MAFS5250 – Computational Methods for Pricing Structured Products

Topic 4 – Exotic structured products

4.1 Participating life insurance policies

- Product nature: bonus distribution mechanism
- Finite difference scheme
- Numerical results and pricing behavior

4.2 Convertible bonds

- Embedded features
- Modeling considerations in convertible bond pricing model
- Finite difference algorithm
- Analysis of pricing behavior

4.1 Participating life insurance policies

Product nature: bonus distribution mechanism

- Annual rate of return guarantee and bonus distribution
  - interest is credited to the policy account balance according to some smoothing surplus distribution mechanism
  - specified claim to a fraction of any excess return (surplus) generated by the investments
- Surrender option held by investors
  - American style feature of early redemption

Contractual terms

- A contract of nominal value P_0 is issued by the insurance company at time zero.
- The contract is acquired by an investor for a single premium V_0, where $V_0 \leq P_0$ (sold at a discount).
- Assuming no mortality risk, there are no further payments from or to the contract prior to expiration time T. At expiry, the contract is settled by a single payment from the issuer to the investor.

The nominal value P_0 is preset while V_0 is determined from the pricing model as the fair value of the contract at time 0.

Reference B. Jensen, P.L. Jørgensen and A. Grosen, "A finite difference approach to the valuation of path dependent life insurance liabilities," *Geneva Papers on Risk and Insurance Theory*, vol. 26 p.57-84 (2001).

Distributed reserve and buffer

- Policy account balance, $P(t)$ — book value of the policy

  To the insurer, $P(t)$ is the amount set aside to cover the contract liability, considered as the distributed reserve.

- Market value of the asset base backing the contract, $A(t)$

- Undistributed reserve or buffer, $B(t)$

Mechanism in place to protect the policy reserve from unfavorable fluctuations in the asset base. Accounting rule gives

$$A(t) = P(t) + B(t).$$

Since pension and life insurance companies typically invest largely in highly liquid assets such as bonds and stocks for which the relevant market prices are easily observable, we can assume that A is tradeable. This justifies the use of a risk neutral pricing measure.

Crediting mechanism of the policy value process

We write $\{P(t)\}_{0\leq t\leq T}$ as the account balance process of the contract. The benefit from the contract at maturity T is denoted by $P(T)$.

The evolution of $P(\cdot)$ between successive time points in the point set

$$\mathcal{T} = \{1, 2, \cdots, T\}$$

is determined by the discretely compounded policy interest rate process, $\{r_P(t)\}_{t\in\mathcal{T}}$. We have

$$P(t) = [1 + r_P(t)]P(t-1), \quad t \in \mathcal{T},$$

so that

$$P(t) = P_0 \prod_{i=1}^{t} [1 + r_P(i)].$$

Time is measured in years, $P(\cdot)$ is updated annually, and $r_P(\cdot)$ is annualized rate. That is, $P(\cdot)$ is held fixed for $(t-1, t)$ and has a jump in value at t^+.

Dynamics of the asset side

$$dA(t) = \mu A(t)\,dt + \sigma A(t)\,dW(t), \quad A(0) = A_0.$$

Here, $W(t)$ is a standard Brownian motion defined on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on the finite time interval $[0, T]$, σ is volatility and μ is the expected rate of return.

Under the risk neutral probability measure Q, all prices discounted by the continually compounded risk free interest rate r are Q-martingales. We have

$$dA(t) = rA(t)\,dt + \sigma A(t)\,dW^Q(t), \quad A(0) = A_0,$$

where $W^Q(t)$ is a standard Brownian motion under measure Q.

Liability side of the balance sheet

Specification of $r_P(t)$ (dynamic distribution of funds to investor's account) based on $P(t^-)$ and $B(t^-) = A(t) - P(t^-)$:

$$r_P(t) = \max\left(r_G, \alpha\left(\frac{B(t^-)}{P(t^-)} - \gamma\right)\right),$$

r_G: annual rate of return guarantee of the contract

γ: target buffer ratio

α: distribution ratio

If the actual/observed buffer relative to the policy account balance at time t^- exceeds the desired level γ, then the insurer distributes a fraction α of the surplus.

$P(t)$: strictly increasing process with jump at discrete time points;

$B(t)$ may be temporarily negative (insolvency with respect to the contract). This occurs when $A(s) < P(t^+)$, where $t < s < t + 1$, since $P(t^+)$ has been fixed at t^+ earlier than s.

Evaluation of the account balance process $P(t)$

In the favorable scenario where the asset return is sufficiently high, specifically, $r_G < \alpha\left(\frac{B(t^-)}{P(t^-)} - \gamma\right)$, we have

$$\begin{aligned} P(t^+) &= P(t^-)\left[1 + \alpha\left(\frac{B(t^-)}{P(t^-)} - \gamma\right)\right] \\ &= P(t^-) + \alpha\left[B(t^-) - B^*(t^-)\right], \end{aligned}$$

where $B(t^-) = A(t) - P(t^-)$ and $B^*(t^-) = \gamma P(t^-)$ is the desired level of buffer over the period $(t-1, t)$. In general

$$\begin{aligned} P(t^+) &= P(t^-)\left[1 + \max\left(r_G, \alpha\left(\frac{B(t^-)}{P(t^-)} - \gamma\right)\right)\right] \\ &= P(t^-)\left[1 + r_G + \max\left(\alpha\left(\frac{A(t) - P(t^-)}{P(t^-)} - \gamma\right) - r_G, 0\right)\right]. \end{aligned}$$

Here, the policy account process $P(\cdot)$ is highly dependent on the path followed by $A(t)$. Note that $P(\cdot)$ and $B(\cdot)$ have jump at each discrete time point due to crediting mechanism while $A(t)$ is assumed to be continuous at all times.

Bonus option

- The guaranteed interest rate implies a bond floor under the final payment from the contract of $P^T_{floor} = P_0(1+r_G)^T$.

- Let $V^E(s)$ denote the time-s value of the contract (European style) and let $D(s)$ denote the time-s value of the bond component.

$$\begin{aligned} V^E(s) &= E^Q\left[e^{-r(T-s)}P(T)|\mathcal{F}_s\right], \quad \text{for all } s \in [0,T], V^E(T) = P(T) \\ D(s) &= e^{-r(T-s)}P_0(1+r_G)^T \quad \text{[annually compounded].} \end{aligned}$$

Value of the bonus option is given by

$$\Gamma(s) = V^E(s) - D(s).$$

Surrender option (American style contract)

This is the right given to the investor to terminate the contract prematurely

$$V^A(s) = \sup_{\tau \in \mathcal{T}_{s,T}} E^Q \left[e^{-r(\tau - s)} P(\tau) | \mathcal{F}_s \right],$$

where the contract is terminated at the investor's discretion at time τ. Here, $\mathcal{T}_{s,T}$ denotes the class of $\mathcal{F}_s$-stopping times taking values in $[s, T]$. In most contracts, the investors may have to pay a penalty charge for premature surrender. Here, we assume zero penalty charge.

- Since the surrender payoff is $P(\tau)$, so

  $$V^A(s) \geq P(s), \quad 0 \leq s \leq T.$$

  However, if there is a surrender charge, which may be either fixed or proportional or combination of both, then the surrender payoff is reduced by the surrender charge. With penalty charges, V^A drops in value as a result.

The contract value with surrender right is the sum of the bond component, bonus option value and surrender option value. Let $\psi(s)$ denote the time-s value of the surrender option. We then have

$$\psi(s) = V^A(s) - D(s) - \Gamma(s).$$

Comment on the optimal surrender policy

For an American style contract, since the value of $P(s)$, $t < s < t+1$, has been fixed at time t^+, it will never be optimal to exercise the contract between two updates of $r_P(\cdot)$. Exercising the contract at time $s, t < s < t+1$, will result in a loss of interest amounting to $\left[e^{r(s-t)} - 1\right] P(t^+) > 0$, compared to exercising at time t^+. On the other hand, during the time interval $(t, t+1)$, it may be possible that $A(s)$ drops drastically. The investor may envision the significant loss of the bonus option while the guaranteed return r_G is below the riskfree rate. It may become advantageous to early surrender.

States of the world

Note that $P(s), t < s < t+1$, does not change during the time interval $(t, t+1); P(s)$ is updated at t^+ based on $P(t^-)$ and $A(t)$.

All relevant information about the states of the world is summarized by $(A(s), P(t^+))$, where $t < s < t+1 \leq T$, observing that $A(s)$ is continuous for all times while $P(s)$ has jump across sampling dates. We write the time-s value of the contract as

$$V_s = V(s, A(s), P(t^+)), \quad t < s < t+1 \leq T, t \in \mathcal{T} \cup \{0\}.$$

Here, $\mathcal{T}$ is the set of sampling dates and the contract initiation date is time zero.

Continuity of value function across a sampling date t

$$V_{t^-} = V_{t_+} \Leftrightarrow V(t^-, A(t), P(t^-)) = V(t^+, A(t), P(t^+)).$$

Any joint realization of (s, A_s, P_s) necessarily changes V in a continuous manner since there is no net cash flow associated with the contract across a sampling date.

- The contract value is the discounted expectation of the terminal value of the policy account and it remains to be continuous across t even when $P(t)$ has a jump. Such a jump of $P(t)$ has been anticipated based on $A(t^-)$ and $P(t^-)$, so the contract value at t^- should have reflected the jump of $P(t)$ across t.

- Since the insurer is NOT required to sell the assets in response to the jump in $P(t)$ across t, so $A(t)$ remains to be continuous. The cash flow to the policy holder occurs only at maturity or surrender, where the assets are sold to provide the cash payment required for settlement of the contract. There is no cash flow to and from the policy holder across the sampling date t.

Model formulation

- $P(\cdot)$ is updated only discretely, so it does not change outside the set of time points $\mathcal{T}$ (sampling dates).

- Between these 'sampling dates', the Black-Scholes equation is to be solved (within the time interval between the consecutive years t and $t+1$, and before the no-jump condition on V is applied):

$$\frac{\partial V}{\partial s} + \frac{\sigma^2}{2} A^2 \frac{\partial^2 V}{\partial A^2} + rA \frac{\partial V}{\partial A} - rV = 0, \quad s \in (0, T]/\mathcal{T},$$

with $V_T = P(T)$.

Over the time interval $(t, t+1)$, we write the numerical option value as

$$V_{t,k}^{i,j} = V(t + 1 - k\Delta s, i\Delta A, j\Delta P),$$

where $t \in \{0, 1, \cdots, T-1\}, 0 \leq k \leq K, 0 \leq i \leq I, 0 \leq j \leq J$. As k increases from 0 to K, the calendar time decreases from $t+1$ to t.

