MAFS5250 – Computational Methods for Pricing Structured Products

Topic 6 – Advanced numerical schemes for pricing path dependent options

6.1 Discretely sampled fixed strike Asian option

- Change of numeraire approach
- Choice of Markovian state variable
- Construction of the Crank-Nicolson scheme

6.2 Discretely sampled lookback options

- Fixed strike lookback call option
- Floating strike lookback option

6.1 Discretely sampled fixed strike Asian options

Change of numeraire approach

Reference:
Andreasen, J., "The pricing of discretely sampled Asian and lookback options: a change of numeraire approach," Journal of Computational Finance (Fall 1998) p.5-30.

- We make use of change of numeraire techniques to obtain the option price as a function of time and a one-dimensional Markovian state variable. The state variable exhibits jumps at the observation points with probability one.

- Between two observation points, the state variable evolves continuously, so the price function is governed by a PDE (consequence of the Feynman-Kac representation Theorem).

- A coupled sequence of PDEs are resulted, the solution of the first PDE between two successive fixing dates generates the terminal condition of the second one.

- The Asian option model is converted into a barrier option pricing problem. Why do we have the "barrier type" behavior? This is because when the state variable goes beyond a certain level, typically in the status of being currently in-the-money, the optionality in the payoff function disappears. It is then possible to derive the risk adjusted expectation of the terminal payoff in closed form.

- The Crank-Nicolson scheme is adopted for the numerical solution of the resulting PDE.

- Highly accurate: within 1% accuracy even with 50 time steps between successive observation time points.

Model formulation: change of numeraire and Markov representation

Under a risk neutral measure Q, the stock price dynamics is given by

$$\frac{dS_t}{S_t} = (r-q)dt + \sigma\, dW_t.$$

Based on the risk neutral valuation principle where the money market account is used as the numeraire, the time-t price of the contingent claim H at time T is given by

$$F_t = E_t^Q[e^{-r(T-t)}H].$$

We apply the change of measure with the stock price as the numeraire, which is defined by

$$\left.\frac{dQ'}{dQ}\right|_{\mathcal{F}_0} = \frac{S_t e^{qt}}{S_0} \Big/ e^{rt} = \frac{S_t}{S_0 e^{(r-q)t}}, \quad t > 0.$$

- One unit of asset at $t=0$ will grow to e^{qt} units at time t, if the dividends are all used to buy additional units of asset.

The solution to S_t is known to be

$$S_t = S_0 e^{(r-q-\frac{\sigma^2}{2})t+\sigma W_t},$$

so we obtain the share measure Q' defined by the Radon-Nikodym derivative

$$\left.\frac{dQ'}{dQ}\right|_{\mathcal{F}_0} = e^{-\frac{\sigma^2}{2}t+\sigma W_t}.$$

Recall the Girsanov theorem: Suppose W_t^P is P-Brownian, and

$$\frac{dP'}{dP} = e^{\left(-\mu W_t^P - \frac{\mu^2}{2}t\right)},$$

then $W_t^{P'} = W_t^P + \mu t$ is P'-Brownian. By choosing μ to be $-\sigma$ in the above formula, we observe that

$$W_t' = W_t - \sigma t$$

is Q'-Brownian when W_t is Q-Brownian.

Accordingly, the dynamics of S_t under Q' is governed by

$$\frac{dS_t}{S_t} = \left(r - q + \sigma^2\right) t + \sigma\, dW_t', \quad t > 0.$$

By applying the change of measure from Q to Q' conditional on $\mathcal{F}_t$ and observing $\dfrac{F_t}{e^{qt} S_t}$ is Q'-martingale, we obtain

$$F_t = E_t^Q\left[e^{-r(T-t)} H\right] = E_t^{Q'}\left[\frac{e^{-r(T-t)} H}{\frac{S_T e^{qT}}{S_t e^{qt}} \Big/ \frac{e^{rT}}{e^{rt}}}\right] = S_t E_t^{Q'}\left[e^{-q(T-t)} \frac{H}{S_T}\right].$$

Remark Here, we use the numeraire change formula:

$$\left.\frac{dQ^M}{dQ^N}\right|_{\mathcal{F}_t} = \frac{M(T)}{M(t)} \Big/ \frac{N(T)}{N(t)},$$

where $M(t)$ and $N(t)$ are numeraires. When the money market account is used as the numeraire, the discount factor is $e^{-r(T-t)}$. With the stock price as the numeraire, the "pseudo" discount factor is $e^{-q(T-t)}$.

Dimension reduction of a path dependent option model

When the contingent payoff H depends on the path history of S_t up to T, then it is necessary to keep track of the path realization of S_t when we compute F_t using the risk neutral valuation principle.

It may be possible to reduce the dimension of the path dependent option model if we can find a Markov process x_t such that under Q'

$$dx_t = \mu_x(x_t, t)\,dt + \sigma_x(x_t, t)\,dW'_t,$$

and together with the satisfaction of the property:

$$\frac{H}{S_T} = \xi(x_T),$$

for some function ξ.

- If the above properties hold true, then the deflated option price

$$f_t = \frac{F_t}{e^{qt} S_t}$$

will be a function of (x_t, t) only. By the Feynman-Kac representation theorem, f satisfies the following pde

$$\frac{\partial f}{\partial t} + \mu_x \frac{\partial f}{\partial x} + \frac{\sigma_x^2}{2} \frac{\partial^2 f}{\partial x^2} - qf = 0,$$

subject to the terminal condition: $f(x, T) = \xi(x)$.

- The solution of the one-dimensional Black-Scholes type pde can be solved by finite difference calculations.

- Andreasen manages to find such a Markov representation for discretely sampled Asian and lookback options with either fixed strike or floating strike. The payoff structures of many structured products and variable annuities resemble the Asian style payoff. The same approach can be adopted provided that the corresponding Markov representation can be found.

- The numerical pricing of discretely sampled path dependent option is considered more difficult compared to the continuously sampled counterparts. We have jump of the path dependent state variable across a fixing date in discretely sampled options.

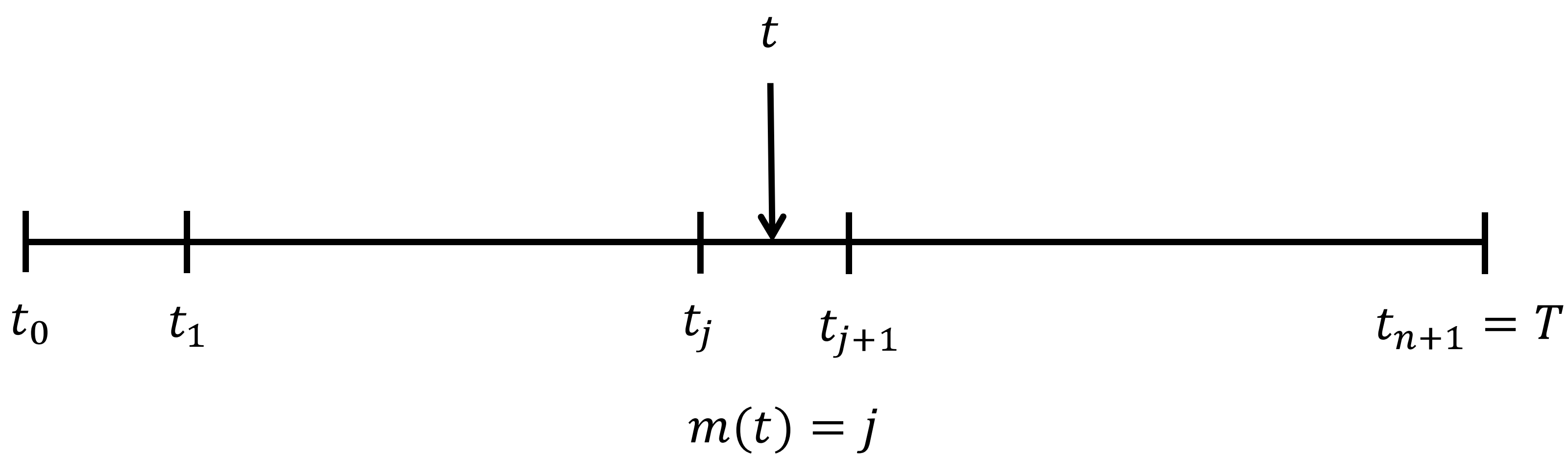
Asian call option with fixed strike (discretely sampled)

Let the fixing dates be $t_1, t_2, ..., t_n$, and let the initial time be $t_0 = 0$ and terminal date be $t_{n+1} = T$. We allow the flexibility that $t_1 = t_0$ and $t_n = T$. Hence, we have

$$0 = t_0 \le t_1 < \cdots < t_n \le t_{n+1} = T.$$

Denote the index of the last fixing date associated with the current time t by

$$m(t) = \text{sup}\{1 \le i \le n : t_i \le t\}.$$



Choice of the Markovian state variable

Define the running sum of discretely sampled stock prices up to time t by

$$I(t) = \sum_{1 \le i \le n,\ t_i \le t} S(t_i) = \sum_{i=1}^{m(t)} S(t_i).$$

The time-t price of the discrete Asian call option with fixed strike K is

$$F_t = S_t E_t^{Q'} \left[e^{-q(T-t)} \left(\frac{\frac{I(T)}{n} - K}{S_T} \right)^+ \right],$$

where the average $A(T) = I(T)/n$. Define the stochastic process

$$x_t = \frac{\frac{1}{n} I(t) - K}{S_t},$$

which jumps by a deterministic amount $\frac{1}{n}$ when the calendar time moves across the fixing date t_i, $i = 1, 2, \ldots, n$.

