p_m &= -\frac{1}{3} - a^2 j^2 \Delta t^2 + 2aj\Delta t \\ p_d &= \frac{1}{6} + \frac{1}{2}(a^2 j^2 \Delta t^2 - aj\Delta t). \end{aligned}$$

The roots of $x^2 - 2x - \frac{1}{3} = 0$ are $1 \pm \sqrt{2/3}$, so p_m remains positive when $1 - \sqrt{2/3} < aj\Delta t < 1 + \sqrt{2/3}$. Note that downward branching is chosen when j is sufficiently positive.

One can readily derive conditions on j for the transition probabilities in the three branching methods to be strictly positive.

(i) For normal branching

$$\frac{-\sqrt{2/3}}{a\Delta t} < j < \frac{\sqrt{2/3}}{a\Delta t};$$

(ii) For upward branching

$$\frac{-1-\sqrt{2/3}}{a\Delta t} < j < \frac{-1+\sqrt{2/3}}{a\Delta t};$$

(iii) For downward branching

$$\frac{1-\sqrt{2/3}}{a\Delta t} < j < \frac{1+\sqrt{2/3}}{a\Delta t}.$$

The left side inequality gives the lower bound on j such that positivity of all probability values is guaranteed. Since $1-\sqrt{2/3} < \sqrt{2/3}$, it is feasible to use either the normal branching or downward branching when $\dfrac{1-\sqrt{2/3}}{a\Delta t} < j < \dfrac{\sqrt{2/3}}{a\Delta t}$.

Number of time steps taken before downward branching starts (maximum number of upward moves)

Let $j_{\text{start}}^{\text{down}} = \left\lceil \dfrac{1-\sqrt{2/3}}{a\Delta t} \right\rceil$, which is the smallest integer greater than $\dfrac{1-\sqrt{2/3}}{a\Delta t}$. When $j \geq j_{\text{start}}^{\text{down}} > \dfrac{1-\sqrt{2/3}}{a\Delta t}$, the downward branching can be adopted with no occurrence of negative transition probabilities.

In actual implementation, it is desirable to adopt downward branching once we reach the $\left(j_{\text{start}}^{\text{down}}\right)^{\text{th}}$ time step so that the most upper node has the value of j that equals $j_{\text{start}}^{\text{down}}$. Due to the mean reversion of R^* (reverting to the zero value), it is not sensible to allow R^* to achieve too high value (too many upward moves) in the trinomial tree of R^*.