Computational domain

Restricted to the finite domain, $(s, A, P) = ([0, T] \times [0, \overline{A}] \times [P_0, \overline{P}])$, where P_0 is the initial policy value, $\overline{A}$ and $\overline{P}$ are sufficiently large constants. Define $P_0 = j_0 \Delta P$ and $\overline{P} = J \Delta P$, so that $\Delta P = \dfrac{\overline{P} - P_0}{J - j_0}$. Let

$$\begin{aligned} I &= \frac{\overline{A}}{\Delta A} = \text{ number of equally spaced steps in the } A\text{-direction} \\ K &= \text{ number of time steps per year} \end{aligned}$$

Discretization considerations

- Choice of the mesh size of the grid in the (A, P)-space.
- Imposition of the auxiliary conditions.
- Implementation of the no-jump condition on the contract value across sampling dates.

Numerical procedure

1. Start at time T and apply the terminal payoff condition on a suitable grid in the (A, P)-space.

2. For every discrete value of P, solve the Black-Scholes equation via a finite difference scheme. This gives $V_{(T-1)^+}$ everywhere in the grid.

3. Apply the no-jump condition on the contract value to obtain $V_{(T-1)^-}$ everywhere in the grid. Actually, we require the procedure of implementing the jump condition on $P(\cdot)$ [jumping from $P(t^-)$ to $P(t^+)$ at time t].

4. Repeat steps 2 and 3 to obtain V_{t^-} from $V_{(t+1)^-}$ everywhere in the grid working backward from $t = T - 1$ to $t = 0$. Minor remark: It is not necessary to apply the no-jump condition at $t = 0$ since there is no crediting of interest at time of initiation of the contract.

We adopt the *fully implicit scheme* (A as the independent state variable and P is fixed):

$$\frac{V_{t,k}^{i,j} - V_{t,k+1}^{i,j}}{\Delta s} + \frac{1}{2}\sigma^2(i\Delta A)^2 \left(\frac{V_{t,k+1}^{i+1,j} - 2V_{t,k+1}^{i,j} + V_{t,k+1}^{i-1,j}}{\Delta A^2} \right)$$
$$+ \, r(i\Delta A) \left(\frac{V_{t,k+1}^{i+1,j} - V_{t,k+1}^{i-1,j}}{2\Delta A} \right) - rV_{t,k+1}^{i,j} = 0,$$

which can be simplified to the following implicit relation:

$$E^i V_{t,k+1}^{i-1,j} + H^i V_{t,k+1}^{i,j} + G^i V_{t,k+1}^{i+1,j} = V_{t,k}^{i,j},$$

where

$$\begin{aligned} E^i &= \frac{r(i\Delta A)}{2}\frac{\Delta s}{\Delta A} - \frac{\sigma^2(i\Delta A)^2}{2}\frac{\Delta s}{(\Delta A)^2}, \\ H^i &= 1 + r\Delta s + \sigma^2(i\Delta A)^2\frac{\Delta s}{(\Delta A)^2}, \\ G^i &= -\frac{r(i\Delta A)}{2}\frac{\Delta s}{\Delta A} - \frac{\sigma^2(i\Delta A)^2}{2}\frac{\Delta s}{(\Delta A)^2}. \end{aligned}$$

Boundary conditions

At $A = 0, i = 0$, the governing differential equation reduces to

$$\frac{\partial V}{\partial s} - rV = 0.$$

There is no equity participation, so the contract value is only influenced by discounting. The corresponding finite difference relation is

$$V_{t,k+1}^{0,j} = (1 - r\Delta s)V_{t,k}^{0,j}.$$

Here, $1 - r\Delta s$ can be visualized as the discrete discount factor over one time step.

The value function is approximately linear in the far field, where A is sufficiently large, so that

$$\frac{\partial^2 V}{\partial A^2} = 0 \quad \text{for } A \to \infty.$$

Applying this condition at $I-1$, this yields the relation:

$$V_{t,k+1}^{I,j} = 2V_{t,k+1}^{I-1,j} - V_{t,k+1}^{I-2,j}.$$

In other words, the numerical boundary value $V_{t,k+1}^{I,j}$ can be computed based on the value functions at the two neighboring interior points. The discretized equation at $I-1$ is seen to be

$$E^{I-1}V_{t,k+1}^{I-2,j} + H^{I-1}V_{t,k+1}^{I-1,j} + G^{I-1}V_{t,k+1}^{I,j} = V_{t,k}^{I-1,j}.$$

Eliminating $V_{t,k+1}^{I,j}$, the last equation becomes

$$(E^{I-1} - G^{I-1})V_{t,k+1}^{I-2,j} + (H^{I-1} + 2G^{I-1})V_{t,k+1}^{I-1,j} = V_{t,k}^{I-1,j}.$$

For the first equation, we have

$$\begin{aligned} H^1V_{t,k+1}^{1,j} + G^1V_{t,k+1}^{2,j} &= V_{t,k}^{1,j} - E^1V_{t,k+1}^{0,j} \\ &= V_{t,k}^{1,j} - E^1(1 - r\Delta s)V_{t,k}^{0,j}. \end{aligned}$$

The imposition of the boundary condition at $i=0$ would affect the first element in the column vector of the tridiagonal system of equations.

The matrix representation of the implicit scheme is given by

$$
\begin{bmatrix}
H^1 & G^1 & 0 & \cdots & \cdots & 0 \\
E^2 & H^2 & G^2 & 0 & & \vdots \\
0 & E^3 & H^3 & G^3 & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & 0 \\
\vdots & & 0 & E^{I-2} & H^{I-2} & G^{I-2} \\
0 & \cdots & \cdots & 0 & (E^{I-1} - G^{I-1}) & (H^{I-1} + 2G^{I-1})
\end{bmatrix}
\begin{pmatrix}
V_{t,k+1}^{1,j} \\ V_{t,k+1}^{2,j} \\ \vdots \\ \\ \\ V_{t,k+1}^{I-1,j}
\end{pmatrix}
$$

$$
= \begin{pmatrix}
V_{t,k}^{1,j} - E^1(1 - r\Delta s)V_{t,k}^{0,j} \\ V_{t,k}^{2,j} \\ \vdots \\ \\ \\ V_{t,k}^{I-1,j}
\end{pmatrix}.
$$

The last row in the coefficient matrix reflects the incorporation of the numerical boundary condition in the far field.

Implementation of the no-jump condition on the contract value

We need to code the relationship between $V_{t,K}^{i,j}$ and $V_{t-1,0}^{i,j}$ according to the no-jump condition. Recall

$$P(t^+) = P(t^-) + \max(r_G P(t^-), \alpha\left\{[A(t) - P(t^-)] - \gamma P(t^-)\right\}).$$

Here, $A(t) - P(t^-)$ is the actual buffer at t^- while $\gamma P(t^-)$ is the desired buffer at t^-.

For each i and j in the grid, we write $P(t^+) = \tilde{j}\Delta P, P(t^-) = j\Delta P$, where j is some integer index. In terms of $\tilde{j}$ and j, the crediting mechanism across a sampling date is modeled by

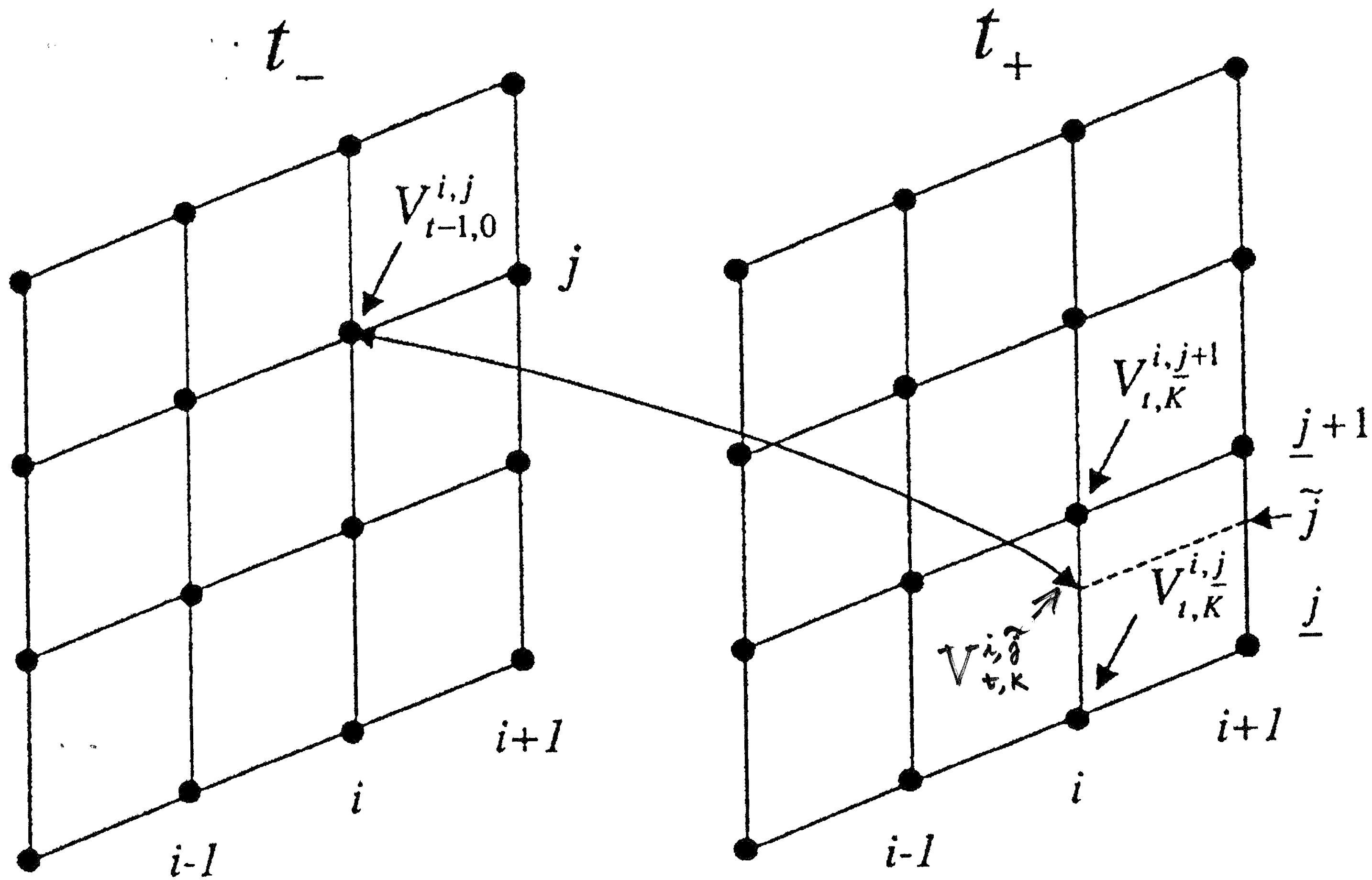
$$\begin{aligned}\tilde{j} &= \frac{j\Delta P + \max\{r_G(j\Delta P), \alpha[(i\Delta A - j\Delta P) - \gamma(j\Delta P)]\}}{\Delta P} \\ &= j + \max\left\{r_G j, \alpha\left[\left(i\frac{\Delta A}{\Delta P} - j\right) - \gamma j\right]\right\}. \qquad (i)\end{aligned}$$

Remark Though there is an upward jump in P, the contract value remains continuous across the date of interest crediting.