- We observe

$$
\begin{aligned}
x(t_i) &= \frac{\frac{1}{n}\sum_{j=1}^{i} S(t_j) - K}{S(t_i)} \\
&= \frac{\frac{1}{n}\sum_{j=1}^{i-1} S(t_j) - K}{S(t_i)} + \frac{1}{n} = x(t_i^-) + \frac{1}{n}, \quad i = 1, 2, \ldots, n.
\end{aligned}
$$

- At times between successive observation dates, the process x_t evolves continuously since $I(t)$ is held fixed and only S_t changes continuously as time evolves. The dynamics of x_t under Q' with jump is

$$
dx(t) = -(r-q)x(t)dt - \sigma x(t)dW'(t) + \frac{1}{n}dm(t).
$$

To show the above dynamics equation of x_t, recall that under Q'

$$
\frac{dS_t}{S_t} = (r - q + \sigma^2)\,dt + \sigma\,dW_t'
$$

so

$$
\begin{aligned}
\frac{d\left(\frac{1}{S_t}\right)}{\frac{1}{S_t}} &= [-(r-q+\sigma^2) + \sigma^2]\,dt - \sigma\,dW_t' \\
&= -(r-q)\,dt - \sigma\,dW_t'.
\end{aligned}
$$

Under Q', the evolution of x_t depends only on x_t itself and known jumps at t_i, $i = 1, ..., n$; so x_t is Markovian. Therefore, F_t/S_t depends on x_t and t only, so we write

$$\frac{F_t}{S_t} = f(x_t, t) = E_t^{Q'}\left[e^{-q(T-t)} x_T^+ | x_t\right], \quad x_T^+ = \max\left(x_T, 0\right).$$

Furthermore, once $x_t \geq 0$, then $x_u \geq 0$ for all $u \geq t$ since the sign of x_t depends on $\frac{1}{n}\ I(t) - K$ and $I(t)$ is non-decreasing in t.

For $x_t \geq 0$, we have

$$f(x_t, t) = E_t^{Q'}\left[e^{-q(T-t)} x_T | x_t\right].$$

When optimality on x_T disappears, the pricing problem reduces to that of a futures contract. One can obtain the following closed form formula:

$$e^{-q(T-t)} E_t^{Q'}[x_T | x_t] = e^{-r(T-t)} x_t + \frac{1}{n} \sum_{i: t < t_i \leq t_n} e^{-r(T-t_i) - q(t_i - t)}.$$

For convenience, we write the above known solution as $g(x_t, t)$.

Proof

We multiply the SDE for $x(t)$ throughout by the integrating factor $e^{(r-q)t}$:

$$d[e^{(r-q)t}x(t)] = -e^{(r-q)t}\sigma x(t)\,dW'(t) + \frac{1}{n}e^{(r-q)t}dm(t).$$

Integrating both sides with respect to u from t to T, we obtain

$$e^{(r-q)T}x_T - e^{(r-q)t}x_t = -\int_t^T e^{(r-q)u}\sigma x(u)\ dW'(u) + \frac{1}{n}\int_t^T e^{(r-q)u}\ dm(u).$$

Note that d$m(u)$, where u runs from t to T, has an infinite jump characterized by $\delta(u-t_i)$ on all the fixing dates t_i between t and T. Therefore, we may write

$$dm(u) = \sum_{i:t<t_i\leq t_n} \delta(u-t_i), \qquad t \leq u \leq T,$$

so that

$$\int_t^T e^{(r-q)u}\ dm(u) = \int_t^T e^{(r-q)u} \sum_{i:t<t_i\leq t_n} \delta(u-t_i)\ du = \sum_{i:t<t_i\leq t_n} e^{(r-q)t_i}$$

Next, we take the expectation under Q' conditional on $\mathcal{F}_t$. Note that the expectation of the stochastic integral equals zero due to the differential Brownian (whose expectation is zero). We then have

$$E_t^{Q'}[x_T] = e^{-(r-q)(T-t)}x_t + \frac{1}{n}\, e^{-(r-q)T} \sum_{i:t<t_i\leq t_n} e^{(r-q)t_i}.$$

Remarks

- Recall $E_t^Q[S_T] = e^{(r-q)(T-t)}S_t$, where $r-q$ is the drift rate of S_t under Q. Here, the drift rate of x_t under Q' is $-(r-q)$, so $E_t^{Q'}[x_T] = e^{-(r-q)(T-t)}x_t$.
- Due to the drift rate $-(r-q)$ under Q', the growth factor applied over (t_i, T) is $e^{-(r-q)(T-t_i)}$. The time-T future value of $\frac{1}{n}$ added to x_t at fixing date t_i is given by $\frac{1}{n}\, e^{-(r-q)T}e^{(r-q)t_i}$. We sum these jump terms for all fixing dates t_i, where $t < t_i \leq t_n$.

Lastly, we obtain

$$\begin{aligned} g(x_t, t) &= e^{-q(T-t)} E_t^{Q'}[x_T] \\ &= e^{-r(T-t)} x_t + \frac{1}{n} \sum_{i:t<t_i \le t_n} e^{-r(T-t_i)-q(t_i-t)}. \end{aligned}$$

When the current level of x_t is below $x = 0$, the process x_t may pass through the level $x = 0$ at some future fixing date t_i, where $m(t) < i \le n$.

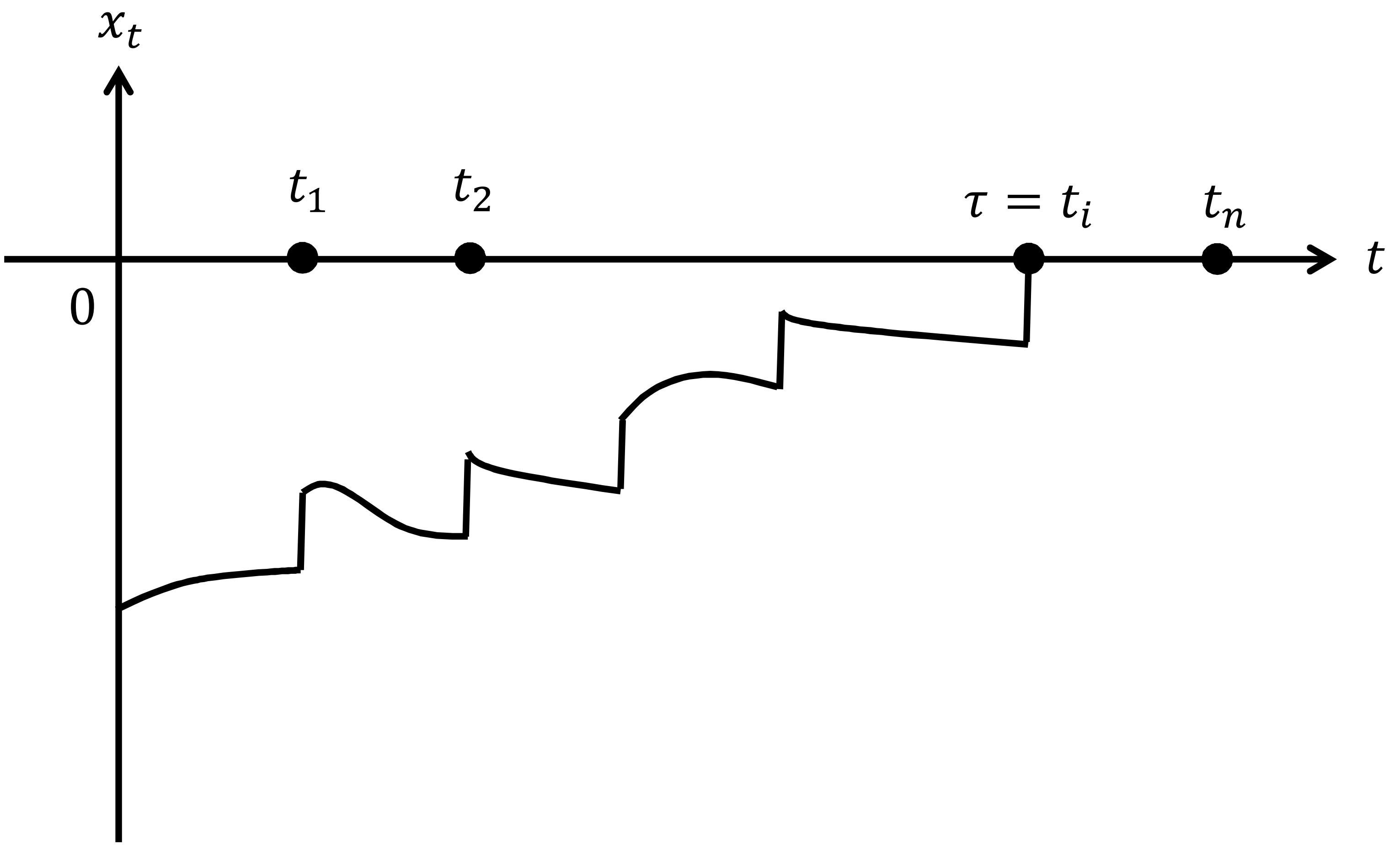
- If crossing does occur at time t_i, $i \le n$, then

$$f(x_{t_i}, t_i) = g(x_{t_i}, t_i).$$

- On the other hand, if crossing never occurs, then the terminal payoff $\max(x_T, 0)$ is zero.