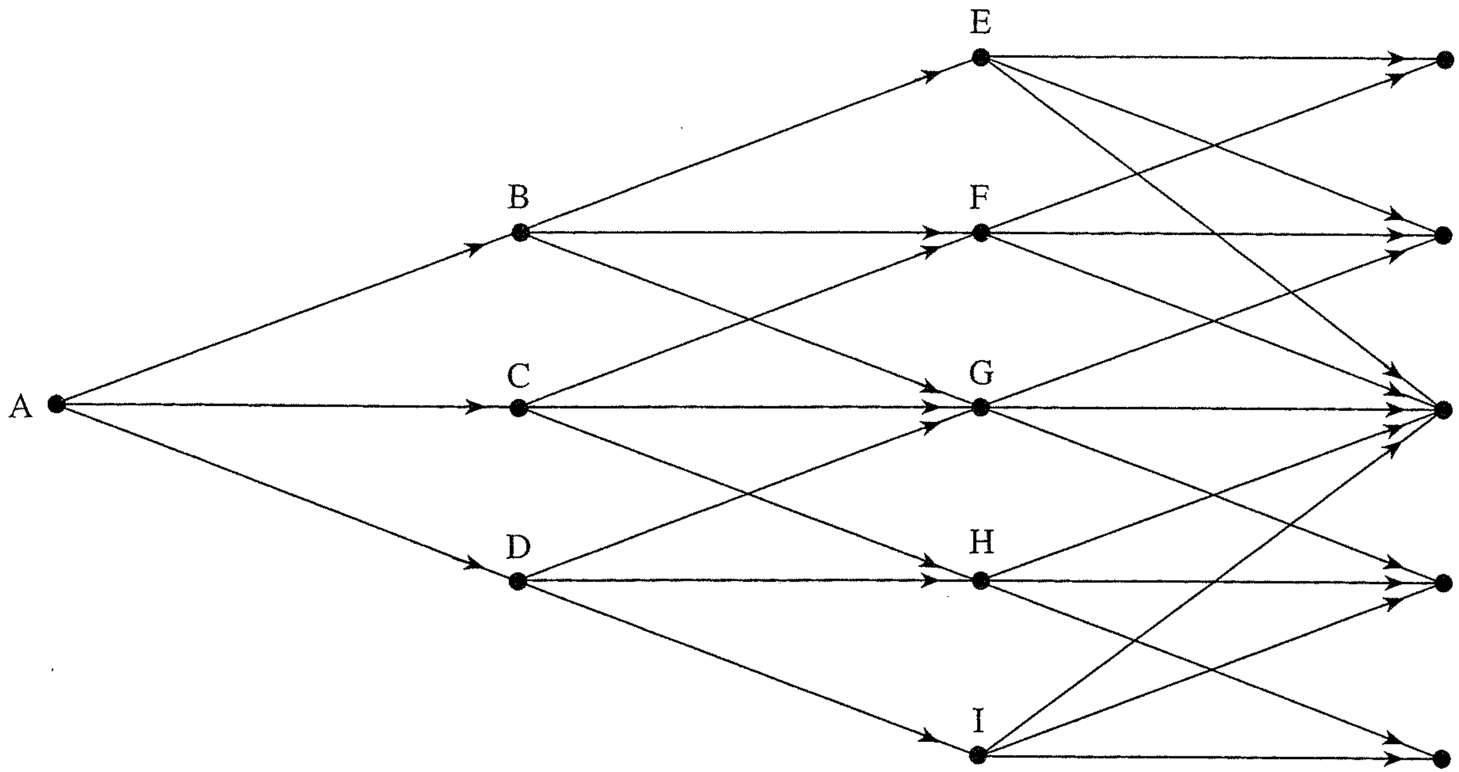
Essentially, $j_{\text{start}}^{\text{down}}$ is the maximum number of upward moves j_{max} in the R^*-tree.

Numerical example – Forward induction procedure

- Suppose that $\sigma = 0.01, a = 0.1$, and $\Delta t = 1$ year. In this case, $\Delta R = 0.01\sqrt{3} = 0.01732$. We set j_{max} to be the smallest integer greater than $\dfrac{1-\sqrt{2/3}}{a\Delta t} = 0.184/0.1$ so that $j_{\text{max}} = 2$. The choice of j_{max} guarantees that all nodal probabilities are positive when upward branching is used. By symmetry, we take $j_{\text{min}} = -j_{\text{max}}$, so $j_{\text{min}} = -2$. The tree is as shown in Figure 2. The probabilities on the branches emanating from each node are calculated using the equations on p.46-48 for p_u, p_m and p_d.

- The probabilities at each node in Figure 2 depend only on j, so the probabilities at node B are the same as the probabilities at node F. Furthermore, the tree is symmetrical. The probabilities at node D are the mirror image of the probabilities at node B; that is, p_u of node D is the same as p_d of node B. Due to the drift term $-aR^*dt$, we have $p_u < p_d$ at B and $p_u > p_d$ at D.

Tree for R^* in Hull–White model (first stage).



Node:	A	B	C	D	E	F	G	H	I
R(%)	0.000	1.732	0.000	−1.732	3.464	1.732	0.000	−1.732	−3.464
p_u	0.1667	0.1217	0.1667	0.2217	0.8867	0.1217	0.1667	0.2217	0.0867
p_m	0.6666	0.6566	0.6666	0.6566	0.0266	0.6566	0.6666	0.6566	0.0266
p_d	0.1667	0.2217	0.1667	0.1217	0.0867	0.2217	0.1667	0.1217	0.8867

Figure 2. Tree for R^* in the Hull-White model (first stage)

Spot rates used in generating the trees for R^* and R in Figures 2 and 3 respectively

Maturity	*Rate* (%)	*time*-0 *price of discount bond*
0.5	3.430	$e^{-0.0343\times 0.5}$
1.0	3.824	$e^{-0.03824\times 1}$
1.5	4.183	$e^{-0.04183\times 1.5}$
2.0	4.512	$e^{-0.04512\times 2} = 0.9137$
2.5	4.812	$e^{-0.04812\times 2.5}$
3.0	5.086	$e^{-0.05086\times 3}$

The n-year spot rate is set to be the yield to maturity of the discount bond maturing n years later.