- We solve for the contract value at finite number of preset values of P, $P = j\Delta P$, $j = j_0, j_0 + 1, \ldots, J$, with $P_0 = j_0\Delta P$ and $\overline{P} = J\Delta P$. The updated P would not fall onto one of these preset values. Linear interpolation between neighboring contract values is then applied.

- Denote the integer part of $\widetilde{j}$ as $\underline{j}$. If $\underline{j} + 1 \leq J$, compute $V^{i,j}_{t-1,0}$ by using the linear interpolation

$$V^{i,j}_{t-1,0} = V^{i,\widetilde{j}}_{t,K} \approx [1 - (\widetilde{j} - \underline{j})]V^{i,\underline{j}}_{t,K} + (\widetilde{j} - \underline{j})V^{i,\underline{j}+1}_{t,K}. \qquad (ii)$$

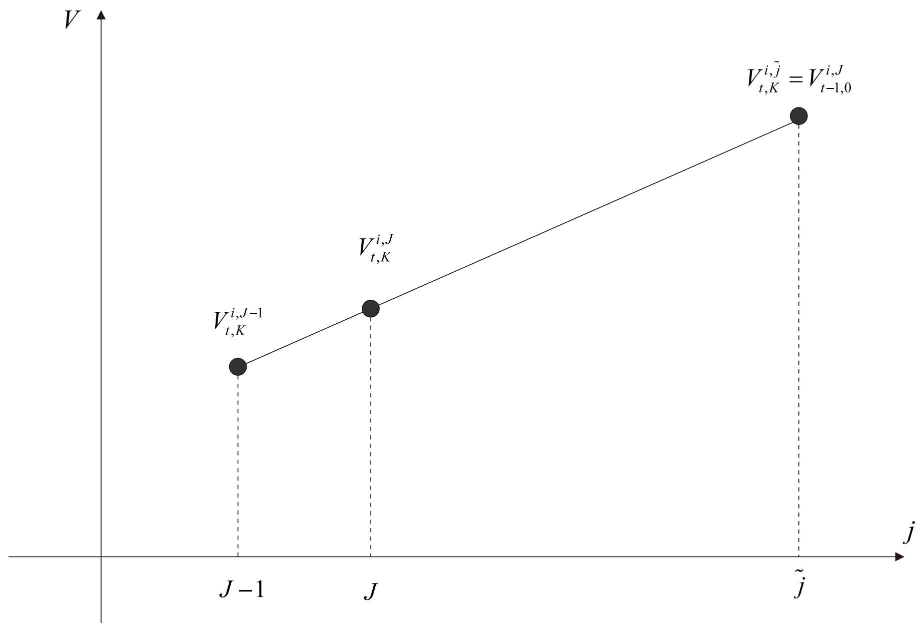


Note that V_t and $A(t)$ are continuous and $P(t)$ has a jump across a sampling date. The "initial" values $V^{i,j}_{t-1,0}$ at the zeroth time step, $k = 0$, in the interval $(t-1, t)$ is given by

$$[1 - (\tilde{j} - \underline{j})]V^{i,\underline{j}}_{t,\bar{K}} + (\tilde{j} - \underline{j})V^{i,\underline{j}+1}_{t,\bar{K}}.$$

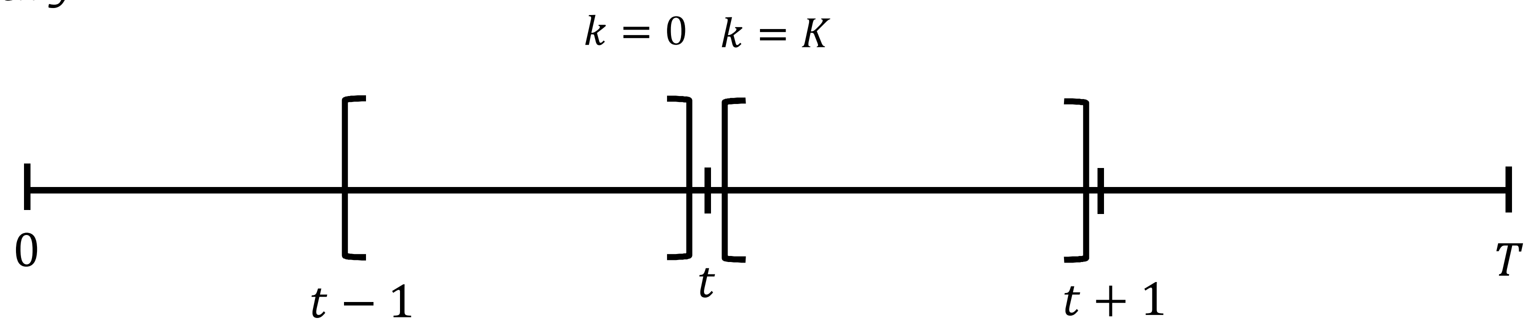
At time t^-, suppose $P(t^-)$ assumes the upper bound value $J\Delta P$. Subsequently, $P(t^+)$ jumps to $\tilde{j}\Delta P > J\Delta P$. The updated value of P falls outside the preset upper bound of P. If $\underline{j}+1 > J$ and hence lies outside the grid, then (ii) cannot be used. Instead, since for large values of P, the contract value V is approximately linear in P, we can apply the linear extrapolation as follows:

$$V_{t-1,0}^{i,J} = V_{t,K}^{i,J} + (\tilde{j} - J)\left(V_{t,K}^{i,J} - V_{t,K}^{i,J-1}\right). \qquad (iii)$$



Extrapolation beyond the computational domain

Summary



- $P(s)$ stays at the same constant value as $P(t^+)$, for $t < s < t+1$.

- For a given fixed value of P, solve the Black Scholes equation numerically between t^+ and $(t+1)^-$.

- $V_{t,K}^{i,\widetilde{j}} = V_{t-1,0}^{i,j}$ across the sampling date t.

  Note that i stays at the same value since the underlying asset price is continuous across the sampling date t. However, j jumps to $\widetilde{j}$ across t.

Numerical results and pricing behavior

A large number of experiments have resulted in the choices of values shown in the table below in the numerical implementation.

Choice of parameter values	
$\overline{A}$	1000
$\overline{P}$	2000
I	800
J	200
K	100

- Setting $\overline{A}$ and $\overline{P}$ involves a tradeoff between covering as much probability mass as possible and avoiding to enlarge the solution region unnecessarily. With $A_0 = P_0 = 100$, setting $\overline{P} = 2000$ and $A = 1000$ should be sufficient.
- It is relatively more important to operate with a fine grid in the A-direction as A is the uncertainty generating factor in the model.

Convergence and computation time

Basic set of parameter values: $A_0 = P_0 = 100, r = 5\%, r_G = 4\%, T =$ 20 years, $\alpha = 0.3, \gamma = 0.1, \sigma = 15\%, K = 100$. The numerical results of the participating policy with surrender right are listed below:

	(I, J)				
	$(100, 100)$	$(200, 200)$	$(400, 400)$	$(800, 800)$	$(1600, 1600)$
Contract value	113.45	111.96	111.36	111.30	111.28
Relative error	1.95%	0.62%	0.08%	0.02%	0.00%
CPU time (sec.)	11	48	202	828	3858

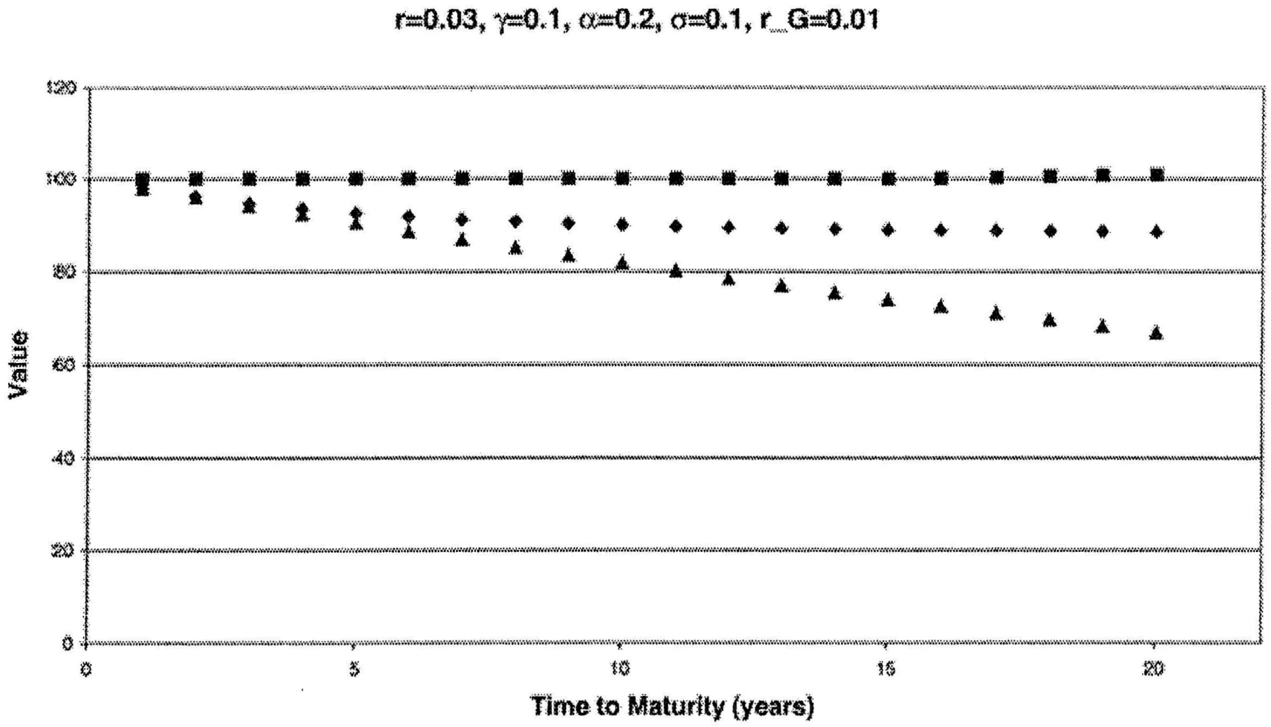
The number of time steps per year K has been fixed at 100. The numerical tests show that gain in accuracy from further increasing K is negligible.

