MAFS 5250

Computational Methods for Pricing Structured Products

Final Examination – 2015 Spring

Time allowed: 2 hours **Course Instructor:** *Prof. Y. K. Kwok*

[points]

1. (a) There has been a debate on the choice of the state variable in the construction of explicit schemes for solving the Black-Scholes equation: either the choice of S or $\ln S$. Consider the Black-Scholes equation:

$$\frac{\partial V}{\partial \tau} = \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV, \quad S > 0, \tau > 0,$$

find the constraints on the stepwidth ΔS and the time step $\Delta \tau$, respectively, so as to avoid the occurrence of negative coefficients in the resulting two-level four-point scheme when the state variable is chosen to be S. In particular, explain the difficulty encountered when the stock price becomes very small in the computational domain. **[4]**

Hint The constraint on ΔS has dependence on S_i while the constraint on $\Delta \tau$ has dependence on the option model parameters and ΔS.

(b) The Crank-Nicolson scheme appears to have second order temporal truncation error while the fully implicit scheme has only first order temporal truncation error. Explain why both schemes exhibit the same order of convergence when combined spatial and temporal truncation errors are considered together. **[2]**

(c) Suppose we use the fully implicit scheme to solve for an American option model under the Black-Scholes framework, explain clearly why the application of the direct dynamic programming procedure of taking the maximum value among the intrinsic value and continuation value at each node does not work. **[2]**

(d) Consider the numerical pricing of the down-barrier proportional step call option whose terminal payoff is given by

$$\exp(-\rho \tau_B^-) \max(S_T - K, 0),$$

where B is the down-barrier, ρ is the killing rate and τ_B^- is defined by

$$\tau_B^- = \int_0^T H(B - S_t)\, dt.$$

The resulting modified Black-Scholes equation is given by

$$\frac{\partial V}{\partial \tau} = \frac{\sigma^2}{2} \frac{\partial^2 V}{\partial x^2} + \left(r - \frac{\sigma^2}{2} \right) \frac{\partial V}{\partial x} - [r + \rho H(B - S)]\, V = 0, \quad -\infty < x < \infty, \tau > 0,$$

where $x = \ln S$. Derive the explicit two-level four-point scheme for pricing the down-barrier proportional step call option. **[2]**

2. (a) Let c_i and $\widetilde{c}_i$ denote the simulated call value in the i^{th} simulation path. Explain why

$$\text{var}\left(\frac{c_i + \widetilde{c}_i}{2}\right) = \frac{1}{2}[\text{var}(c_i) + \text{cov}(c_i, \widetilde{c}_i)].$$

Explain why the antithetic variates method in pricing a European call option can achieve variance reduction in Monte Carlo simulation. **[3]**

Hint Explain why the above negative correlation property, $\text{cov}(c_i, \widetilde{c}_i) < 0$, is in general valid.

(b) The Monte Carlo simulation method was believed to be unable to price American options due to the nature of forward simulation of the asset price paths. *Outline* the key steps in the linear regression method in pricing an American option, in particular, stating the technique used in the determination of the optimal stopping rule. **[4]**

3. In the paticipating policy model, there are two state variables. The dynamics of the asset $A(t)$ under a risk neutral measure Q is governed by the Geometric Brownian motion:

$$dA(t) = rA(t)\ dt + \sigma A(t)\ dW^Q(t).$$

The updating of the policy account $P(t)$ is based on

$$P(t^+) = P(t^-) + \max(r_G P(t^-), \alpha\{[A(t) - P(t^-)] - \gamma P(t^-)\})$$

across a fixing date, where

r_G : guaranteed rate of return
γ : target buffet ratio
α : distribution ratio.

Explain the key considerations in the incorporation of the jump condition in $P(t)$ (crediting mechanism) in the finite difference calculations. **[5]**

Hint Write $P(t^+) = \widetilde{j}\Delta P$ and $P(t^-) = j\Delta P$. Let $A = i\Delta A$, where ΔA is the stepwidth for A. Explain why

$$\widetilde{j} = j + \max\left\{r_G j, \alpha\left[\left(i\frac{\Delta A}{\Delta P} - j\right) - j\gamma\right]\right\}.$$

In particular, describe the interpolation scheme adopted when $0 < \widetilde{j} \leq J$, where J is the total number of discretization steps in P. How do we implement the extrapolation scheme when $\widetilde{j} > P$?

4. In the numerical evaluation of the swap on discrete realized variance, the procedure reduces to evaluating N expectations of squared returns of the form

$$E_Q\left[\left(\frac{S_T - S_{T-\widetilde{\Delta}}}{S_{T-\widetilde{\Delta}}}\right)^2\right]$$

for some fixed time interval $\widetilde{\Delta}$ and $T = i\widetilde{\Delta}$, $i = 1, 2, \ldots, N$. The path dependence on $S_{T-\widetilde{\Delta}}$ is captured by a path dependent state variable I_t with the property that

$$I_t = \begin{cases} 0 & t < T - \widetilde{\Delta} \\ S_{T-\widetilde{\Delta}} & t \geq T - \widetilde{\Delta} \end{cases}.$$

Explain how to define I_t via the Dirac function and derive the corresponding pricing equation for the contingent claim. What is the terminal condition in the pricing model? **[4]**

Hint Be careful that discounting is not required when we compute the fair strike of a forward contract.

5. (a) Describe the hard call protection clause and soft call protection clause in a convertible bond and explain the purposes of embedding these provisions in the convertible bond contract. **[3]**

(b) The pricing equation for a defaultable convertible bond can be formulated as

$$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2}S^2\frac{\partial^2 V}{\partial S^2} + (r - q + h)S\frac{\partial V}{\partial S} - [r + (1 - R)h]V + c(t) = 0,$$

where h is the hazard rate of default and R is the recovery rate.

(i) Explain using financial argument why the drift rate is modified to $r - q + h$, where h can be interpreted as the additional negative dividend yield. **[1]**

(ii) Note that the risky discount rate becomes $r + (1 - R)h$. Explain why the bond's credit spread is given by $(1 - R)h$. Give your comment on the calibration of R and h. **[2]**

(iii) Recall that the coupon rate $c(t)$ can be used to model the set of discrete coupon payments c_i paid at the coupon date t_i, $i = 1, 2, \ldots, N$, by setting

$$c(t) = \sum_{i=1}^{N} c_i \delta(t - t_i).$$

Explain the modeling consideration behind the above formulation. Explain how do we incorporate the jump condition across a discrete coupon payment date t_i, $i = 1, 2, \ldots, N$, in a finite difference scheme. **[3]**

6. Consider the pricing of a discretely monitored fixed strike Asian call option with fixing dates: $t_1, t_2, \ldots, t_n$. Define the running sum of the discretely sampled stock prices up to time t by

$$I(t) = \sum_{i=1}^{m(t)} S(t_i),$$

where $m(t) = \sup\{1 \leq i \leq n; t_i \leq t\}$. Define the Markov stochastic process

$$x_t = \frac{\frac{1}{n}I(t) - K}{S_t},$$

where K is the fixed strike of the Asian call option.

(a) Explain why x_t jumps by a deterministic amount $\frac{1}{n}$ when the calendar time moves across a fixing date. **[1]**

(b) Explain why the crossing of x_t across $x = 0$ can only occur at one of the fixing dates. **[1]**

(c) Determine the terminal payoff of the fixed strike Asian call option in terms of X_T. Explain why the Asian option loses optionality and closed form price formula can be derived once X_t shoots above the zero value. Briefly explain why this property of loss in optionality is highly relevant in the design of the numerical algorithm. **[3]**

— *End* —