Recall that $R(t)$ is the Δt-period rate known at time t. With $\Delta t = 1$, so $R(0) = 0.03824$.

- The second stage in the tree construction is to convert the tree for R^* into a tree for R. This is accomplished by displacing the nodes on the R^*-tree so that the initial term structure of discount bond prices is exactly matched.
- Define
$$\alpha(t) = R(t) - R^*(t).$$
We calculate the α's iteratively so that the initial term structure of discount bond prices is matched exactly.
- Define α_i as $\alpha(i\Delta t)$, the value of R at time $i\Delta t$ on the R-tree minus the corresponding value of R^* at time $i\Delta t$ on the r^*-tree.
- Define $Q_{i,j}$ as the present value of a security that pays off $1 if node (i, j) is reached and zero otherwise. The discrete mean reversion level α_i and the discrete Arrow-Debreu price $Q_{i,j}$ can be calculated using forward induction in such a way that the initial term structure of discount bond prices is matched exactly.

Formulas for α's and Q's

By the definition of discrete Arrow Debreu price, we observe

$$E\left[D(t_0, t_m)\mathbf{1}_{\{R(t_m)=\alpha_m+j\Delta R\}}\middle|\mathcal{F}_{t_0}\right] = Q_{m,j}.$$

To express the approach more formally, we suppose that the Arrow Debreu prices $Q_{m,j}$ have been determined. The next step is to determine α_m so that the tree correctly prices a zero-coupon bond maturing at $(m+1)\Delta t$. The Δt-period spot rate R at node (m,j) is $\alpha_m + j\Delta R$, so that the price P_{m+1} of a zero-coupon bond maturing at time $(m+1)\Delta t$ is given by

$$P_{m+1} = \sum_{j=-n_m}^{n_m} Q_{m,j} \exp\left(-(\alpha_m + j\Delta R)\Delta t\right), \tag{A}$$

where n_m is the number of nodes on each side of the centered node at time $m\Delta t$.

To guarantee \$1 at t_{m+1}, we purchase $\exp(-(\alpha_m + j\Delta R)\Delta t)$ units of the (m,j)-state security, for all j. When the j^{th} state occurs, we deposit $\exp(-(\alpha_m + j\Delta R)\Delta t)$ dollar received over the period $[t_m, t_{m+1}]$. This procedure is performed for all j.

To show the formula, we consider

$$P_{m+1} = E\left[D(t_0, t_{m+1})|\mathcal{F}_{t_0}\right],$$

where $D(t_0, t_{m+1})$ is the discount factor from t_0 to t_{m+1}.

Recall $E[X|\mathcal{F}] = \sum_{j=1}^{n} E[X|B_j]\mathbf{1}_{B_j}$, where $\{B_1, B_2, \ldots, B_n\}$ is the finite partition that generates $\mathcal{F}$. Using this formula, we obtain

$$\begin{aligned}
& E[D(t_m, t_{m+1})|\mathcal{F}_{t_m}] \\
= & \sum_j \mathbf{1}_{\{R(t_m)=\alpha_m + j\Delta R\}} E\left[D(t_m, t_{m+1})\middle| R(t_m) = \alpha + j\Delta R\right] \\
= & \sum_j \mathbf{1}_{\{R(t_m)=\alpha_m + j\Delta R\}} \exp(-(\alpha_m + j\Delta R)\Delta t).
\end{aligned}$$

Applying the tower rule, we obtain

$$
\begin{aligned}
& E[D(t_0, t_{m+1})|\mathcal{F}_{t_0}] \\
= \; & E[E[D(t_0, t_m)D(t_m, t_{m+1})|\mathcal{F}_{t_m}]|\mathcal{F}_{t_0}] \\
= \; & E[D(t_0, t_m)E[D(t_m, t_{m+1})|\mathcal{F}_{t_m}]|\mathcal{F}_{t_0}] \\
= \; & E\left[D(t_0, t_m)\sum_j \mathbf{1}_{\{R(t_m)=\alpha_m+j\Delta R\}} \exp\left(-(\alpha_m + j\Delta R)\Delta t\right)|\mathcal{F}_{t_0}\right] \\
= \; & \sum_j Q_{m,j} \exp(-(\alpha_m + j\Delta R)\Delta t).
\end{aligned}
$$