Operation counts (CPU time)

Doubling I and J increases CPU time roughly by four-folds.

▲ value of bond element (decreases with increasing maturity since $r_G < r$)

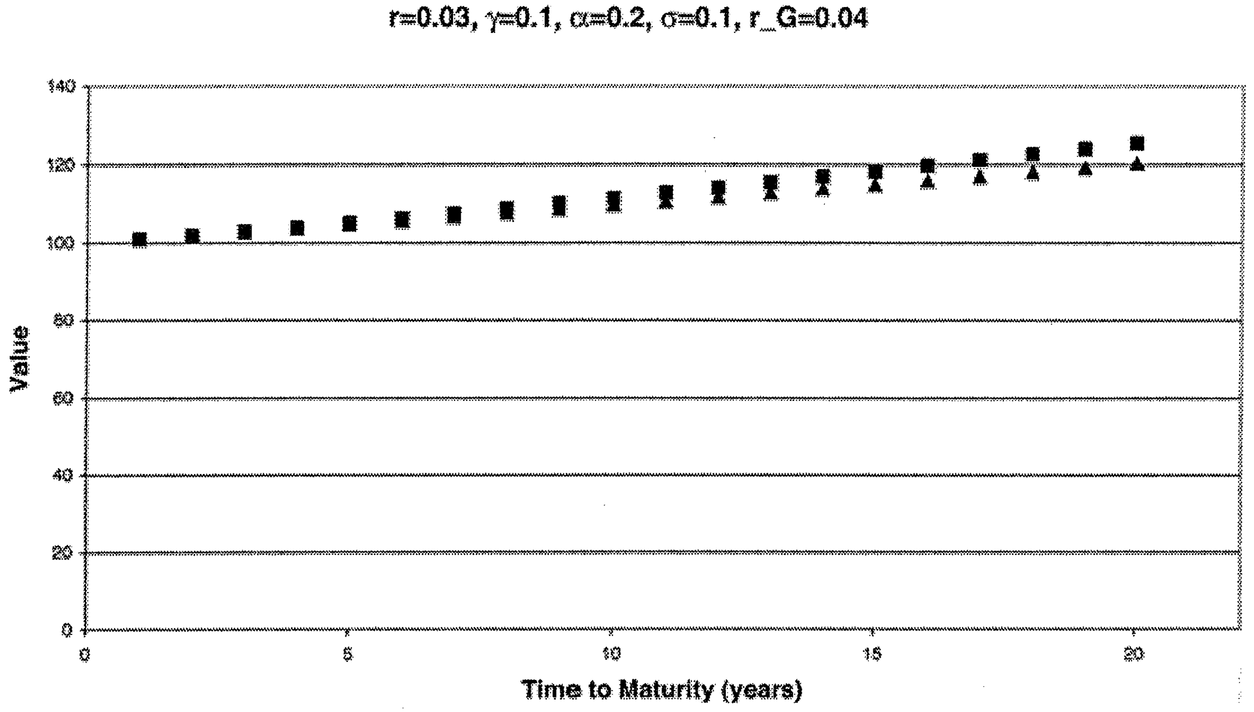
♦ value of the European contract ■ value of the American contract



(a)

- Value of sum of option elements = value of contract − value of bond element. Value of each option element increases as the time to maturity increases.

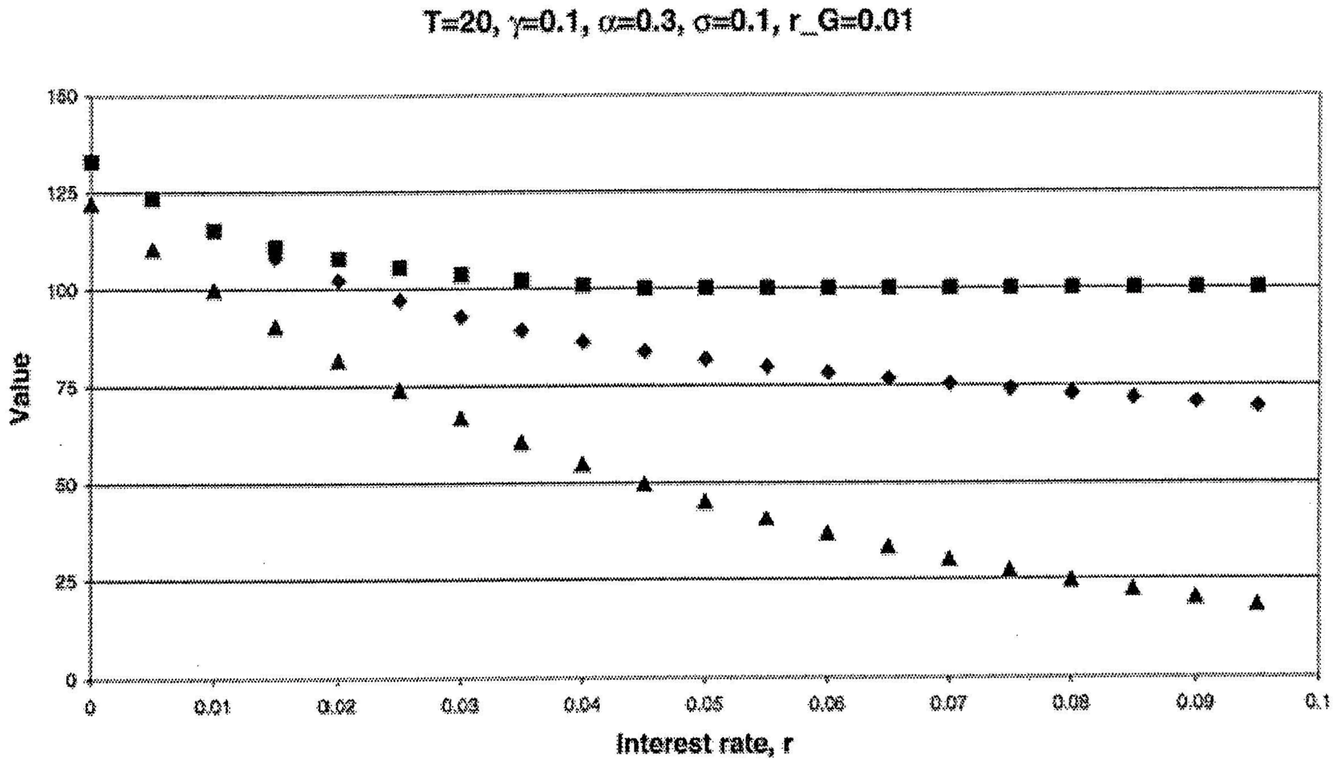
The guaranteed interest rate is raised from being smaller than the riskless interest rate r to being larger than r $[r_G = 0.04, r = 0.03]$.



(b)

- The surrender option has virtually no value since the guaranteed interest rate is higher than the riskless interest rate. There is very little incentive to terminate the contract prematurely.
- The bonus option has dropped in value (lower value is attached to the equity participation) and the contract value mainly consists of the bond element.

Effect on contract value with varying values of riskless rate r.



- A higher interest rate r implies higher values of the bonus and surrender options. The value of the bond component declines.
- There is a critical interest rate above which the American contracts should be immediately exercised (as evidenced by values of American contracts staying the same when r is above some threshold value). At sufficiently high r, the funds are better reinvested in riskless bonds to realize the higher return.

More discussion on the pricing behavior

- Raising r_G implies smaller values of the bonus and surrender options.

- An increase in T yields a larger bonus option value.

- $\sigma = 0$ is not sufficient to make the values of the option elements equal zero.

  (i) When $r < r_G$ and $\sigma = 0$, the issuer will never be able to build reserves for bonus payments and the contracts are in effect above par riskless bonds.

  (ii) When $r_G < r$ and $\sigma = 0$, the company will surely be able to build bonus reserves and be forced to distribute part of this in the form of bonus.

4.2 Convertible bonds

Combination of bonds and equities – bond plus conversion option

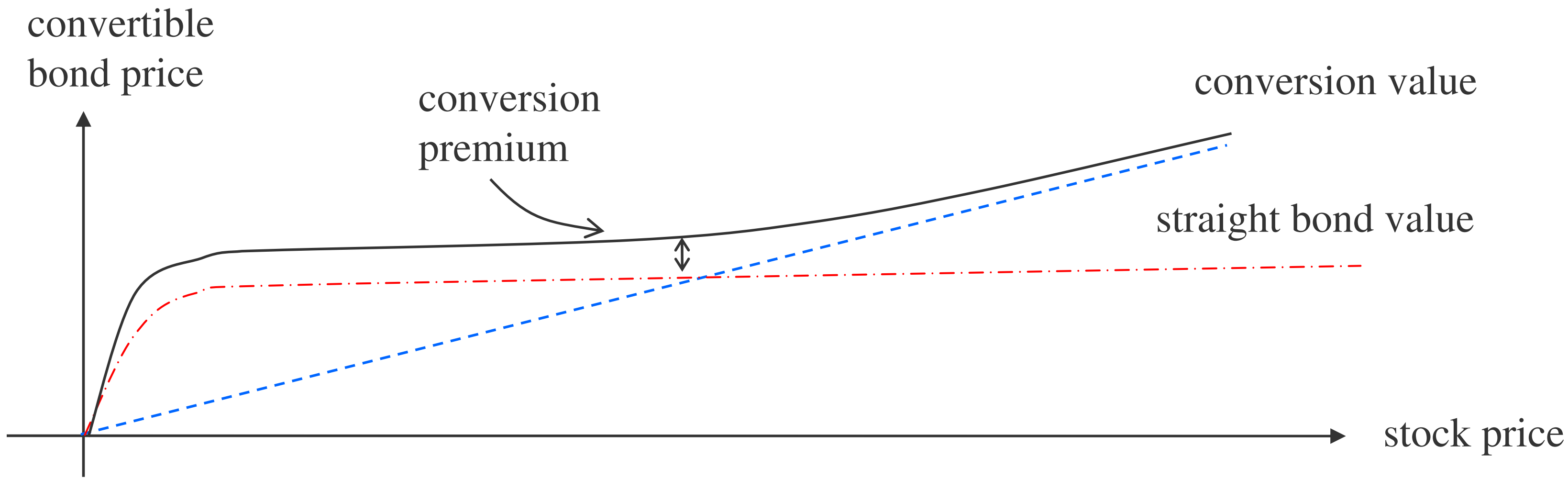
- Bondholder has the right to convert the bond into common shares of issuer's company at some contractual price (conversion number may change over time with dependence on the realization of the stock price path).

