MAFS 5250

Computational Methods for Pricing Structured Products

Final Examination – 2017 Spring

Time allowed: 2 hours **Course Instructor:** *Prof. Y. K. Kwok*

[points]

1. (a) The Backward-Time Centered-Space finite difference scheme is known to be first order time accurate, where

$$V_j^n(\Delta\tau) = V_j^n(0) + K\Delta\tau + O(\Delta\tau)^2.$$

Here, $V_j^n(0)$ is visualized as the exact solution of the continuous model and K is some constant independent of $\Delta\tau$. Suppose we achieve smooth numerical finite difference solutions that well observe the linear rate of convergence in time. Show how to perform extrapolation to obtain good estimate of $V_j^n(0)$ using numerical solution of $V_j^n(\Delta\tau)$ and $V_j^n\left(\frac{\Delta\tau}{2}\right)$. **[2]**

(b) The nondifferentiability of the terminal payoff of a call/put option may cause erratic convergence of the numerical solution due to quantization error in approximating the terminal payoff numerically. In pricing an American call/put option, discuss an appropriate technique of reducing quantization error arising from nondifferentiability of the terminal payoff. **[2]**

2. Consider the pricing of the participating policy, where the crediting mechanism of the policy value $P(t)$ across the sampling date t is governed by

$$P(t^+) = P(t^-) + \max(r_G P(t^-), \alpha\{[A(t) - P(t^-)] - \gamma P(t^-)\}).$$

Here, α and γ are the parameters in the bonus formula, r_G is the guaranteed minimum return, $A(t)$ is the asset value process.

(a) Let $[P_0, P_{\max}]$ be the computational domain and ΔP be the stepwidth, so that $P_0 = j_0\Delta P$ and $P_{\max} = J_{\max}\Delta P$. In the discretized scheme, we set

$$P(t^-) = j\Delta P,\ A(t) = i\Delta A \ \text{ and } \ P(t^+) = \widetilde{j}\Delta P.$$

(i) Deduce the formula for the jump of the index j to $\widetilde{j}$. **[1]**

(ii) How would you compare this procedure with the correlated evolution function in the Forward Shooting Grid algorithm? **[1]**

(b) Suppose the participating policy allows surrender on a sampling date, how would you construct the corresponding dynamic programming procedure so as to incorporate this surrender feature. **[3]**

Hint Recall that it is optimal to surrender right after the sampling date and the contract value is continuous across the sampling date if the contract remains alive.

(c) Suppose $P(t^+)$ jumps to $\widetilde{j}\Delta P > J_{\max}\Delta P$, where $J_{\max}$ is the upper bound of the index j. For large values of P, the contract value is almost linear in P, so this justifies the use of linear extrapolation. Derive the corresponding linear extrapolation formula in terms of $V_{t,K}^{i,J_{\max}}$ and $V_{t,K}^{i,J_{\max}-1}$. **[2]**

3. The pricing equation for a defaultable convertible bond $V(S,t)$ can be formulated as

$$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2}S^2\frac{\partial^2 V}{\partial S^2} + (r-q+h)S\frac{\partial V}{\partial S} - [r+(1-R)h]V + c(t) = 0,$$

where h is the hazard rate of default, R is the recovery rate and $c(t)$ is the continuous coupon rate.

(a) Let $N(t)$ be the total number of discrete coupons collected from time 0 to t so that

$$\int_0^t c(u)\ \mathrm{d}u = \sum_{i=1}^{N(t)} c_i H(t-t_i),$$

where discrete coupon c_i is received at time t_i, $i = 1, 2, \ldots, N(t)$.

(i) Express $c(t)$ in terms of c_i and t_i, $i = 1, 2, \ldots, N(t)$. **[1]**

(ii) Suppose the explicit finite difference scheme takes the form

$$V_j^{m+1} = P_u V_{j+1}^m + P_m V_j^m + P_d V_{j-1}^m - [r+(1-R)h]V_j^m$$

at a time level that does not correspond to a coupon payment date. Here, $x = j\Delta x$, $\tau = m\Delta\tau$ and $x = \ln S$. How would you add an extra term in the above finite difference scheme in order to incorporate the discrete coupons payment feature? **[2]**

(b) Describe the soft call provision and how it is related to the Parisian feature of knock-out. An extra state variable is needed to capture the excursion time of the stock price shooting above the trigger price B. How would you modify the above finite difference scheme in order to capture the soft call provision? **[4]**

4. Let Σ be a $n \times n$ correlation matrix, which is symmetric and semi-positive definite. We compute the Cholesky decomposition of Σ such that $AA^T = \Sigma$. Suppose n uncorrelated standard normal variables $x_1, x_2, \ldots, x_n$ with zero mean and unit variance have been generated.