The solution for α_m from eq.(A) is

$$
\alpha_m = \frac{\ln \sum_{j=-n_m}^{n_m} Q_{m,j} e^{-j\Delta R\Delta t} - \ln P_{m+1}}{\Delta t}. \tag{B}
$$

Once α_m has been determined, $Q_{m+1,j}$ can be calculated using

$$
Q_{m+1,j} = \sum_k Q_{m,k} q(k,j) \exp\left(-(\alpha_m + k\Delta R)\Delta t\right), \tag{C}
$$

where $q(k,j)$ is the probability of moving from node (m,k) to node $(m+1,j)$ and the summation is taken over all values of k for which this is nonzero.

Illustration of the Second Stage

- The value of $Q_{0,0}$ is 1.0. The value of α_0 is chosen to give the right price for a zero-coupon bond maturing at time Δt. That is, α_0 is set equal to the initial Δt-period interest rate, where $\alpha_0 = R(0)$.

- We take $\Delta t = 1$ in this example, $\alpha_0 = 0.03824$ (as derived from the zero rates on p.53). This defines the position of the initial node on the R-tree in Figure 3.

- The next step is to calculate the values of $Q_{1,1}, Q_{1,0}$, and $Q_{1,-1}$. There is a probability of $p_u = 0.1667$ that the $(1,1)$ node is reached from the $(0,0)$ node and the discount rate for the first time step is 3.824%. Based on eq.(C) on P.57, the value of $Q_{1,1}$ is therefore $0.1667e^{-0.03824} = 0.1604$ since only one node $(0,0)$ that goes to $(1,1)$. Similarly, based on $p_m = 0.6666$, we have $Q_{1,0} = 0.6666e^{-0.03824} = 0.6417$. By symmetry, $Q_{1,-1} = Q_{1,1} = 0.1667e^{-0.03824} = 0.1604$.

Once $Q_{1,1}, Q_{1,0}$, and $Q_{1,-1}$ have been calculated, we are in a position to determine α_1. This is chosen to give the right price for a zero-coupon bond maturity at time $2\Delta t$ as observed at time zero. From $\Delta R = 0.01\sqrt{3} = 0.01732$ and $\Delta t = 1$, the price of this discount bond as seen at node B is $e^{-(\alpha_1+0.01732)}$ since one year $R(1)$ at B is $\alpha_1 + 0.01732$.

Similarly, the bond price at $t = 2$ as seen at node C is $e^{-\alpha_1}$ and the price as seen at node D is $e^{-(\alpha_1-0.01732)}$. The discount bond price as seen at the initial node A is therefore [see the right hand side of eq.(A) on P.55]

$$Q_{1,1}e^{-(\alpha_1+0.01732)} + Q_{1,0}e^{-\alpha_1} + Q_{1,-1}e^{-(\alpha_1-0.01732)}.$$

From the initial term structure of spot rates, this bond price should be

$$P_2 = e^{-0.04512\times 2} = 0.9137.$$

Substituting for the Q's in the above equation, we obtain

$$0.1604e^{-(\alpha_1+0.01732)} + 0.6417e^{-\alpha_1} + 0.1604e^{-(\alpha_1-0.01732)} = 0.9137$$