Analogy as an up-and-in barrier option

(first passage time problem for a Markov process)



The Markov process x_t has a finite known jump at $\frac{1}{n}$ at each sample point. Here, τ is the first sample time that $x_\tau \geq 0$.

The crossing of x_t across $x = 0$ can only occur at one of the sample points. If crossing has not occurred prior to a sample point, the sign of $\frac{1}{n}I(t) - K$ remains negative at times between 2 sample points. Hence, x_t remains to be negative at least before reaching the next sample point.

We observe continuity of y across t_i while x has a jump of $\frac{1}{n}$ across t_i. To get rid of the discontinuous dynamics of $x(t)$, we define

$$y(t) = x(t) - \frac{m(t)}{n}.$$

We then have

$$dy(t) = -(r-q)\left[y(t) + \frac{m(t)}{n}\right] dt - \sigma \left[y(t) + \frac{m(t)}{n}\right] dW_t'.$$

By virtue of the Feynman-Kac representation theorem, we deduce that the governing equation for f is given by

$$qf = \frac{\partial f}{\partial t} - (r - q)\left[y(t) + \frac{m(t)}{n}\right]\frac{\partial f}{\partial y} + \frac{\sigma^2}{2}\left[y(t) + \frac{m(t)}{n}\right]^2 \frac{\partial^2 f}{\partial y^2}.$$

What is the domain of definition in the (y,t)-plane for $t_{i-1} < t < t_i$?

Note that $\{(x,t) : x < 0, 0 < t < T\}$ is the domain of definition in the (x,t)-plane since the discrete fixed strike Asian option can be visualized as an up-and-out barrier option with the upper barrier at $x = 0$.

At $t_0 < t < t_1$, $y(t) = x(t)$; when $t_{n-1} < t < t_n$, $y(t) = x(t) - \frac{n-1}{n}$ since there are $n-1$ jumps at $t_1, t_2, \ldots, t_{n-1}$, each jump has magnitude $\dfrac{1}{n}$. In general, when $t_{i-1} < t < t_i$, $y(t) = x(t) - \frac{i-1}{n}$, $i = 1, 2, ..., n$.

We perform the numerical calculations backward in the calendar time, starting from the last period: $t_n < t \le t_{n+1} = T$.

In the last time period: $t_n < t \le t_{n+1}$ (between the last monitoring instant and maturity date), we have $y = x - 1$. Recall that

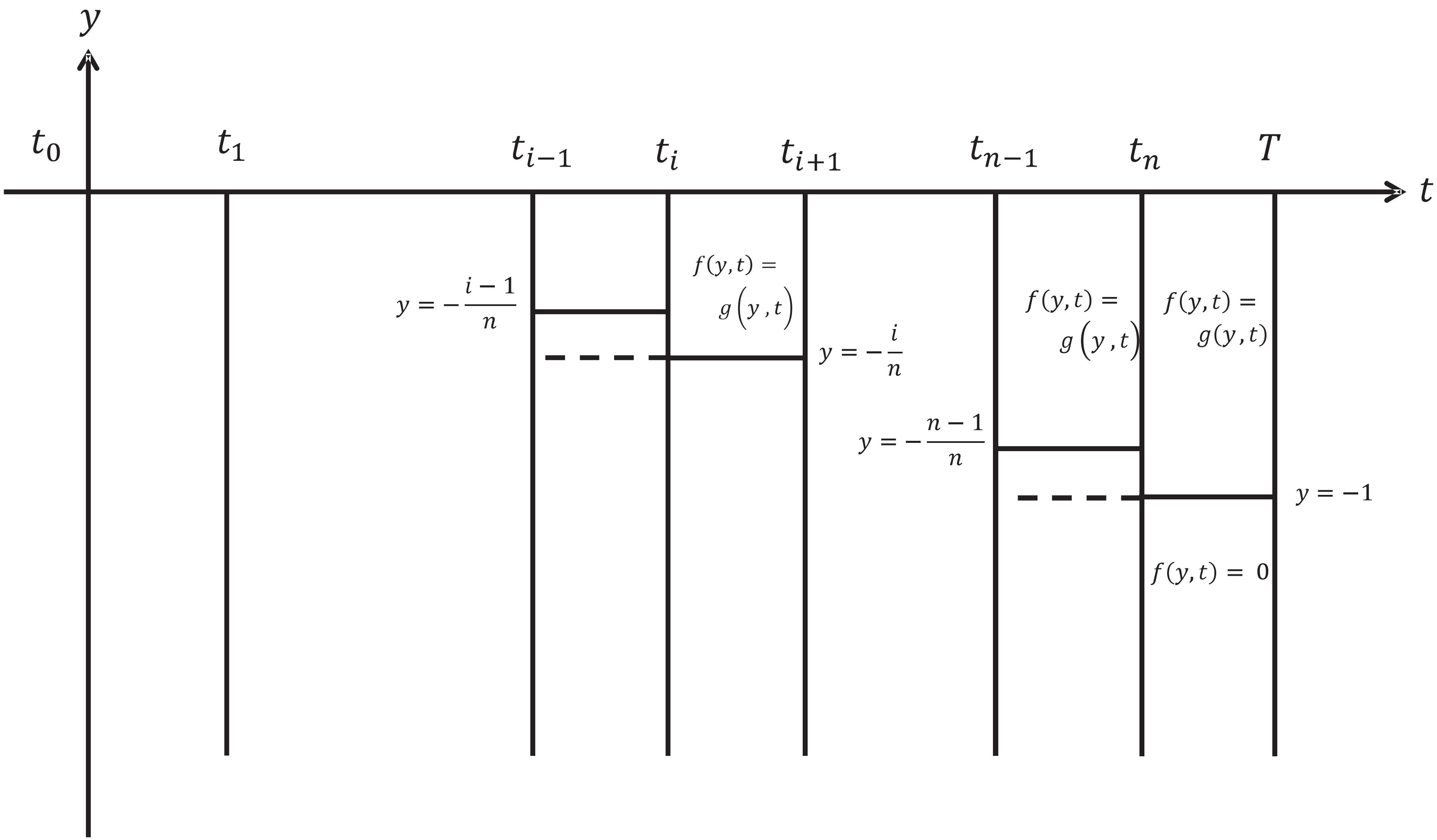
$$f(x_t, t) = g(x_t, t) \quad \text{for} \quad x_t \ge 0 \Leftrightarrow y_t \ge -1.$$

Suppose $x_t < 0$ in the last time period, then the option is sure to expire out-of-the-money. We then have

$$f(y, t) = 0 \quad \text{for} \quad y < -1 \text{ and } t \ge t_n.$$

In general, for $t_{i-1} < t < t_i$, the terminal condition is given by

$$f(y, t_i^-) = \begin{cases} f(y, t_i^+) & \text{if } y < -\frac{i}{n} \\ g(y, t_i) & \text{if } -\frac{i}{n} \le y < -\frac{i-1}{n} \end{cases}.$$



For $t_{i-1} < t < t_i$, $x = y + \frac{i-1}{n}$; and $x < 0 \Leftrightarrow y < -\frac{i-1}{n}$.

- Note that the state space of y within which the numerical solution of f is sought changes as time progresses.

- The state space is kept constant for all t between 2 sample points. Running backward in time, the newly added region is subject to the terminal condition as specified by the known function $g(\cdot,\cdot)$, where $f(y,t_i^-) = g(y,t_i)$, $-\dfrac{i}{n} \leq y < -\dfrac{i-1}{n}$.

- Between 2 sample points, say within the time interval $[t_{i-1}, t_i]$, the pde is solved numerically using the Crank-Nicolson scheme.

When y lies within $\left[-\frac{i}{n}, -\frac{i-1}{n}\right)$ at t_i^-, the Asian option becomes in-the-money when t advances across t_i (due to the jump of $\dfrac{1}{n}$ in x_t at $t = t_i$). By virtue of continuity of option value, we have $f(y,t_i^-) = g\,(y,t_i)$. on the other hand, when $y < -\frac{i}{n}$, we have continuity of $f(y,t)$ across $t = t_i$.