Given A, show how to obtain n correlated normal variables with correlated structure that agrees with Σ from the n uncorrelated normal variables generated from Monte Carlo simulation. Check mathematically that the new set of normal variables do have Σ as their correlation matrix. **[3]**

5. The control variate method is based upon using a random variable which is associated with the quality we have to estimate, but which has a known or easily computable expected value, to adjust our estimator and reduce its variance.

Let Z be correlated with the random variable Y whose expectation we wish to find. Assume that $E[Z]$ is known.

Let $\widehat{\theta} = Y$ be the usual estimator and define another estimator

$$\widehat{\theta}_c = Y + c(Z - E[Z])$$

for some number c.

(a) Show that $\text{var}(\widehat{\theta}_c)$ is minimized at $c = c^*$, where

$$c^* = -\frac{\text{cov}(Y, Z)}{\text{var}(Z)}.$$ **[1]**

(b) Find $\text{var}(\widehat{\theta}_{c^*})$ and show that reduction in variance is achieved provided $\text{cov}(Y, Z) \neq 0$. **[2]**

6. Suppose we are interested in computing

$$\theta = E_f[h(X)],$$

where X has a probability distribution f. Let g be another probability distribution with $g(x) \neq 0$ wherever $f(x) \neq 0$. We then have

$$\theta = E_g[h^*(X)],$$

where $h^*(X) = \dfrac{h(X)f(X)}{g(X)}$.

(a) Show that

$$\text{var}_f(h(X)) - \text{var}_g(h^*(X)) = \int h^2(x) \left[1 - \frac{f(x)}{g(x)}\right] f(x)\, \mathrm{d}x.$$ **[2]**

(b) In order to achieve a variance reduction, suppose there is a region $\mathcal{R}$ where $h(x)f(x)$ is large, explain why it is desirable to have a density g that puts more weight on $\mathcal{R}$. **[2]**

7. In the pricing of a fixed strike Asian option with n fixing dates, we define

$$I(t) = \sum_{i=1}^{m(t)} S(t_i),$$

where $m(t) = \sup\{1 \leq i \leq n : t_i \leq t\}$. For a given strike price K, we define

$$x_t = \frac{\frac{1}{n}I(t) - K}{S_t}.$$

(a) Explain why x_t exhibits a jump of amount $\dfrac{1}{n}$ at each fixing date. Show that the crossing of x_t across $x = 0$ can only occur at one of the sampling points. **[2]**

(b) Explain why the discrete fixed strike Asian option can be visualized as an up-and-out barrier option with an upper barrier at $x = 0$ with domain of definition: $\{(x, t) : x < 0 \text{ and } 0 < t < T\}$. Briefly explain the simplification that can be applied to the pricing problem (without detailed derivation of formulas) when $x_t > 0$ at a given t. **[3]**

8. In the discrete fixed strike lookback call option model, we define

$$x(t) = \frac{\overline{S}(t)}{S(t)} \quad \text{for} \quad t \geq t_1,$$

where $\overline{S}(t) = \sup_{1 \leq i \leq m(t)} S(t_i)$, $m(t) = \sup\{1 \leq i \leq n : t_i \leq t\}$.

(a) Explain why

$$x(t_i) = \begin{cases} 1 & \text{if } x(t_{i-}) \leq 1 \\ x(t_{i-}) & \text{if } x(t_{i-}) > 1 \end{cases}.$$ **[1]**

(b) Under the share measure Q', we define

$$f(x(t), t) = E_t^{Q'}[e^{-q(T-t)} x(T) \big| x(t)].$$

Suppose $\overline{S}(t) \geq K$ and $t \geq t_1$, explain why

$$F(S(t), t) = S(t) f(x(t), t) - e^{-r(T-t)} K.$$ **[2]**

(c) Suppose $\overline{S}(t) < K$, explain why the lookback call option can be formulated as an up-and-in barrier option. Let τ^* be the random first passage time defined by

$$\tau^* = \inf_{i=1,2,\ldots,n} \{t_i : \; S(t_i) \geq K\},$$

how to express the lookback option price function as an expectation in terms of the price function in part (b) and τ^*. **[4]**

— End —