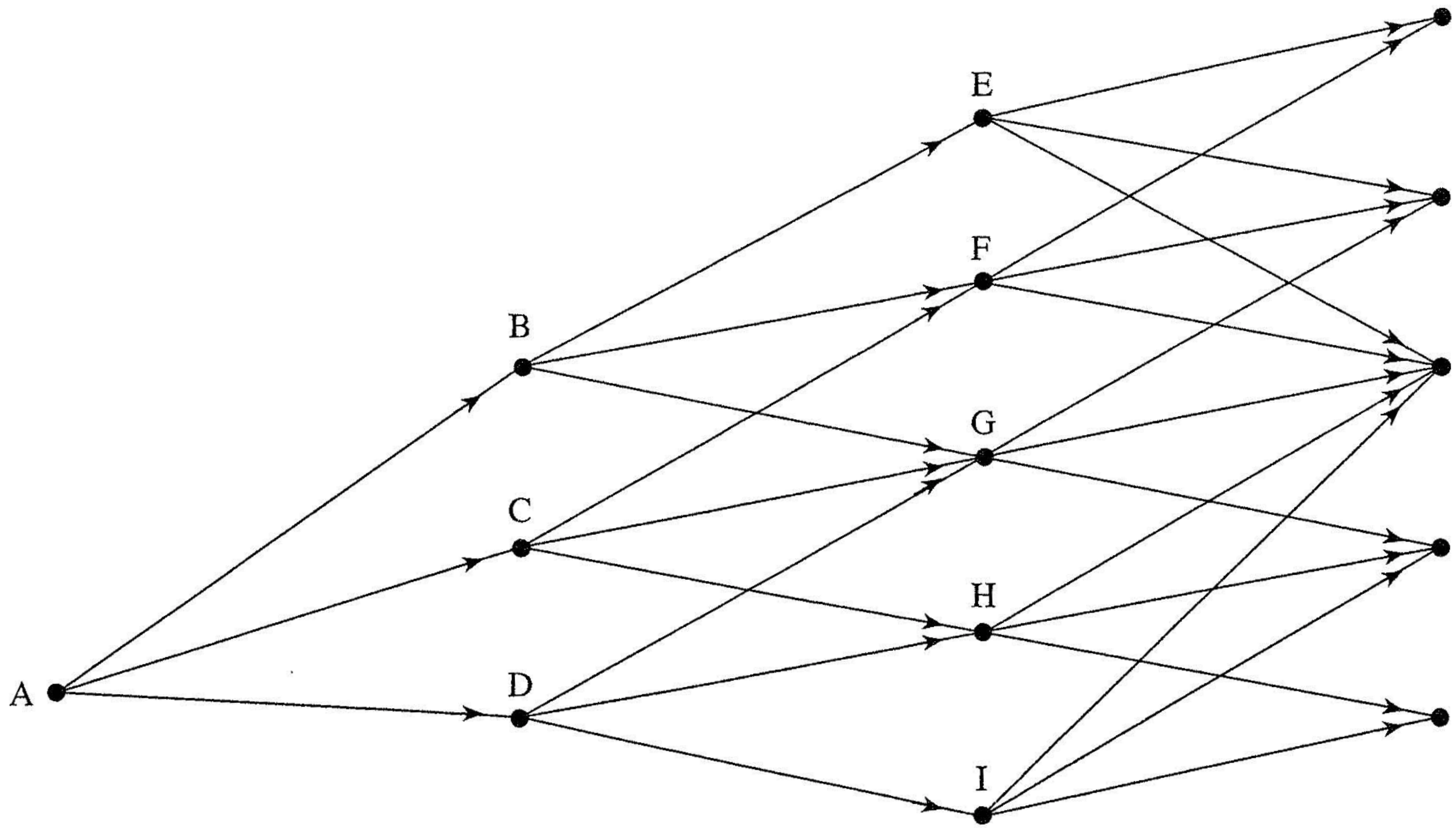
or

$$e^{-\alpha_1}(0.1604e^{-0.01732} + 0.6417 + 0.1604e^{0.01732}) = 0.9137.$$

This gives [see eq.(B)]

$$\alpha_1 = \ln\left(\frac{0.1604e^{-0.01732} + 0.6417 + 0.1604e^{0.01732}}{0.9317}\right) = 0.05205.$$

This means that the central node at time Δt in the tree for R corresponds to an interest rate of 5.205% (see Figure 3).