Construction of the Crank-Nicolson scheme

Rewrite the governing partial differential equation between two successive fixing dates as

$$\left[-q+\frac{\partial}{\partial t}+\mu(y)\frac{\partial}{\partial y}+v(y)\frac{\partial^2}{\partial y^2}\right]f=0,\ t_{i-1}<t<t_i,\ y<-\frac{i-1}{n},$$

where

$$\mu(y)=-(r-q)\left[y+\frac{m(t)}{n}\right] \quad \text{and} \quad v(y)=\frac{\sigma^2}{2}\left[y+\frac{m(t)}{n}\right]^2.$$

We suppress the dependence of μ and v on t since $m(t)$ is constant in each interval $[t_{i-1},t_i]$.

Approximate the differential operators in the partial differential equation by centered differences at $\left(y, t + \frac{\Delta t}{2}\right)$. We obtain the Crank-Nicolson scheme:

$$\left[-\frac{q}{2} + \frac{1}{\Delta t} + \frac{\mu(y)}{2}\delta_y + \frac{v(y)}{2}\delta_{yy}\right] f(y, t + \Delta t)$$
$$= \left[\frac{q}{2} + \frac{1}{\Delta t} - \frac{\mu(y)}{2}\delta_y - \frac{v(y)}{2}\delta_{yy}\right] f(y, t),$$

where the difference operators are

$$\delta_y h(y) = \frac{h(y + \Delta y) - h(y - \Delta y)}{2\Delta y},$$
$$\delta_{yy} h(y) = \frac{h(y + \Delta y) - 2h(y) + h(y - \Delta y)}{\Delta y^2}.$$

The artificial boundary conditions: $\delta_{yy} f = 0$ is applied at the lower boundary y_{min} and $\dfrac{\partial f}{\partial t} = qf$ at the upper boundary [known solution $g(x, t)$ at $y = y_{\text{max}}$ is subject to the discount rate q when x_t is held fixed and time dependence is only exhibited only in those terms that involve $e^{-q(t_i - t)}$].

For the interval $[t_{i-1}, t_i]$, we limit our state space to the discrete grid: $\{(y_l, s_k)\}_{k=0,1,\ldots,K;l=0,1,\ldots,L}$ with

$$s_k = t_{i-1}\frac{K-k}{K} + t_i\frac{k}{K} \quad \text{and} \quad y_l = y_{\text{min}}\frac{L-l}{L} + y_{\text{max}}\frac{l}{L}.$$

Here, $\Delta t = \frac{t_i - t_{i-1}}{K}$ and $\Delta y = \frac{y_{\text{max}} - y_{\text{min}}}{L}$. We have

$$y_{\text{max}} = -\frac{i-1}{n}$$

and we set $y_{\text{min}} = -2$ (for maturity less than one year).

It is convenient to choose the same stepwidth Δy at all time intervals. Since the state space of y increases in size in the backward induction procedure, so L increases accordingly with an increase in y_{max}.

- If we set $y_{\text{min}} = -2$ for all time intervals, we observe the advantage of having a smaller computational domain in later time intervals (saving of computational efforts).

For the interior nodes, $l = 1, 2, ..., L$, we have

$$\left(\frac{q}{2} + \frac{1}{\Delta t}\right) f(s_k, y_l) - \frac{\mu(y_l)}{2} \frac{f(s_k, y_{l+1}) - f(s_k, y_{l-1})}{2\Delta y}$$
$$- \frac{v(y_l)}{2} \frac{f(s_k, y_{l+1}) - 2f(s_k, y_l) + f(s_k, y_{l-1})}{\Delta y^2}$$
$$= \left(-\frac{q}{2} + \frac{1}{\Delta t}\right) f(s_{k+1}, y_l) + \frac{\mu(y_l)}{2} \frac{f(s_{k+1}, y_{l+1}) - f(s_{k+1}, y_{l-1})}{2\Delta y}$$
$$+ \frac{v(y_l)}{2} \frac{f(s_{k+1}, y_{l+1}) - 2f(s_{k+1}, y_l) + f(s_{k+1}, y_{l-1})}{\Delta y^2}.$$

Applying $\delta_{yy} = 0$ at the $(k, L)^{th}$ node, we obtain

$$f(s_k, y_{L+1}) = 2f(s_k, y_L) - f(s_k, y_{L-1}),$$

where $f(s_k, y_{L+1})$ is a fictitious point outside the computational domain.

We are marching backward in time, starting from $k = K - 1$ down to $k = 0$. The values at $k = K$ (terminal conditions) are inferred from the last time interval (by continuity of value function).

The matrix system of equations of the Crank-Nicolson scheme:

$$\begin{pmatrix} - & \mathbf{A}_0 & - \\ - & \mathbf{A}_1 & - \\ & \vdots & \\ & \vdots & \\ - & \mathbf{A}_{L-1} & - \\ - & \mathbf{A}_L & - \end{pmatrix} \begin{pmatrix} f(s_k, y_0) \\ f(s_k, y_1) \\ \vdots \\ \vdots \\ f(s_k, y_{L-1}) \\ f(s_k, y_L) \end{pmatrix} = \begin{pmatrix} - & \mathbf{B}_0 & - \\ - & \mathbf{B}_1 & - \\ & \vdots & \\ & \vdots & \\ - & \mathbf{B}_{L-1} & - \\ - & \mathbf{B}_L & - \end{pmatrix} \begin{pmatrix} f(s_{k+1}, y_0) \\ f(s_{k+1}, y_1) \\ \vdots \\ \vdots \\ f(s_{k+1}, y_{L-1}) \\ f(s_{k+1}, y_L) \end{pmatrix}.$$

Here, both coefficient matrices are essentially tridiagonal. We would like to find the explicit representation of the rows $\boldsymbol{A}_\ell$ and $\boldsymbol{B}_\ell$; and special care is taken at $\ell = 0$ and $\ell = L$ so as to incorporate the numerical boundary conditions.

At $\ell = L$, the numerical scheme becomes

$$\left(\frac{q}{2} + \frac{1}{\Delta t}\right) f(s_k, y_L) - \frac{\mu(y_L)}{2\Delta y}[f(s_k, y_L) - f(s_k, y_{L-1})]$$
$$= \left(-\frac{q}{2} + \frac{1}{\Delta t}\right) f(s_{k+1}, y_L) + \frac{\mu(y_L)}{2\Delta y}[f(s_{k+1}, y_L) - f(s_{k+1}, y_{L-1})].$$

The last two components in the row vector $\boldsymbol{B}_L$ are

$$B_{L,L-1} = -\frac{\mu(y_L)}{2\Delta y}, \quad B_{L,L} = -\frac{q}{2} + \frac{1}{\Delta t} + \frac{\mu(y_L)}{2\Delta y};$$

and the last two components in the row vector $\boldsymbol{A}_L$ are

$$A_{L,L-1} = \frac{\mu(y_L)}{2\Delta y}, \quad A_{L,L} = \frac{q}{2} + \frac{1}{\Delta t} - \frac{\mu(y_L)}{2\Delta y}.$$

At $y = y_{\text{max}}$ (node $\ell = 0$), the differential equation becomes $\dfrac{\partial f}{\partial t} = qf$ so that the discretized scheme at y_0 is

$$\frac{f(s_{k+1}, y_0) - f(s_k, y_0)}{\Delta t} = q\frac{f(s_k, y_0) + f(s_{k+1}, y_0)}{2}$$

giving

$$\left(\frac{q}{2} + \Delta t\right) f(s_k, y_0) = \left(-\frac{q}{2} + \Delta t\right) f(s_{k+1}, y_0).$$

Hence, the two row vectors $\boldsymbol{A}_0$ and $\boldsymbol{B}_0$ each contains only one component, namely,

$$A_{0,0} = \frac{q}{2} + \frac{1}{\Delta t} \quad \text{and} \quad B_{0,0} = -\frac{q}{2} + \frac{1}{\Delta t}.$$

Since we have set $\delta_{yy} = \delta_y = 0$ at y_0, it is not surprising that $\mu(y_0)$ and $v(y_0)$ do not appear in $\boldsymbol{A}_0$ and $\boldsymbol{B}_0$.