- Issuer's call and holder's put

  - Hard call and Parisian soft call provision; notice period requirement

Holder's perspective: take advantage of the future potential growth of issuer's company

Issuer's perspective: raise capital at a lower cost by the provision of conversion privilege to the bondholders



- High default risk when the stock price level is very low.

- Conversion premium = value above the bond floor.

The conversion option is similar to a call option on the underlying stock, where the call's strike price equals the bond floor value.

Equity perspective on convertibles

- To take advantage of the upside potential growth of the underlying stock (participation into equity).
- Swapping the variable stock dividends in return for fixed coupon payments until the earlier of the maturity date and conversion date.

Fixed income perspective on convertibles

- Provides the "bond floor" value.
- Conversion option that allows the investor to exchange the straight bond for a fixed number of shares.

Call terms

Issuer has the right to call back the bond at a pre-specified call price prior to final maturity, usually with a notice period requirement. Upon call, the holder can either convert the bond or redeem at the call price.

Issuer's perspective on the call right

- To have the flexibility to call if they think they can refinance the debt more cheaply at a lower interest rate.

- To force bondholders to convert debt into equity, which can reduce the company's debt level and result in a beneficial effect on the balance sheet. The issuer has the flexibility to shift debt into equity to reduce the leverage of the firm. It is used as a tool by the issuer for possible future equity financing – managing the debt / equity balance.

Call protection

Hard (or absolute):

To protect the bond from being called for a certain period of time.

Soft (or provisional):

The issuer is allowed to call only when certain conditions are satisfied. Say the closing price of the stock has been in excess of 150% of the conversion price on any 20 trading days within 30 consecutive days.

Role of the call protection

To preserve the value of the equity option for the bondholders. The premium of the conversion right has been paid upfront at the time of purchase. While waiting for the stock price to increase, convertibles typically provide more income than the stock. Without the call protection, this income stream could be called away too soon. Hard call protection of a longer time period is more desirable for the investors.

Put feature

Allows the holder to sell back the bond to the issuer in return for a fixed sum. Usually, the put right lasts for a much shorter time period than the life of the bond.

- The holder is compensated for the lesser amount of coupons received in case the equity portion of the convertible has a low value.
- It helps immunize the holder against the risk of rising interest rates by effectively reducing the year to maturity. With a smaller value of duration*, the convertible price becomes less sensitive to interest rates.

⋆ Duration D is the weighted average of the times of cash flow stream, weighted according to the present value of the cash flow amount. The percentage change in bond price P is proportional to negative yield change, where the proportional constant is the duration: $\frac{\Delta P}{P} \approx -D\times$ yield change.

Convertible bond issued by the Bank of East Asia

US$250,000,000

2.00 percent Convertible Bonds due 2003

Issue date	July 19, 1996
Issue price	100 percent of the principal amount of the Bonds, plus accrued interest, if any, from July 19, 1996 (in denominations of US$1,000 each)
Conversion period	From and including September 19, 1996 up to and including July 7, 2003

Conversion feature

Conversion price	HK$31.40 per Share and with a fixed rate of exchange on conversion of HK$7.7405 = US$1.00.
Dilution protection clause	The Conversion Price will be subject to adjustment for, among other things, subdivision or consolidation of the Shares, bonus issues, right issues and other dilutive events.

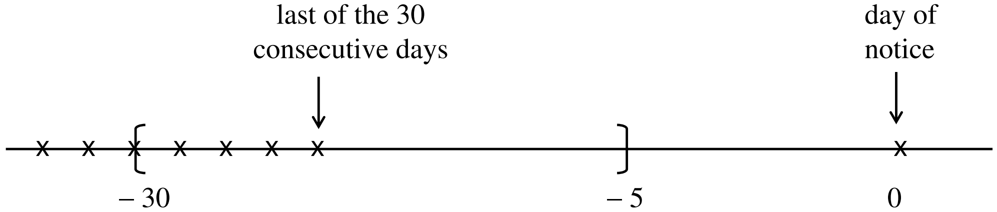
Put feature

Redemption at the option of the bondholders	On July 19, 2001, the Bonds may be redeemed at the option of the Bondholders in US dollars at the redemption price equal to 127.25 percent of the principal amount of the Bonds, together with accrued interest.

The investors are protected to have 27.25% returns on the bond investment upon early redemption by the holder.

Call feature

Redemption at the option of the issuer	On or after July 19, 1998, the Issuer may redeem the Bonds at any time in whole or in part at the principal amount of each Bond, together with accrued interest, if for each of 30 consecutive Trading Days, the last of which Trading Days is not less than five nor more than 30 days prior to the day upon which the notice of redemption is first published, the closing price of the Shares as quoted on the Hong Kong Stock Exchange shall have at least 130 percent of the Conversion Price in effect on such Trading Day.



Soft call protection – Parisian feature

The closing price has to be above 130 percent of the conversion price on consecutive 30 trading days.

- On the date of issuance of the notice of redemption (taken as day 0), the Issuer looks back 5 to 30 days (corresponds to $[-30, -5]$ time interval) to check whether the history of the stock price path satisfies the Parisian constraint. That is, the last of the 30 trading days (with closing price above 130% of the conversion price) falls in $[-30, -5]$ time interval.

- From Issuer's perspective, when the Parisian constraint has been satisfied, the Issuer has 5 to 30 days to decide on redemption or not.

Modeling considerations in convertible bond pricing models

Reference "The valuation of convertible bonds with credit risk," by E. Ayache, P.A. Forsyth and K.R. Vetzal (2003) Working paper of University of Waterloo.

Choices of the underlying state variables

- Firm value versus stock price

  Earliest works use the value of the issuing firm as the underlying state variable. From corporate finance perspective, the firm value model can incorporate the balance sheet information on the firm's liabilities.

  The firm's debt and equity are claims contingent on the firm value, and options on its debt and equity become compound options on this variable.

Advantage:
Dilution effect on equity upon conversion of the bond into shares can be modeled directly.

Disadvantage:
Since the value of the firm is not a traded asset, parameter estimation is difficult. Any other liabilities of the firm that are more senior than the convertible must be simultaneously valued.

- Most pricing models use the issuing firm's stock price.
  - Since stock is a traded asset, so parameter estimation is easy. Also, the use of risk neutral valuation principle is more convincing. Hedging ratios can be computed easily.
  - There is no need to estimate the values of other more senior claims.

How about stochastic riskfree interest rate?

- Addition of the stochastic interest rate as an additional state variable increases the dimensionality of the pricing model.

- Practitioners often regard a convertible bond primarily as an equity instrument, where the main risk factor is the stock price. The random nature of the riskfree interest rate is of second order importance as the role of interest rate serves as the discount rate but not in the payoff structure.

Brennan and Schwartz (1980) conclude that "for a reasonable range of interest rates the errors from the non-stochastic interest rate model are likely to be slight."

Why does interest rate fluctuation have lower impact on convertible bond value? When interest rate increases, the conversion option increases in value due to the drop in the bond floor value. Here, the bond floor can be visualized as the strike price of the conversion option (the underlying call option increases in value when the strike price decreases). The drop in value of the bond component is compensated by the increase in value of the conversion option.

- Quite often, the interest rate r and stock price S are negatively correlated. When the correlation coefficient ρ_{Sr} is negative, an increase in r leads to a drop in the bond component and a drop in S (lowering of the equity component). Hence, high negative value of correlation may lead to higher sensitivity of the convertible bond value to interest rate fluctuation.

Other considerations in modeling

- Modeling of the default risk of the issuer
  - arrival of the default event

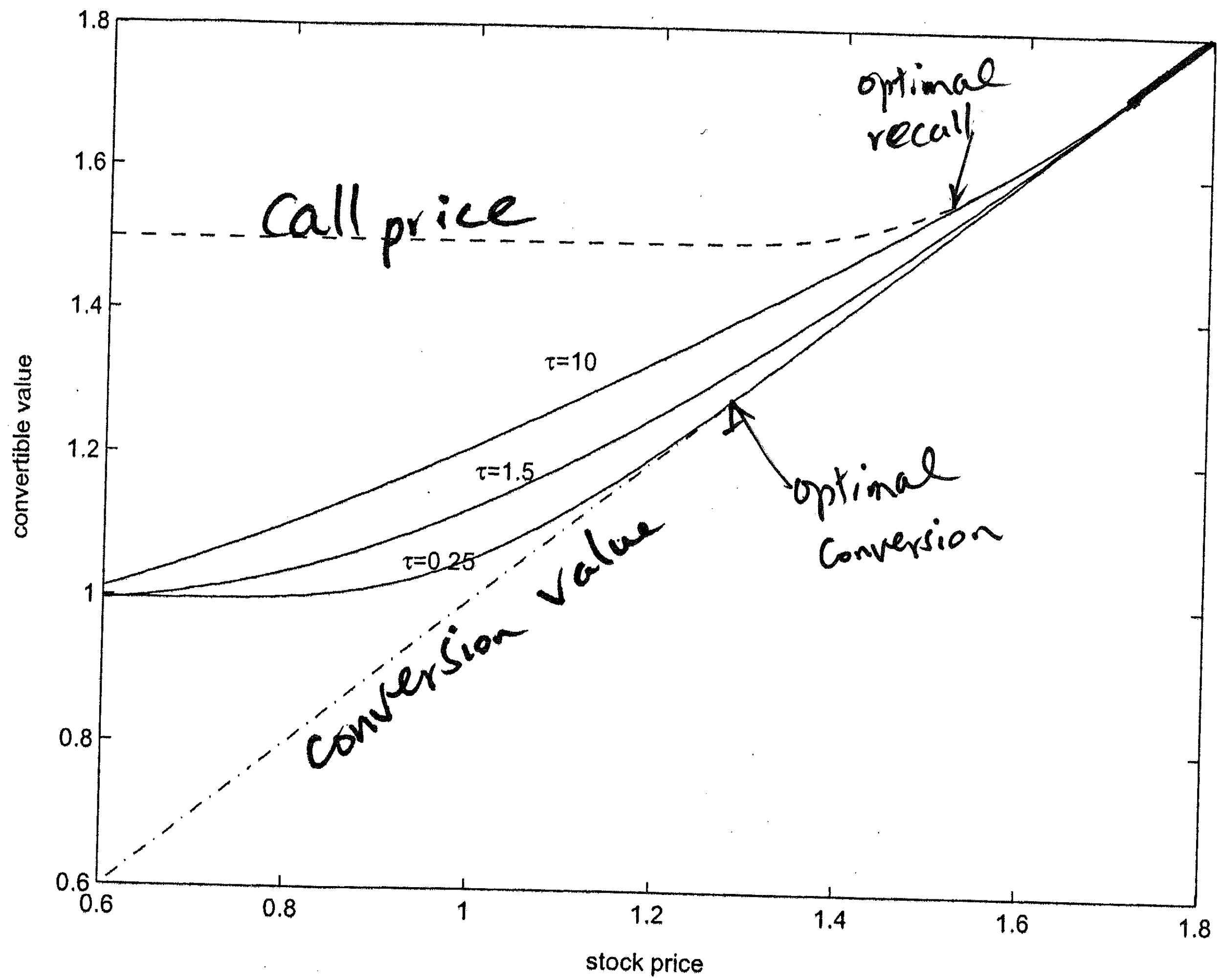
    Structural approach versus reduced form approach (Poisson arrival with specified hazard rate)
  - loss upon default

    Under the stock price model, what would be the drop of the stock price upon default?