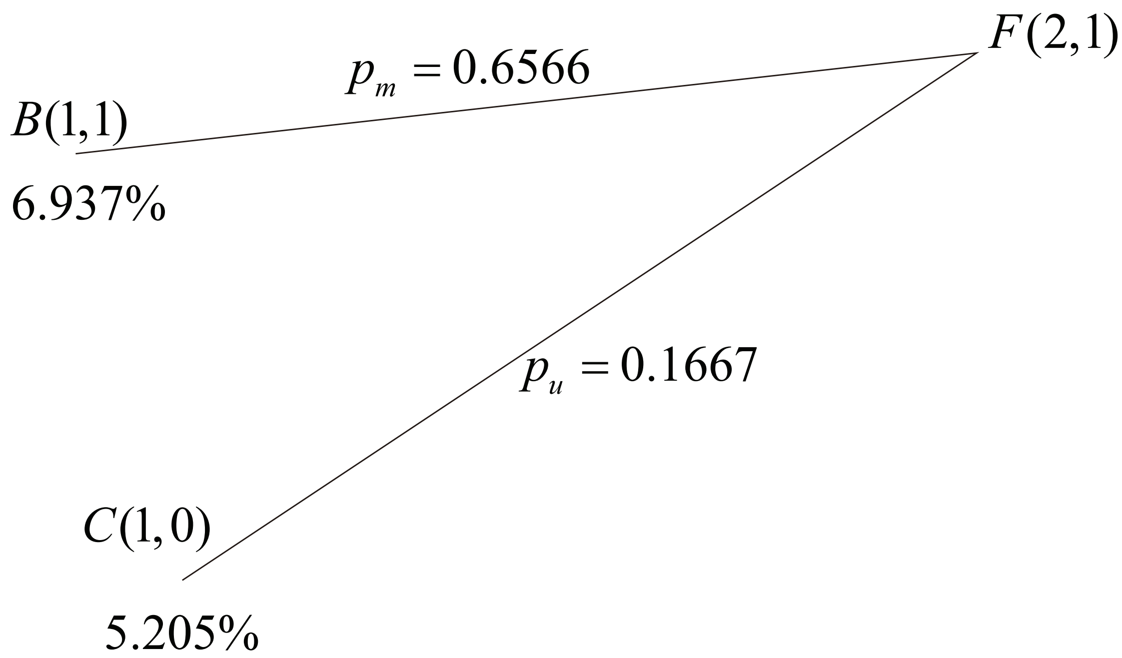


Node:	A	B	C	D	E	F	G	H	I
R(%)	3.824	6.937	5.205	3.473	9.716	7.984	6.252	4.520	2.788
p_u	0.1667	0.1217	0.1667	0.2217	0.8867	0.1217	0.1667	0.2217	0.0867
p_m	0.6666	0.6566	0.6666	0.6566	0.0266	0.6566	0.6666	0.6566	0.0266
p_d	0.1667	0.2217	0.1667	0.1217	0.0867	0.2217	0.1667	0.1217	0.8867

Figure 3. Tree for R in the Hull-White model (second stage)

Next, we calculate the Arrow-Debreu prices $Q_{2,2}, Q_{2,1}, Q_{2,0}, Q_{2,-1}$, and $Q_{2,-2}$.

Consider $Q_{2,1}$ as an example. This is the time-0 value of a security that pays off \$1 if node F is reached and zero otherwise. Node F can be reached only from nodes B and C. The Δt-rates at these nodes are $R_{11} = \alpha_1 + \Delta R = 6.937\%$ and $R_{10} = \alpha_1 = 5.205\%$, respectively. The probabilities associated with the B-F and C-F branches are 0.6566 and 0.1667, respectivley.



The value at node B of a security that pays \$1 at node F is therefore $0.6566e^{-0.06937}$. The value at node C is $0.1667e^{-0.05205}$. These two values contribute to $Q_{3,1}$. According to eq.(C) on P.57, we have

$$Q_{2,1} = 0.6566e^{-0.06937}Q_{1,1} + 0.1667e^{-0.05205}Q_{1,0} = 0.1998.$$

Similarly, since node G can be reached from the three nodes B, C and D, we have

$$\begin{aligned} Q_{2,0} &= 0.2217e^{-0.06937}Q_{11} + 0.6666e^{-0.05205}Q_{10} \\ &\quad + 0.2217e^{-0.03473}Q_{1,-1} \\ &= 0.4736. \end{aligned}$$

The next step in producing the R-tree in Figure 3 is to calculate α_2. Using eq.(A), we have

$$\begin{aligned} & Q_{22}e^{-(\alpha_2+2\times 0.01732)} + Q_{21}e^{-(\alpha_2+0.01732)} + Q_{20}e^{-\alpha_2} \\ + \;& Q_{2,-1}e^{-(\alpha_2-0.01732)} + Q_{2,-2}e^{-(\alpha_2-2\times 0.01732)} = e^{-0.05086\times 3}. \end{aligned}$$

After that, the $Q_{3,j}$'s can then be computed. We can then calculate α_3; and so on.

Extension to other models

The procedure that has just been outlined can be extended to more general models of the form

$$df(r) = [\theta(t) - af(r)]\,dt + \sigma\,dZ, \quad a > 0.$$

The family of models has the property that they can fit a given term structure of discount bond prices.