We solve a sequence of matrix equations from $k = K-1, K-2, ..., 0$:

$$A\boldsymbol{f}(s_k) = B\boldsymbol{f}(s_{k+1}),$$

where $\boldsymbol{f}$ is the vector

$$\boldsymbol{f}(s_k) = (f(s_k, y_0), \cdots, f(s_k, y_L))^T$$

and A and B are $(L+1)$-dimensional tridiagonal matrices with the l^{th} rows given by

$$\boldsymbol{A}_l = \left(0, \cdots, 0, \frac{\mu(y_l)}{4\Delta y} - \frac{v(y_l)}{2(\Delta y)^2}, \frac{q}{2} + \frac{1}{\Delta t} + \frac{v(y_l)}{(\Delta y)^2}, -\frac{\mu(y_l)}{4\Delta y} - \frac{v(y_l)}{2(\Delta y)^2}, 0, \cdots, 0\right)$$

$$\boldsymbol{B}_l = \left(0, \cdots, 0, -\frac{\mu(y_l)}{4\Delta y} + \frac{v(y_l)}{2(\Delta y)^2}, -\frac{q}{2} + \frac{1}{\Delta t} - \frac{v(y_l)}{(\Delta y)^2}, \frac{\mu(y_l)}{4\Delta y} + \frac{v(y_l)}{2(\Delta y)^2}, 0, \cdots, 0\right)$$

for $l = 1, \cdots, L-1$.

and

$$
\begin{aligned}
\boldsymbol{A}_0 &= \left(\frac{q}{2} + \frac{1}{\Delta t}, 0, 0, \cdots, 0\right), \\
\boldsymbol{A}_L &= \left(0, \cdots, 0, \frac{\mu(y_L)}{2\Delta y}, \frac{q}{2} + \frac{1}{\Delta t} - \frac{\mu(y_L)}{2\Delta y}\right), \\
\boldsymbol{B}_0 &= \left(-\frac{q}{2} + \frac{1}{\Delta t}, 0, 0, \cdots, 0\right), \\
\boldsymbol{B}_L &= \left(0, \cdots, 0, -\frac{\mu(y_L)}{2\Delta y}, -\frac{q}{2} + \frac{1}{\Delta t} + \frac{\mu(y_L)}{2\Delta y}\right).
\end{aligned}
$$

Backward induction procedure

- We start at time t_n^-. By applying the known analytic solution at t_n, we obtain $\boldsymbol{f}(t_n^-)$. We then numerically solve backward in time to t_{n-1}^+ by solving the matrix system: $A\boldsymbol{f}(s_k) = B\boldsymbol{f}(s_{k+1})$.

- At time t_{n-1}^+, the numerical solution together with the function $g(\cdot, t_{n-1})$ serves as the terminal condition for the numerical solution over the next time interval (t_{n-2}, t_{n-1}).

The finite difference algorithm is surprisingly accurate even with 50 time steps per each time interval. The maximum relative error compared with the Monte Carlo simulation results is approximately 0.4%.

The parameters are: $r = 0.05$, $q = 0.0$, $\sigma = 0.2$, $T = 1.0$, $t = 0.0$, $n = 10$, $S(0) = 100.0$, $t_i = 0.1i$. MC refers to Monte Carlo solution, and FD refers to finite difference solution. The different I's refer to the number of time steps. We used $I/10$ number of steps per jump size $1/n$ in the y direction. The Monte Carlo prices are based on 10^5 simulations with a control variate technique. The standard deviations of the Monte Carlo option prices are approximately 3.0×10^{-3}.

K	Asian option prices			
	MC	FD ($I = 500$)	FD ($I = 100$)	FD ($I = 50$)
90.0	12.98	12.99	12.99	12.98
92.5	11.05	11.05	11.05	11.05
95.0	9.27	9.27	9.27	9.27
97.5	7.67	7.66	7.66	7.66
100.0	6.24	6.23	6.23	6.23
102.5	5.01	5.00	5.00	5.00
105.0	3.96	3.95	3.95	3.95
107.5	3.08	3.07	3.07	3.07
110.0	2.36	2.35	2.35	2.36
CPU	46.0s	0.65s	0.06s	0.04s

6.2 Discretely sampled lookback options

Fixed strike lookback call option

For $t \geq 0$, we define the discretely monitored realized maximum of the asset price $S(t)$ by

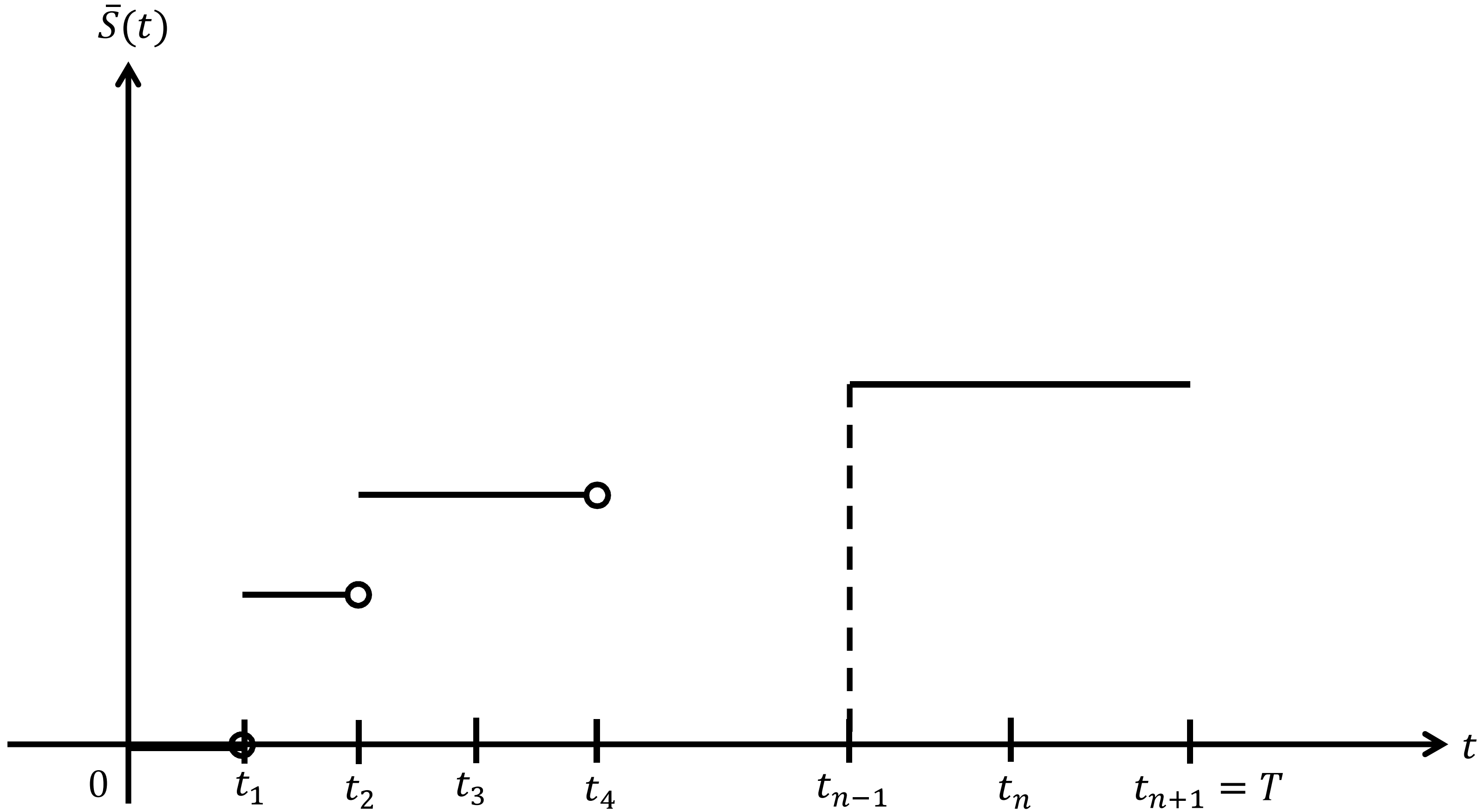
$$\bar{S}(t) = \sup_{1 \leq i \leq m(t)} S(t_i)$$

with $\bar{S}(t) = 0$ for $t < t_1$ for notational convenience.

Terminal payoff $= (\bar{S}(T) - K)^+$.

First, we solve for the time-t option value when $\bar{S}(t) \geq K$.

Next, we solve for the case $\bar{S}(t) < K$. This is done by treating the option pricing problem as a first passage problem of S to the level K, where the payoff at the first passage time is equal to the option value corresponding to $\bar{S}(t) \geq K$.



1. $\bar{S}(t) = 0$ for $0 \leq t < t_1$ and $\bar{S}(t) = S(t_1)$ for $t_1 \leq t < t_2$. Here, t_n is the last observation date and $T = t_{n+1}$ is the maturity date.

2. $\bar{S}(t)$ is updated to a higher value on an observation date if a new maximum asset price is realized.

Once $\bar{S}(t) \geq K$, $t < T$, the discrete fixed strike lookback call is guaranteed to be in-the-money. The time-t option price is given by

$$\begin{aligned} F(S(t), t) &= E_t^Q \left[e^{-r(T-t)} (\bar{S}(T) - K) \right] \\ &= S_t E_t^{Q'} \left[e^{-q(T-t)} \frac{\bar{S}(T)}{S(T)} \right] - e^{-r(T-t)} K. \end{aligned}$$

This is consistent with the observation that $F_t / e^{qt} S_t$ is Q'-martingale. Define $x(t) = \frac{\bar{S}(t)}{S(t)}$ for $t \geq t_1$. Note that we have adopted the change of measure from Q to Q' since it is more convenient to work with $x(t)$.