- Issuer's call provision
  - soft call requirement, trigger prices
  - call notice period

- Dilution upon holders' conversion – more shares are issued

- Holder's put right



When $\tau = 0.25$ and $\tau = 10$, the bond price curves intersect the conversion value line (shown as dotted-dashed line) and the cap value curve $\widetilde{c}_n(S, \tau_n)$ (shown as dashed line), respectively. Here, τ_n is the length of the notice period. When $\tau = 1.5$, the price curve ends at the intersection point of the conversion value line and the cap value curve.

Properties of the solution curves for the convertible bond price

1. The solution curves are bounded between the upper obstacle function (call price which has stock price dependence due to the notice period requirement) and the conversion value (given by conversion number $\times$ stock price).

2. At $\tau = 10$, optimal conversion may be delayed since it is far from expiry. The convertible bond value may shoot above the call price if not called. In this case, the convertible bond may be terminated prematurely by issuer's call at sufficiently high value of stock price (provided that the hard call and soft call provisions are satisfied).

3. At $\tau = 0.25$, bondholder's optimal conversion may occur when the stock price reaches some threshold value. Beyond the threshold stock price, the convertible bond value would fall below the conversion value if unconverted.

Credit spread: dependence on various aspects of default risks

The credit spread of the bond component should reflect the combination of default probability and loss upon default. The proper modeling of default must include

(i) probability of arrival of default

(ii) recovery rate upon default.

Later, we show that the risky discount rate is $r + (1 - R)h$, where h is the hazard rate and R is the recovery rate. The credit spread equals zero when either $R = 1$ or $h = 0$.

Game between holder and issuer: conversion and calling

conv = value of stocks received if conversion takes place
call = call price
roll = value given by the rollback (neither converted nor recalled)

Six possible permutations on their relative values

(i) $conv < call$

$$\begin{array}{ccccc} conv & < & call & < & roll \\ conv & < & roll & < & call \\ roll & < & conv & < & call \end{array}$$

(ii) $call < conv$ (this occurs when the stock price is very high, so convertible value = *conv* for sure)

$$\begin{array}{ccccc} call & < & conv & < & roll \\ call & < & roll & < & conv \\ roll & < & call & < & conv \end{array}$$

What would happen when $call < conv$?

- This occurs when the stock price level is sufficiently high, that is, the conversion option is sufficiently deep-in-the-money. Since the convertible bond value is always equal or above $conv$, so the issuer initiates calling immediately.

- Upon calling, the holder chooses to convert into stocks since $conv > call$.

- This represents a straightforward case since convertible value = $conv$ for sure, and there is no need to perform any numerical calculations to find the convertible bond value.

Optimal premature conversion into shares would strike the tradeoff between the stock dividends and the coupons.

Dynamic programming procedures

First choice

$$\max(\min(roll, call), conv).$$

At each node, the optimal strategy of the holder is exemplified by taking the maximum of $\min(roll, call)$ and $conv$.

- The maximum reflects the conversion right, which persists with or without recall by the issuer.
- The bond value before potential conversion is seen to be $\min(roll, call)$ since the issuer would initiate calling when the roll value shoots beyond the call price.

Second choice

$$\text{min}(\text{max}(roll, conv), \text{max}(call, conv))$$

- When the issuer does not call, max$(roll, conv)$ represents the optimal strategy of the holder.

- Upon recall, the holder chooses to accept the call price or convert into shares. This can be represented by max$(call, conv)$.

The issuer chooses to recall or to abstain from recalling in order to minimize the option value.

These two procedures can be shown to be mathematically equivalent if we apply the distributive rule of sequencing the order on the "max" and "min" operations.

Third choice

One should rule out the trivial scenario where $call < conv$. Given that $conv < call$, the second dynamic programmic procedure can be reduced to

$$\min(call, \max(conv, roll)).$$

Under the scenario $conv < call$, the holder has the optionality to convert into shares but the convertible bond value is always capped by $call$.

The 3 choices of dynamic programming procedures give the same set of outcomes

	outcome
$conv < call < roll$	$call$
$conv < roll < call$	$roll$
$roll < conv < call$	$conv$

Summary

Recall that when call price $<$ conversion value, then the convertible bond value is simply given by the conversion value (straightforward case). We are interested to find the price function within the range of stock prices that observe

$$\text{call price} > \text{conversion value.}$$

That is, we are interested to compute the price function whose value is capped by "call price". Hence, the dynamic programming procedure

$$\min(call, \max(conv, roll))$$

makes good sense.

- The convertible bond value lies between the "call price" (upper obstacle function) and conversion value (lower obstacle function).

- The call price has "stock price" dependence if we include the consideration of the notice period requirement.

Delayed call phenomenon

Empirical studies show that most convertible bonds are recalled by issuers only when the stock price is sufficiently well above the "theoretical" critical recall price. In this case, the holders are almost sure to make the "so called" forced conversion into shares. This is consistent with one of the corporate finance considerations – delayed equity financing.

- How to incorporate this behavior into the pricing model?

- Optimality of recall may not simply be defined by the rule: avoid convertible bond value to shoot above the call price if not recalled. Corporate treasurer may have to judge the impact on the stock price upon the announcement of recall.

Pricing model of a convertible bond with credit risk

- We adopt the one-factor contingent claims model with stock price as the underlying state variable.

- We assume constant interest rate and model the arrival of default by a Poisson arrival process with constant hazard rate.

- The stock price S_t under the risk neutral valuation framework is assumed to follow the Geometric Brownian motion

$$\frac{dS_t}{S_t} = (r - q)dt + \sigma_S \; dZ_t,$$

  where r is the riskless interest rate, q and σ_S are the constant dividend yield and volatility of the stock price, respectively, and Z_t is the standard Brownian motion.

Conditional on no prior default up to time t, the probability of default within the time period $(t, t+dt)$ is $h\ dt$, where h is the constant hazard rate of arrival of default.

Assume that upon default the bondholder receives the fraction R (recovery rate) of the bond value and the stock price drops to zero instantaneously, the corresponding governing equation for the convertible bond price function $V(S,t)$ is given by

$$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} + (r - q + h) S \frac{\partial V}{\partial S} - [r + (1 - R)h]\, V + c(t) = 0,$$
$$0 < S < S^*(\tau), 0 < t < T.$$

Intuitive observations on the governing equation

1. With probability $h\,dt$ over the instantaneous time interval $(t, t+dt)$, the stock price drops to zero due to default. Under expectation calculations, we may consider h as the expected negative yield. Therefore, the drift rate of the stock price process is modified to $r - q + h$.

2. The convertible bond value is discounted at the risky rate $r + (1 - R)h$. Here, $(1 - R)h$ is commonly recognized as the credit spread. Investors demand the rate of return to be higher than the riskfree interest rate by a certain spread due to potential loss upon default. Both r and $(1 - R)h$ represent the rate of decrease in the bond value over time.

3. The bond price function satisfies the governing equation only in the continuation region $= \{(S,t) : 0 < S < S^*(t), 0 < t < T\}$, where the bond remains alive. Here, $S^*(t)$ denotes the critical stock price at which the bond ceases to exist either due to either early conversion or calling, and T is the bond maturity date.

4. The source term $c(t)$ as the coupon rate arises from the coupon payment stream. The external cash payout rate may be represented by

$$c(t) = \sum_{i=1}^{N} c_i \delta(t - t_i),$$

where c_i is the coupon payment paid on the discrete coupon payment dates $t_i, i = 1, 2, \cdots, N$. The Dirac function $c_i\delta(t - t_i)$ indicates the discrete nature of a coupon payment. The coupon rate becomes infinite at t_i when a discrete coupon amount is collected across t_i.

Derivation of the governing equation

As usual, consider the portfolio $\Pi = -V + \Delta S$, where Δ units of the underlying stock are held to hedge against the short position of one unit of the convertible bond. Using Ito's lemma and considering the portfolio value under no-default and default, the expectation of $d\Pi$ subject to default risk is given by

$$\begin{aligned} E[d\Pi] \;&=\; (1 - h\,dt)\left[-\left(\frac{\partial V}{\partial t} + \frac{\sigma^2}{2}S^2\frac{\partial^2 V}{\partial S^2}\right)dt - \frac{\partial V}{\partial S}\,dS \right. \\ &\quad \left. +\; \Delta\,dS + \Delta q S\,dt - c\,dt\right] - h\,dt[-(1-R)V + \Delta S]. \end{aligned}$$

Here, $h\ dt$ is the probability of default in the next differential time instant dt. The loss in portfolio value upon default is $\Delta S + (1-R)(-V)$ (note that the portfolio shorts one unit of V). The expected change in value due to default is $-h\ dt$ times the above loss amount. Now, we take $\Delta = \frac{\partial V}{\partial S}$ so that the stochastic terms involving dS vanish. Assuming that the default risk is firm-specific, there will be no expected excess return for bearing this risk. Hence, the *expected* rate of return of the portfolio is the same as the riskfree interest rate. That is,

$$E[d\Pi] = r\Pi\,dt.$$

Collecting the terms, we obtain

$$
\begin{aligned}
&-\left(\frac{\partial V}{\partial t}+\frac{\sigma^2}{2}S^2\frac{\partial^2 V}{\partial S^2}\right)+qS\frac{\partial V}{\partial S}-hS\frac{\partial V}{\partial S}+h(1-R)V-c(t)\\
=\ & r\left(-V+S\frac{\partial V}{\partial S}\right)
\end{aligned}
$$

so that

$$
\frac{\partial V}{\partial t}+\frac{\sigma^2}{2}S^2\frac{\partial^2 V}{\partial S^2}+(r-q+h)S\frac{\partial V}{\partial S}-[r+(1-R)h]V+c(t)=0.
$$

For the source term arising from the discrete coupons, we equate the total coupon collected over the time interval $(0,t)$ to obtain

$$
\int_0^t c(u)\,du=\sum_{i=1}^{N_t}c_iH(t-t_i),\quad \text{where } H(t-t_i)=\begin{cases}1 & t\ge t_i\\ 0 & t<t_i\end{cases}.
$$

Here, N_t is the number of discrete coupons received within $(0,t)$. By differentiating with respect to t on both sides, we have

$$
c(t)=\sum_{i=1}^{N_t}c_i\delta(t-t_i).
$$

Auxiliary conditions

(i) Terminal payoff on the maturity date T

The terminal value of V is given by

$$V(S,T) = (c_N + P)\mathbf{1}_{\{P+c_N \geq nS\}} + nS\mathbf{1}_{\{P+c_N < nS\}},$$

where $\mathbf{1}_A$ is the indicator function for the event A. Here, P denotes the par value of the bond, c_N is the last coupon payment and n is the number of units of stock to be exchanged for the bond upon conversion. The last coupon is not paid if the bondholder chooses to convert at maturity.