Under the discrete setting, we assume that the Δt-period rate R follows the same process as r:

$$df(R) = [\theta(t) - af(R)]\,dt + \sigma\,dZ, \quad a > 0.$$

1. Build the tree for x^*

   We start by setting $x = f(R)$, so that

   $$dx = [\theta(t) - ax]\, dt + \sigma\, dZ.$$

   The first stage is to build a tree for a variable x^* that follows the same process as x except that $\theta(t) = 0$ and the initial value is zero. The procedure here is identical to the procedure already outlined for building a tree like that in Figure 2.

2. *Determination of α_i and $Q_{i,j}$ by fitting the initial term structure of discount bond prices*

   As in Figure 3, we then displace the nodes at time $i\Delta t$ by an amount α_i to provide an exact fit to the initial term structure of discount bond prices. The equations for determining α_i and $Q_{i,j}$ inductively are slightly different from those for the $f(R) = R$ case.

The value of Q at the first node, $Q_{0,0}$, is set equal to 1. Suppose that the $Q_{i,j}$ have been determined for $1 \leq m$ ($m \geq 0$). The next step is to determine θ_m so that the tree correctly prices an $(m+1)\Delta t$ zero-coupon bond.

Define g as the inverse function of f so that $R = f^{-1}(x) = g(x)$. The Δt-period interest rate at the j^{th} node at time $m\Delta t$ is $g(\theta_m + j\Delta x)$. For example, if $f(R) = \ln R$, then $g(x) = e^x$.

The price of a zero-coupon bond maturing at time $(m+1)\Delta t$ is given by

$$P_{m+1} = \sum_{j=-n_m}^{n_m} Q_{m,j} \exp\left(-g(\theta_m + j\Delta x)\Delta t\right).$$

Since the inverse function g is involved, we cannot pull out θ_m explicitly as in eq.(B) on p.57. The above equation for θ_m is solved using a numerical procedure such as the Newton-Raphson method. The value θ_0 of θ when $m = 0$, is $x(0) = f(R(0))$.

Once θ_m has been determined, the Arrow Debreu prices $Q_{i,j}$ for $i = m+1$ can be calculated using