- Unlike the Asian option model, the numerical calculation of the closed form analytic solution to $E_t^{Q'} \left[e^{-q(T-t)} \frac{\overline{S}(T)}{S(T)} \right]$ is more cumbersome than its numerical solution via finite difference scheme since the closed form formula involves n-dimensional cumulative normal distribution functions.

Potential updating of recorded realized maximum value across a sampling date

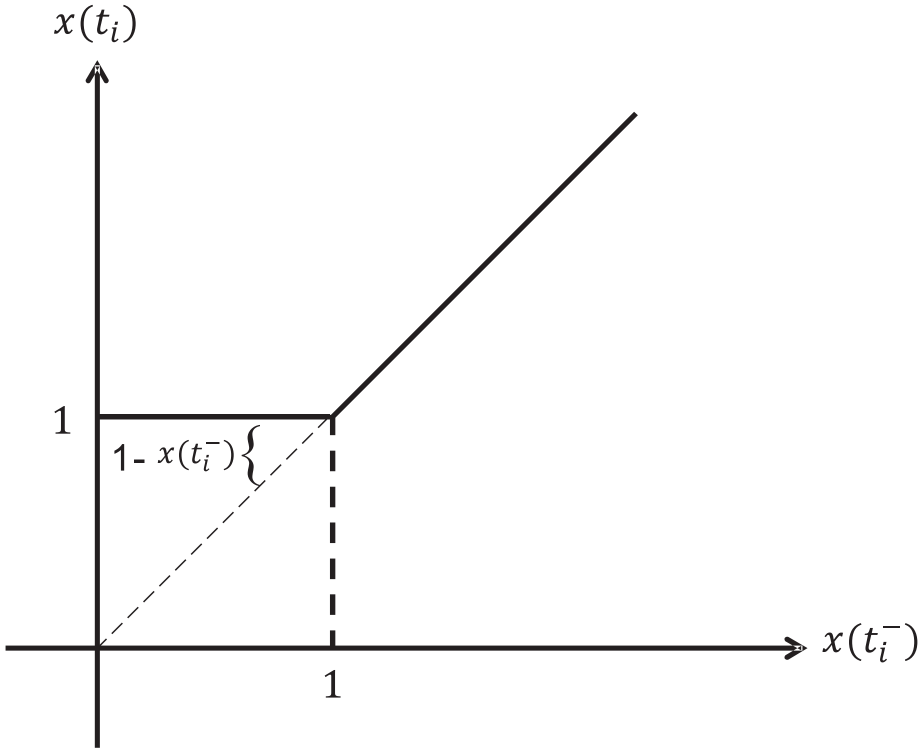
It may occur that $S(t_i^-) \geq \bar{S}(t_i^-) = \bar{S}(t_{i-1})$. That is, the stock price at t_i^- is equal to or higher than the recorded realized maximum on the last fixing date. In this case, we have $x(t_i) = 1$ since the updated realized maximum $\bar{S}(t_i)$ becomes $S(t_i)$ [by virtue of continuity of $S(t)$ across t_i].

If otherwise, suppose we have $S(t_i^-) < \bar{S}(t_i^-)$ and so $x(t_i^-) > 1$. There will be no updated realized maximum. By continuity, we have $x(t_i) = x(t_i^-)$.

The jump amount on $x(t)$ across t_i^- is equal to (i) zero if $S(t_i^-) < \overline{S}(t_i^-)$. If otherwise, the jump amount is $1 - x(t_i^-)$ when $x(t_i^-) \leq 1 \Leftrightarrow S(t_i^-) \geq \overline{S}(t_i^-)$.

In summary, for $1 \leq i \leq n$,

$$x(t_i) = \begin{cases} 1 & \text{if} \quad x(t_i^-) \leq 1 \\ x(t_i^-) & \text{if} \quad x(t_i^-) > 1 \end{cases}. \tag{i}$$



On the observation date t_i, we always have $x(t_i) \geq 1$. After then, $x(t)$ may drop below 1 when $S(t)$ increases beyond $\overline{S}(t_i)$.

The process x_t may jump by an amount $(1-x(t^-))^+$ when t falls on an observation date. Such jump occurs when a new realized maximum of the stock price is recorded on that observation date. In between two successive observation dates, the evolution of x_t is continuous.

Under Q', the dynamics of x_t is governed by

$$\begin{aligned} dx(t) &= -(r-q)x(t^-)dt - \sigma x(t^-)dW'(t) + (1-x(t^-))^+ dm(t) \\ x(t_1) &= 1. \end{aligned}$$

Therefore, x_t is a Markov process with domain on $x > 0$.

We define

$$f(x(t),t) = E_t^{Q'}\left[e^{-q(T-t)}\frac{\bar{S}(T)}{S(T)}\right] = E^{Q'}\left[e^{-q(T-t)}x(T)|x(t)\right].$$

When $t \geq t_1$, suppose $\bar{S}(t) \geq K$, then

$$F(S(t), t) = S(t)f(x(t), t) - e^{-r(T-t)}K. \tag{A}$$

At the first time $t_i (i \leq n)$ with S staying at or above K, $x(t_i) = 1$ since a newly recorded maximum must occur and so gives a jump of $x(t)$ to the value one. We then obtain the payoff

$$F(S(t_i), t_i) = S(t_i)f(1, t_i) - e^{-r(T-t_i)}K. \tag{i}$$

- Note that $\bar{S}(t) - K$ can change sign only on the observation dates at which potential updating of the recorded maximum value may occur.

How about when $\bar{S}(t) < K$? In general, for $t \geq 0$ with $\bar{S}(t) < K$, we deduce from eq.(i) that

$$F(S(t),t) = E^Q\left[e^{-r(\tau^*-t)}\{S(\tau^*)f(1,\tau^*) - e^{-r(T-\tau^*)}K\}\mathbf{1}_{\tau^*\leq t_n}|S(t)\right], \tag{B}$$

where τ^* is the first passage time defined by

$$\tau^* = \inf_{i=1,\cdots,n}\{t_i : S(t_i) \geq K\},$$

with the convention: $\inf \phi = \infty$.

Like an up-and-in barrier option, the option expires with zero terminal payoff unless $S(t)$ crosses K at some random first passage time τ^*.

Numerical procedure

1. First, solve for $f(x,u)$ for all (x,u) with $u \geq \max(t,t_1)$.

2. If $\bar{S}(t) \geq K$, then the option price is given by eq. (A). Otherwise, solve the first passage time problem as depicted in eq. (B).

Numerical solution for $f(y,t)$, corresponding to the scenario $\overline{S}(t) \geq K$

Let $y = \ln x$. For the jump dynamics, it is seen that when $y < 0$ (which is equivalent to $x < 1$) at t_i^-, then y jumps to 0 across t_i. When $y \geq 0$ at t_i^-, y remains continuous across t_i.

By virtue of the Feynman-Kac representation theorem, for $t_{i-1} < t < t_i$, $i = 2, 3, \ldots, n+1$, and the incorporation of the jump dynamics in y via the auxiliary condition on an observation date, f is governed by

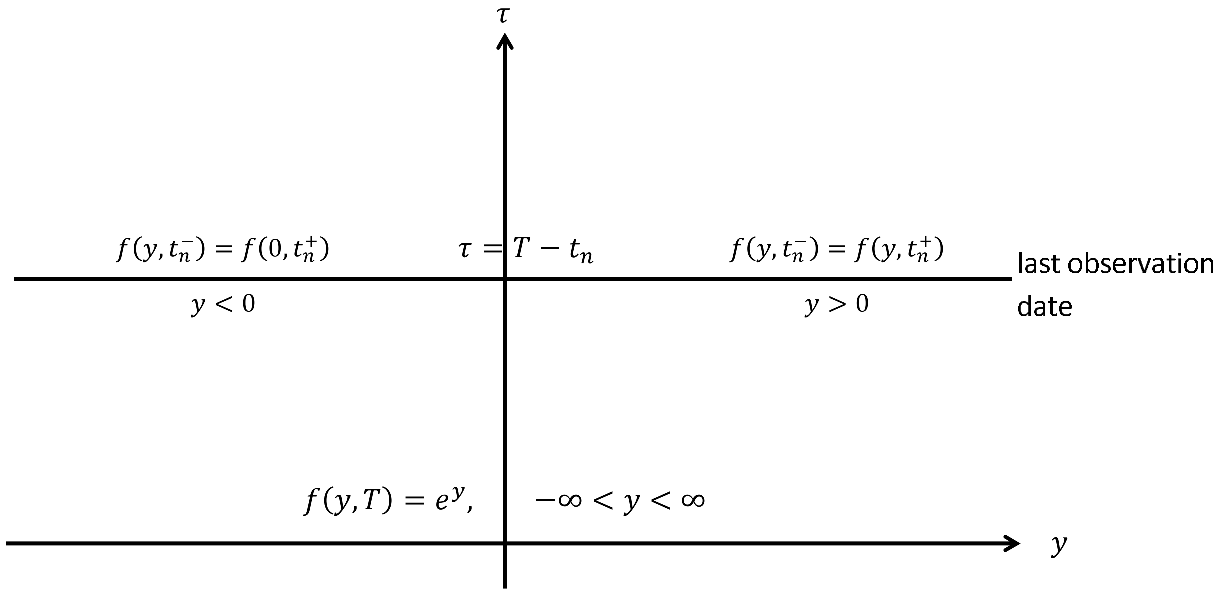
$$qf = \frac{\partial f}{\partial t} - \left(r - q + \frac{\sigma^2}{2}\right)\frac{\partial f}{\partial y} + \frac{\sigma^2}{2}\frac{\partial^2 f}{\partial y^2},$$

subject to

$$f(y, t_i^-) = \begin{cases} f(0, t_i^+) & \text{if} \quad y < 0 \\ f(y, t_i^+) & \text{if} \quad y \geq 0 \end{cases}$$
$$f(y, T) = e^y.$$

We solve for $f(y,t)$ for $t_1^+ < t < T$. Note that $f(y,t)$ is not defined for $t < t_1$ since no maximum value has yet recorded.