(ii) Conversion policy

Since the bondholders have the right to convert the bond into n units of stock at any time, the convertible bond always stays equal or above the conversion value. Upon voluntary conversion, the value of the bond equals the conversion value identically. We then have

$$V(S,t) > nS \quad \text{when the convertible bond remains alive,}$$
$$V(S,\bar{t}) = nS \quad \text{when the convertible bond is converted,}$$

where $\bar{t}$ is the optimal time of conversion chosen by the bondholders.

- The screw clause stipulates that the accrued interest will not be paid upon voluntary conversion. This clause may inhibit bondholders to convert optimally when a coupon date is approaching. More precisely, optimal conversion occurs only at time instants right after coupon payment dates.

(iii) Calling policy

- The convertible bond indenture usually contains the hard call provision where the bond cannot be called for redemption or conversion by the bond issuer in the early life of the bond. This serves as a protection for the bondholders so that the privilege of awaiting growth of the equity component will not be called away too soon.
- Let $[T_c, T]$ denote the callable period, that is, the bond cannot be called during the earlier part of the bond life $[0, T_c)$.
- Upon calling, the bondholders can decide whether to redeem the bond for cash or convert into shares at the end of the notice period of t_n days.

Notice period requirement

- Let $\widehat{t}$ denote the date of call so that $\widehat{t}+t_n$ is the conversion decision date for the bondholders.

- The bondholders essentially replace the original bond at time $\widehat{t}$ by a new derivative that expires at the future time $\widehat{t}+t_n$ and with terminal payoff $\max(nS, K+\widehat{c}) = K+\widehat{c}+\max(nS-(K+\widehat{c}), 0)$, where $\widehat{c}$ is the accrued interest from the last coupon date to the time instant $\widehat{t}+t_n$, and K is the pre-specified call price of the convertible bond.

We write $V_{new}(S, t; K, t_n)$ as the value of this new derivative derived from the notice period requirement. When there is no soft call requirement (a constraint that is related to stock price movement over a short period prior to calling), the convertible bond value should be capped by V_{new}. The convertible bond should be called once its value reaches $V_{new}(S, t; K, t_n)$.

$$
\begin{aligned}
V(S,t) &\leq V_{new}(S,t;K,t_n) \quad \text{within the callable period,} \\
V(S,\widehat{t}) &= V_{new}(S,\widehat{t};K,t_n) \quad \text{at the calling moment.}
\end{aligned}
$$

When there is a soft call requirement, it is possible that $V(S, t)$ stays above $V_{new}(S, t; K, t_n)$.

(iv) Coupon payments

By no arbitrage argument, there is a drop in bond value of amount that equals the coupon payment c_i across a coupon payment date $t_i, i = 1, 2, \cdots, N$. We have

$$V(S, t_i^+) = V(S, t_i^-) - c_i, \quad i = 1, 2, \cdots, N.$$

Remark

The interaction of the optimal conversion and calling policies determines the early termination of the convertible bond. This leads to a game option model in which one has to solve for a set of interactive optimal stopping decisions made by the two counterparties. The synergy of these two features can be treated effectively via a dynamic programming procedure in the numerical schemes.

Finite difference algorithms

We adopt the log-transformed variable $x = \ln S$, and define time to expiry $\tau = T - t$. Let V_j^m denote the numerical approximation of $V(x, \tau)$ at the grid point $x = j\Delta x$ and $\tau = m\Delta t$, where Δx and Δt are the respective stepwidth and time step.

The explicit finite difference scheme takes the following basic form

$$V_j^{m+1} = p_u V_{j+1}^m + p_m V_j^m + p_d V_{j-1}^m - [r + (1-R)h]V_j^m + c_i \mathbf{1}_{\{E_i\}}.$$

The probabilities of upward jump, zero jump and downward jump of the logarithm of the stock price, $x = \ln S$, are given by

$$\begin{aligned} p_u &= \frac{1}{2\lambda^2} + \frac{\left(r - q + h - \frac{\sigma_S^2}{2}\right)\sqrt{\Delta t}}{2\lambda\sigma_S}, \\ p_m &= 1 - \frac{1}{\lambda^2}, \quad p_d = \frac{1}{2\lambda^2} - \frac{\left(r - q + h - \frac{\sigma_S^2}{2}\right)\sqrt{\Delta t}}{2\lambda\sigma_S}, \end{aligned}$$

respectively, and $\Delta x = \lambda\sigma_S\sqrt{\Delta t}$ for some parameter λ.

Here, E_i denotes the event that the coupon payment c_i is paid at t_i. When the payment date t_i is bracketed between two successive time levels $m\Delta t$ and $(m+1)\Delta t$, the bond values V_j^{m+1} are increased by an extra amount c_i due to the discrete coupon payment.

Upon optimal conversion, the holder can convert into M shares of the stock. The "initial" values V_j^0 at time level $n = 0$, which correspond to the terminal payoff values of the bond, are given by

$$V_j^0 = \begin{cases} c_N + P & \text{if} \qquad x_j \leq \ln \dfrac{c_N + P}{M} \\ Me^{x_j} & \text{otherwise} \end{cases} .$$

Interaction of the callable and convertible features

Apply the full dynamic programming procedure

$$\min(\max(V_{cont}, V_{conv}), \max(V_{call}, V_{conv}))$$

at those nodes where holder's conversion and issuer's calling are allowed.

When the calling right is non-operative (say, during the period under the hard call constraint) where the conversion right exists only, the full dynamic programming procedure reduces to the partial dynamic programming procedure

$$V_j^n = \max(V_{cont}, V_{conv}).$$

Soft call requirement

To incorporate the soft call requirement, we model the associated Parisian feature using the forward shooting grid approach, where an extra dimension is added to capture the excursion of the stock price beyond some predetermined trigger level B. With the inclusion of the path dependence of the stock price associated with the soft call requirement, the finite difference scheme is modified as follows:

$$\begin{aligned} V_{j,k}^{m+1} &= p_u V_{j+1,g(k,j+1)}^m + p_m V_{j,g(k,j)}^m + p_d V_{j-1,g(k,j-1)}^m \\ &\quad -[r + (1-R)h] V_{j,g(k,j)}^m + c_i \mathbf{1}_{\{E_i\}}. \end{aligned}$$

Cumulative Parisian feature to activate calling

The grid evolution function assumes the form

$$g_{cum}(k, j) = k + \mathbf{1}_{\{x_j > \ln B\}}$$

for cumulative counting of number of days that the stock price has been staying above the level B.

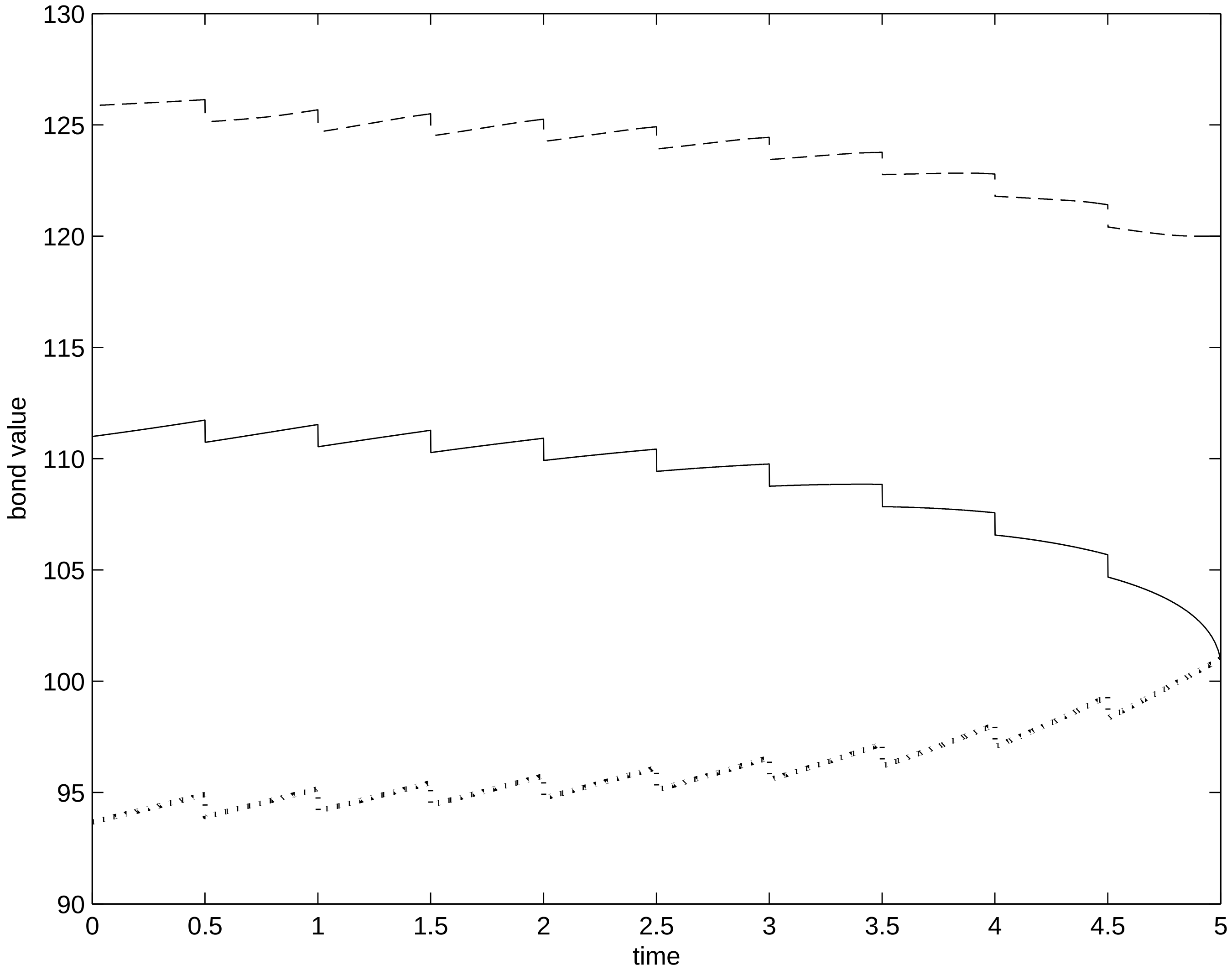
Suppose the condition of M_0 cumulative days of breaching is required in order to activate the calling right, the full dynamic programming procedure with issuer's recall right is applied only when the condition $g_{cum} \geq M_0$ has been satisfied.