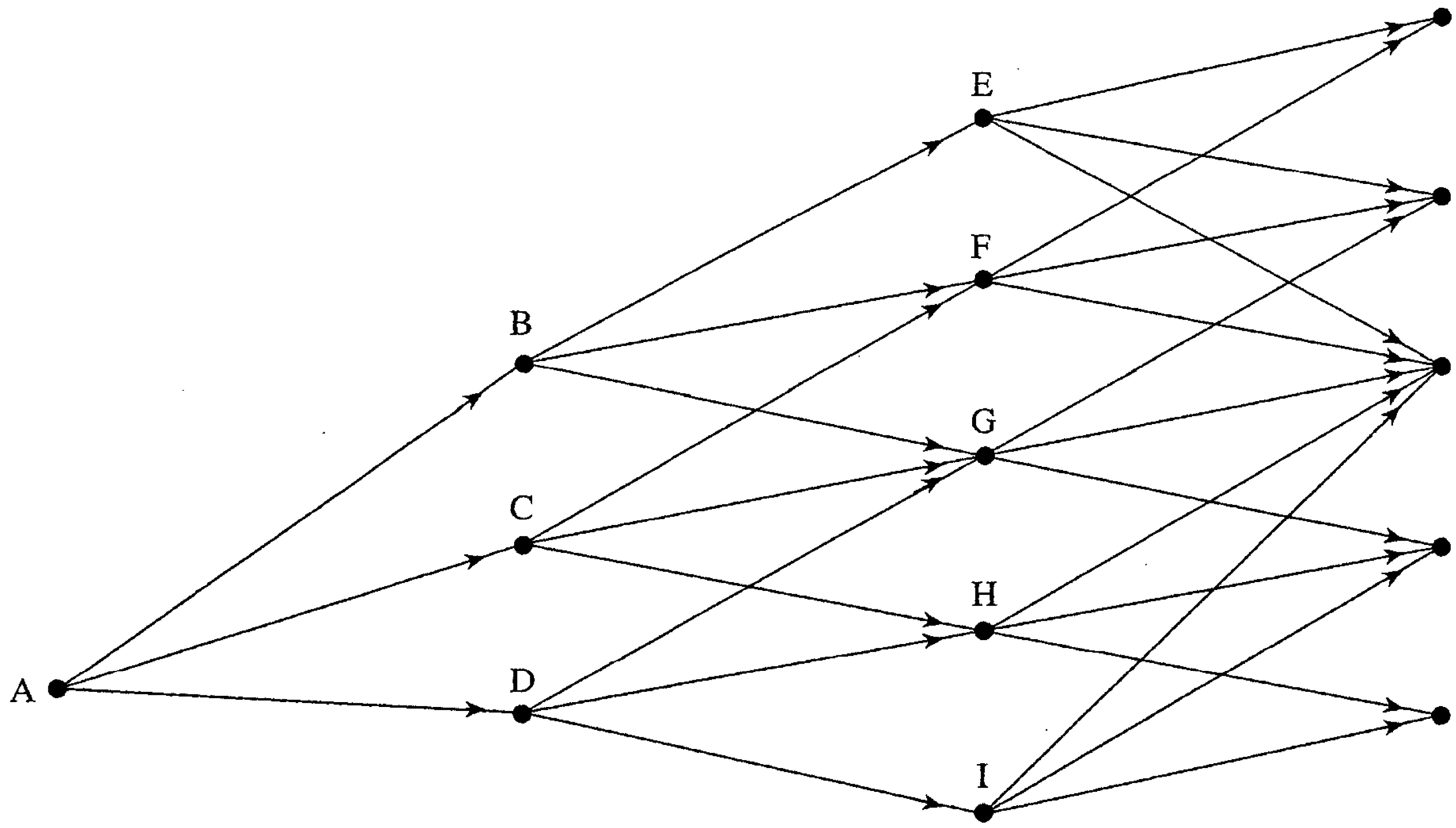
$$Q_{m+1,j} = \sum_k Q_{m,k} q(k,j) \exp\left(-g(\theta_m + k\Delta x)\Delta t\right),$$

where $q(k,j)$ is the probability of moving from node (m,k) to node $(m+1,j)$ and the summation is taken over all values of k where the probability is nonzero.

Figure 4 shows the results of applying the procedure to the lognormal model

$$d \ln r = [\theta(t) - a \ln r]\, dt + \sigma\, dZ$$

With the choices of parameter values: $a = 0.22, \sigma = 0.25, \Delta t = 0.5$, and the spot rates are as in the earlier table.



Node:	A	B	C	D	E	F	G	H	I
x	−3.373	−2.875	−3.181	−3.487	−2.430	−2.736	−3.042	−3.349	−3.655
R (%)	3.430	5.642	4.154	3.058	8.803	6.481	4.772	3.513	2.587
p_u	0.1667	0.1177	0.1667	0.2277	0.8609	0.1177	0.1667	0.2277	0.0809
p_m	0.6666	0.6546	0.6666	0.6546	0.0582	0.6546	0.6666	0.6546	0.0582
p_d	0.1667	0.2277	0.1667	0.1177	0.0809	0.2277	0.1667	0.1177	0.8609

Figure 4. Tree for lognormal model

Various choices of $f(R)$

When $f(r) = r$, we obtain the Hull-White model.

When $f(r) = \ln r$, we obtain the Black-Karasinski model. In most circumstances, these two models appear to perform about equally well in fitting market data on actively traded interest rate derivatives such as caps and European swaptions.

The main advantage of the $f(r) = r$ model is its analytic tractability. We can determine α_m without the root finding procedure. Its main disadvantage is that negative interest rates are possible.

Hull-White model versus Black-Karasinski model

In most circumstances, the probability of negative interest rates occurring under the Hull-White model is very small. However, some analysts are reluctant to use a model where there is any chance at all of negative interest rates. The chance of such occurrence becomes high when the prevailing interest rates are low.

The $f(r) = \ln r$ model has no analytic tractability to find θ_m. However, it has the advantage that interest rates are always positive. Another advantage is that traders naturally think in terms of σ's arising from a lognormal model (as in the Black-Scholes model) rather than σ's arising from a normal model.

The choice of model that appears to work well is one where $f(r)$ is chosen so that rates are lognormal for r less than 1% and normal for r greater than 1%.