Calculations for $f \; [\bar{S}(t) \geq K]$;
time marching in $\tau = T - t$, where τ is the time to expiry.



$$qf = \frac{\partial f}{\partial t} - \left(r - q + \frac{\sigma^2}{2}\right)\frac{\partial f}{\partial y} + \frac{\sigma^2}{2}\frac{\partial^2 f}{\partial y^2},$$

$$-\infty < y < \infty, \quad t_1 < t < T = t_{n+1}.$$

jump condition at t_i : $f(y, t_i^-) = \begin{cases} f(0, t_i^+) & \text{if} \quad y < 0 \\ f(y, t_i^+) & \text{if} \quad y \geq 0 \end{cases}, \qquad i = n-1, n-2, \cdots, 1;$

terminal payoff: $f(y, T) = e^y$.

Lookback option value $F(S(t), t) = S(t) f(y, t) - e^{-r(T-t)} K$, $y_t = \ln \dfrac{\overline{S}(t)}{S(t)}$.

Numerical solution for $g(y,t)$, corresponding to the scenario $\overline{S}(t) < K$

For $\bar{S}(t) < K$, we may define $y = \ln \frac{S(t)}{K}$, and $g = \frac{F}{K}$. Note that g is governed by

$$rg = \frac{\partial g}{\partial t} + \left(r - q - \frac{\sigma^2}{2}\right)\frac{\partial g}{\partial y} + \frac{\sigma^2}{2}\frac{\partial^2 g}{\partial y^2}$$

on $\{(y,t) : t_{i-1} < t < t_i, i = 1, 2, \cdots, n; -\infty < y < \infty\}$, subject to the auxiliary conditions:

$$g(y, t_i^-) = \begin{cases} g(y, t_i^+) & \text{if } \ y < 0 \\ e^y f(0, t_i) - e^{-r(T-t_i)} & \text{if } \ y \geq 0 \end{cases}$$
$$g(y, t_n) = 0.$$

Note that if $\bar{S}(t) < K$ for $t \geq t_n$, then $F(S(t), t) = 0$ since the lookback option is sure to be out-of-the-money at expiry.

- We may treat f and g in the same grid and simultaneously solve for f and g at each time step in that respective order. Be aware that the coefficients in the finite difference schemes are different since the governing equations are not the same. Also, the definitions of the independent variable y in the two formulations are different.

  Remark
  For $y \geq 0$ at $t = t_i$, this corresponds to the first time that $\overline{S}(t)$ reaches K or above. The option value normalized by K is given by [see eq.(i)]

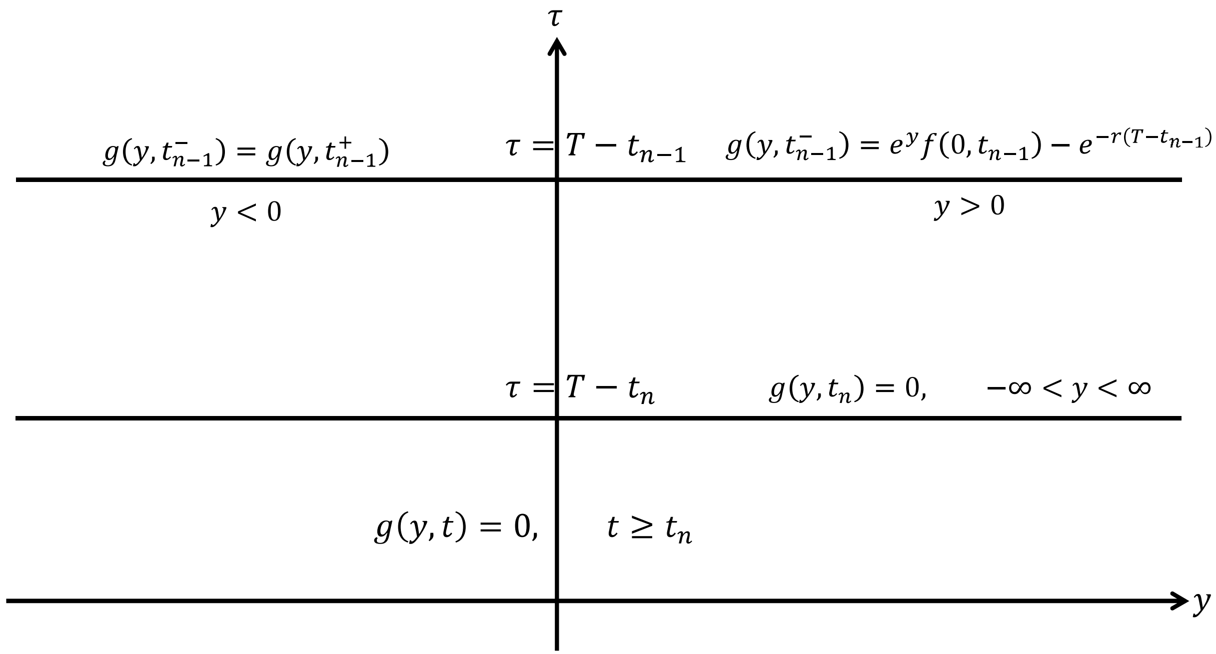
$$g(y, t_i) = \frac{F(y, t_i)}{K} = e^y f(0, t_i) - e^{-r(T-t_i)},$$

  where $x(t_i) = 1 \Leftrightarrow y = 0$ and $\dfrac{S(t_i)}{K} = e^y$.

- At current time t, values of options of different strikes are generated by

$$F(S(t), t) = Kg\left(\frac{S(t)}{K}, t\right).$$

Calculations for g $[\bar{S}(t) < K]$; τ is the time to expiry.



$$rg = \frac{\partial g}{\partial t} + \left(r - q - \frac{\sigma^2}{2}\right)\frac{\partial g}{\partial y} + \frac{\sigma^2}{2}\frac{\partial^2 g}{\partial y^2}, \quad -\infty < y < \infty, \quad 0 < t < t_n.$$

$$g(y,t_i^-) = \begin{cases} g(y,t_i^+) & \text{if} \quad y < 0 \\ e^y f(0,t_i) - e^{-r(T-t_i)} & \text{if} \quad y \geq 0 \end{cases}, \quad i = n-1, n-2, \cdots, 1;$$

$$g(y,t_n) = 0.$$

Lookback option value $F(S(t),t) = Kg(y,t)$, where $y_t = \ln \frac{S(t)}{K}$.

The parameters are: $r = 0.05$, $q = 0.0$, $\sigma = 0.2$, $T = 1.0$, $t = 0.0$, $n = 10$, $S(0) = 100.0$, $t_i = 0.1i$. MC refers to Monte Carlo solution, and FD refers to finite-difference solution. The different I's refer to the number of time steps *and* also to the number of steps in the y-direction. The Monte Carlo prices are based on 10^5 simulations with a control variate technique. The standard error on the estimated Monte Carlo option prices is approximately 3.0×10^{-3}.