Anatomy of the embedded features

Using the one-factor defaultable convertible bond pricing model, we explore the dependence of the convertible bond value on the coupon payment streams, conversion ratio and soft call constraint.

We examine the interaction of the callable and conversion features and show how the notice period requirement affects the critical stock price at which the convertible bond is terminated prematurely either by issuer's optimal calling or holder's voluntary conversion.

par value	100
annualized volatility	20%
dividend yield per year	1%
maturity date	5 years
coupon rate	2% per annum, paid semi-annually
conversion ratio	1
call period	starting 1.0 years from now till maturity
converison period	throughout the life
call price	140
riskless interest rate	flat at 5% per annum
hazard rate	0.02
recovery rate	0.8



Plot of convertible bond value against time at different fixed values of stock price (dotted curve corresponds to $S = 70$, solid curve corresponds to $S = 100$ and dashed curve corresponds to $S = 120$).

At low stock price level ($S = 70$)

- The convertible bond behaves like a conventional coupon bond. Its value increases with time since the riskless interest rate is higher than the coupon rate (see the lower dotted curve). There is a drop in value across a coupon payment date.

- At maturity, the bond value matches the total value of par plus the last coupon.

At intermediate stock price level ($S = 100$)

- At the stock price level $S = 100$ (same as conversion price), the convertible bond drops in value within the last coupon period (see the middle solid curve).

- The more noticeable drop in value is attributed primarily to the higher rate of decrease in the value of the conversion option at times close to maturity.

At high stock price level ($S = 120$)

- At a higher stock price level $S = 120$ (20% above the conversion price), the bond value shows a trend of slight decrease with increasing time (see the upper dashed curve). The conversion option decreases in value with increasing time at a rate faster than the rate of increase in value of the bond component.

- The bond value stays almost at constant value within the last coupon period. This is because the value of a deep-in-the-money convertible bond is dominated by its equity component since the bond is almost sure to be converted into shares at maturity, so the time dependent effect of accrued interest of the bond component is negligible.

Price sensitivity with respect to conversion number and stock price level

stock price	conversion number						
	0.7	0.8	0.9	1.0	1.1	1.2	1.3
50	85.30	85.67	86.29	87.19	88.41	89.97	91.87
100	94.10	99.47	105.90	113.18	121.12	129.56	138.37
120	101.93	110.18	119.49	129.56	140.16	151.14	162.37
130	106.59	116.29	126.98	138.37	150.21	162.37	174.73
140	111.67	122.77	134.81	147.45	160.48	173.77	187.23
150	117.08	129.56	142.88	156.73	170.91	185.30	199.81

The entries in the table are convertible bond values corresponding to different conversion numbers and stock price levels.

- At a low stock price level, the bond value is not quite sensitive to an increase in conversion number.
- The bond value is less sensitive to an increase in stock price when the conversion number is low.
- Both phenomena are due to the low value of the equity component of the convertible bond. The data also reveal that the delta of the bond value increases with higher conversion number, due to an increased weight in the equity component.

Parisian soft call provision

- The current stock price is taken to be 130 and the annualized dividend yield to be 1%. We specify that the issuer can initiate the call only if the stock price stays above the trigger price consecutively or cumulatively for 30 days.

- For the purpose of comparison, the convertible bond value is found to be equal to 144.17 if there is no call feature (infinite trigger price) and equal to 135.71 if there is no soft protection requirement (zero trigger price). These two values serve as the respective upper and lower bound for the value of the bond subject to the soft call requirement.

Price sensitivity with respect to the Parisian soft call provision

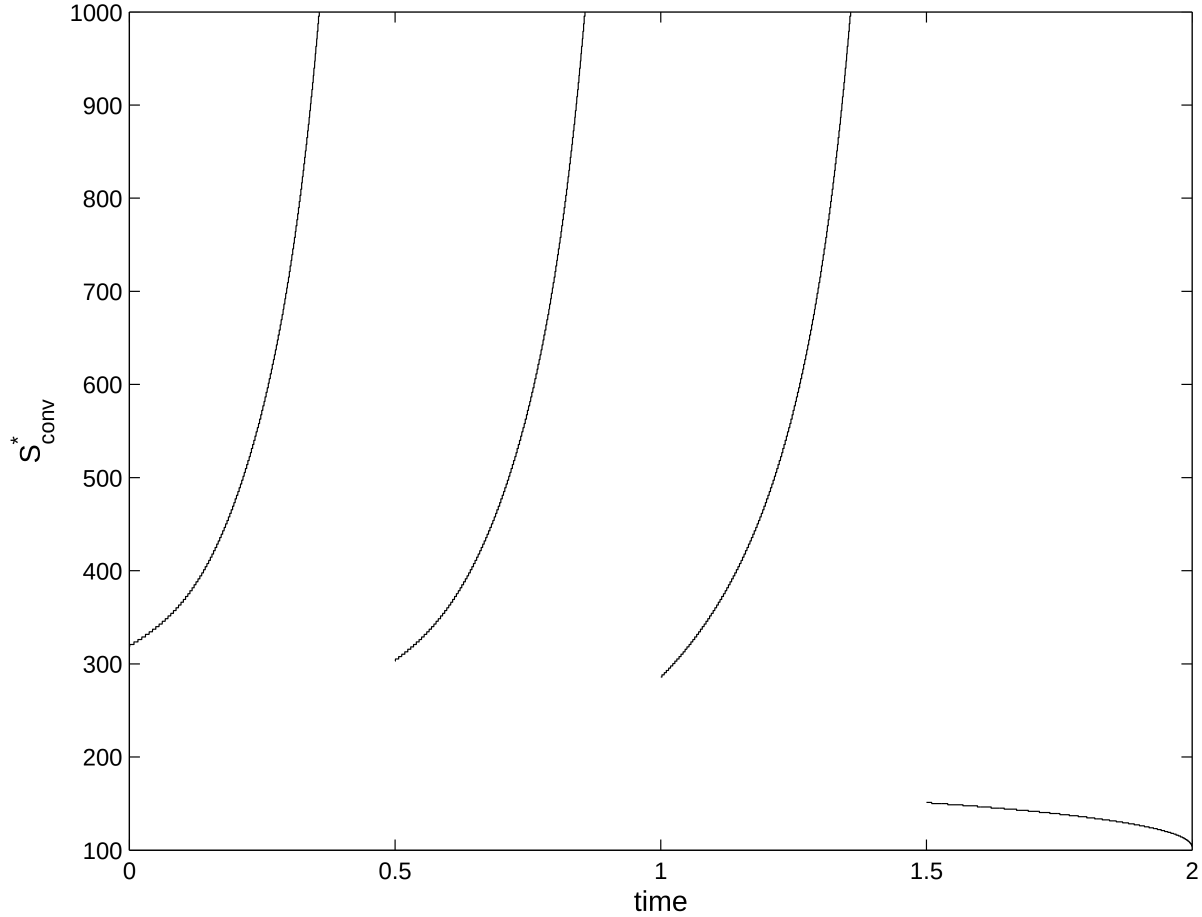
trigger price	consecutive counting	cumulative counting
130	136.01	135.83
140	136.64	136.08
150	137.89	137.13
160	138.93	138.32
180	140.65	140.30
200	141.81	141.60

The entries in the right two columns are values of convertible bond subject to varying levels of trigger price and under the rules of consecutive counting and cumulative counting of the number of days of breaching the trigger price.

1. The bond value increases with an increasing trigger price. This is obvious since it becomes harder for the issuer to initiate the call when calling is constrained by a higher trigger price.

2. The bond value becomes higher when the soft call requirement is more stringent. This is because bondholders have better protection against calling by issuer. Also, this explains why the convertible bond has a higher value under the rule of consecutive counting when compared with that under cumulative counting.

Two-year convertible, coupons paid semi-annually



Assuming that the issuer cannot call, the curves show the plot of the critical conversion price S^*_{conv} against time. Within the last coupon payment period, S^*_{conv} decreases with time. At times right before a coupon date, S^*_{conv} tends to infinite value.

Delayed call phenomenon

- In the earlier theoretical works on analyzing the optimal calling policies, Ingersoll (1977) and Brennan and Schwartz (1977) claimed that the bond issuer should call the bond whenever the convertible bond value reaches the call price.

- The notice period requirement may have profound impact on the critical call price S^*_{call} since the bondholder receives upon calling a more valuable short-lived option (whose maturity date coincides with the ending date of the notice period), rather than the cash amount that equals the sum of the call price plus accrued interest.

$X(t)$ is the critical stock price without the notice period requirement

average value of $S^*_{call}(t)/X(t)$		notice periods (days)		
		15	30	45
volatility	20%	1.049	1.073	1.093
	30%	1.061	1.093	1.122
	40%	1.057	1.101	1.136
interest rate	2%	1.043	1.069	1.088
	5%	1.061	1.093	1.122
	8%	1.077	1.112	1.145
coupon rate	1%	1.106	1.161	1.208
	3%	1.073	1.110	1.145
	5%	1.045	1.077	1.102
price rate	120	1.061	1.093	1.122
	150	1.090	1.135	1.174
	180	1.108	1.158	1.199
hazard rate	0.01	1.065	1.103	1.135
	0.03	1.051	1.079	1.108
	0.05	1.046	1.068	1.086
recovery rate	0.2	1.078	1.118	1.15
	0.5	1.068	1.107	1.135
	0.8	1.061	1.093	1.122

We examine the impact of the notice period requirement on the theoretical critical call price, S^*_{call}. The time-averaged values of the ratio S^*_{call}/X are obtained under varying length of the notice period and different set of parameter values.

- The sample calculations reveal that the so called “delayed call phenomena” may be largely attributed to the under estimation of the critical call price at which the issuer should call the bond optimally.

- A large portion of the “amount of call delay” may be eliminated when more careful contingent claims pricing calculations are performed.

- There may be other rationales from corporate finance perspectives (say, taxes) which explain why issuers choose to delay their calls.

- In future empirical studies on assessing the amount of call delay due to corporate finance considerations, the more accurate theoretical critical stock price should be computed.