K	Fixed strike lookback option prices			
	MC	FD ($I = 500$)	FD ($I = 100$)	FD ($I = 50$)
90.0	24.41	24.39	24.27	23.81
92.5	22.07	22.06	21.93	21.47
95.0	19.78	19.77	19.64	19.18
97.5	17.57	17.56	17.43	16.96
100.0	15.48	15.47	15.34	14.87
102.5	13.53	13.52	13.39	12.95
105.0	11.75	11.74	11.62	11.22
107.5	10.14	10.14	10.03	9.67
110.0	8.70	8.71	8.62	8.30
CPU	46.0s	0.68s	0.06s	0.03s

Floating strike lookback option

Terminal payoff $= (\bar{S}(T) - \alpha S(T))^+$. The fair option price is given by

$$
\begin{aligned}
F(S(t),t) &= S(t) E_t^{Q'} \left[e^{-q(T-t)} \left(\frac{\bar{S}(T)}{S(T)} - \alpha \right)^+ \right] \\
&= S(t) E_t^{Q'} \left[e^{-q(T-t)} (x(T) - \alpha)^+ | x(t) \right] \\
&= S(t) f(x(t), t) \quad \text{if } t \geq t_1.
\end{aligned}
$$

It suffices to solve for $f(x,t)$ where $t \geq t_1$. For $t < t_1$, there is no recorded maximum value yet. Observe that $F(t_1) = S(t_1) f(1, t_1)$ since $\dfrac{\overline{S}(t_1)}{S(t_1)} = 1$ for sure. Since $f(1,t_1)$ is independent of asset price dynamics, so

$$
\begin{aligned}
F(S(t),t) &= e^{-r(t_1-t)} E_t^Q [S(t_1)] f(1,t_1) \\
&= e^{-r(t_1-t)} e^{(r-q)(t_1-t)} S(t) f(1,t_1) \\
&= e^{-q(t_1-t)} S(t) f(1,t_1), \quad t < t_1.
\end{aligned}
$$

Again, this result arises from the property that $F_t / e^{qt} S_t$ is Q'-martingale.

With $y = \ln x$, f solves

$$qf = \frac{\partial f}{\partial t} - \left(r - q + \frac{\sigma^2}{2}\right)\frac{\partial f}{\partial y} + \frac{\sigma^2}{2}\frac{\partial^2 f}{\partial y^2}$$

where $t_{i-1} < t < t_i$, $i > 1$. At $t = t_i^-$, suppose $S(t_i) > \bar{S}(t_i) = \bar{S}(t_{i-1})$, this corresponds to

$$x(t_i^-) = \frac{\bar{S}(t_i^-)}{S(t_i^-)} < 1 \Leftrightarrow y(t_i^-) < 0.$$

Right at the moment $t = t_i$, y jumps to the "zero" value. By continuity, $f(y, t_i^-) = f(0, t_i)$, $y < 0$. When $y \geq 0$, we expect to have continuity of f across t_i. The auxiliary conditions become

$$f(y, t_i^-) = \begin{cases} f(0, t_i) & \text{if} \quad y < 0 \\ f(y, t_i) & \text{if} \quad y \geq 0 \end{cases}, \qquad i = n, n-1, \cdots, 1;$$

$$f(y, t_{n+1}^-) = f(T, y) = (e^y - \alpha)^+.$$

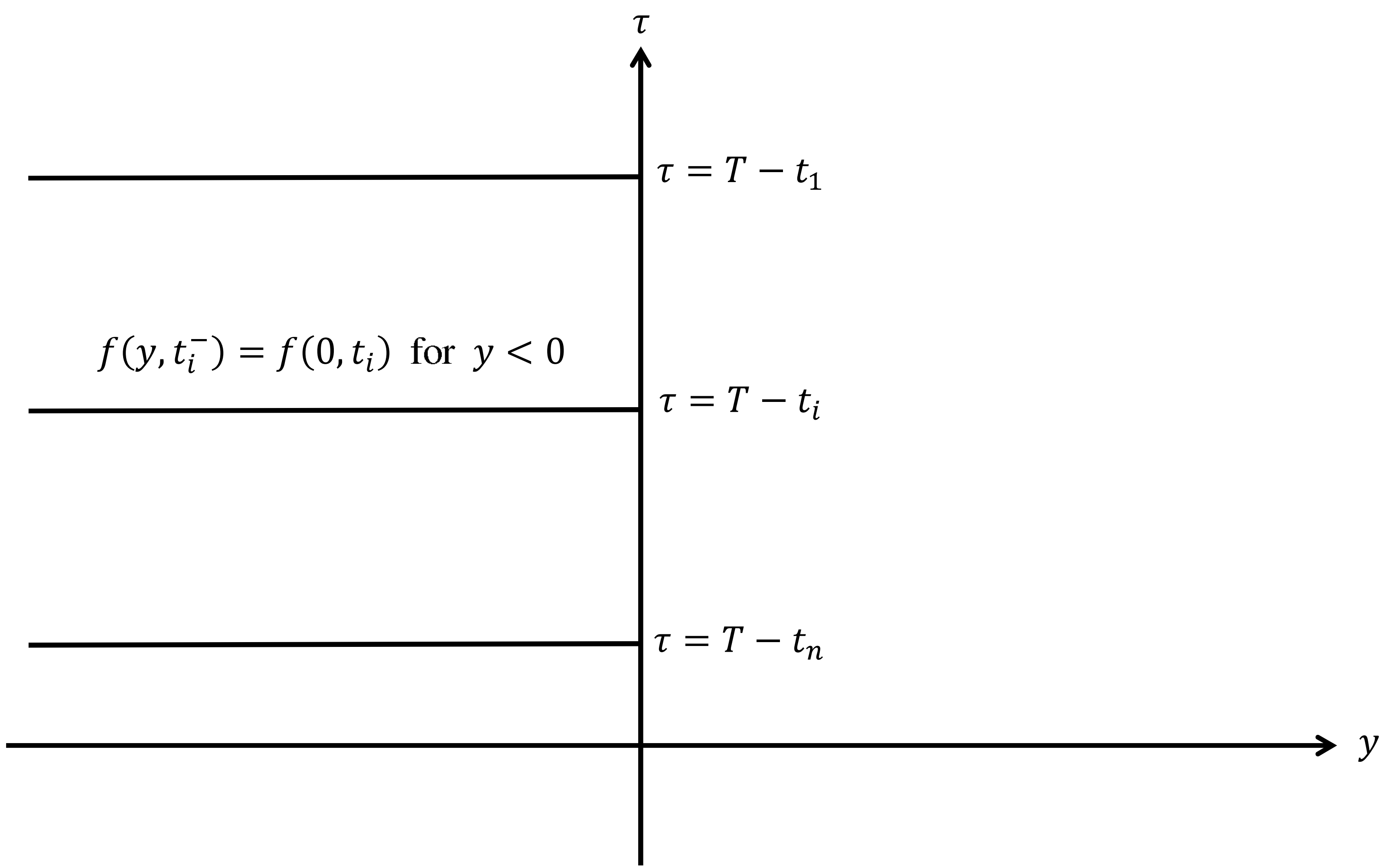
The formulation for floating strike lookback option is very similar to that of the fixed strike lookback option under $\overline{S}(t) \geq K$, except that the terminal condition at T is changed to $(e^y - \alpha)^+$.

Remark

All options with $\alpha \leq 1$ are guaranteed to be "in-the-money". For $\alpha < 1$, the *European* option contract has a value that equals the value of the counterpart with $\alpha = 1$ plus $S(t)(1-\alpha)e^{-q(T-t)}$. This is easily seen since

$$\begin{aligned} F(t) &= e^{-r(T-t)} E_t^Q[\overline{S}(T) - \alpha S(T)] \\ &= e^{-r(T-t)} E_t^Q\left[\left\{\overline{S}(T) - S(T)\right\} + (1-\alpha)S(T)\right] \\ &= e^{-r(T-t)} E_t^Q[\overline{S}(T) - S(T)] + (1-\alpha)S(t)e^{-q(T-t)}. \end{aligned}$$

In other words, once the lookback option price for $\alpha = 1$ is known, the option price for $\alpha < 1$ can be obtained by adding an extra term $(1-\alpha)S(t)e^{-q(T-t)}$.



We perform the usual finite difference calculations as those of the vanilla option models, except that we impose the auxiliary condition: $f(y, t_i^-) = f(0, t_i)$ for $y < 0$, $i = n, n-1, \cdots, 1$.

The parameters are: $r = 0.05$, $q = 0.0$, $\sigma = 0.2$, $T = 1.0$, $t = 0.0$, $n = 10$, $S(0) = 100.0$, $t_i = 0.1i$. MC refers to Monte Carlo solution, and FD refers to finite-difference solution. The different I's refer to the number of time steps *and* also to the number of steps in the y-direction. The Monte Carlo prices are based on 10^5 simulations with a control variate technique. The standard deviation of the Monte Carlo option prices is approximately 3.0×10^{-3}.

α	Floating strike lookback option prices			
	MC	FD ($I = 500$)	FD ($I = 100$)	FD ($I = 50$)
1.000	**10.01**	10.00	9.96	9.86
1.025	8.27	8.26	8.23	8.14
1.050	6.77	6.76	6.74	6.68
1.075	5.51	5.50	5.47	5.43
1.100	**4.46**	4.45	4.42	4.39
1.125	3.59	3.58	3.56	3.53
1.150	2.88	2.87	2.85	2.81
1.175	2.30	2.29	2.28	2.25
1.200	**1.83**	1.82	1.81	1.79
CPU	46.0s	2.43s	0.14s	0.06s

- Option prices drop drastically with increasing value of α.