MAFS 5250

Computational Methods for Pricing Structured Products

Final Examination – 2016 Spring

Time allowed: 2 hours **Course Instructor:** *Prof. Y. K. Kwok*

[points]

1. (a) Deduce the implicit backward finite difference scheme in the *interior* computational nodes for solving the Black-Scholes equation for the European call price function:

$$\frac{\partial V}{\partial \tau} = \frac{\sigma^2}{2}\frac{\partial^2 V}{\partial x^2} + \left(r - \frac{\sigma^2}{2}\right)\frac{\partial V}{\partial x} - rV, \ -\infty < x < \infty, \ \tau > 0.$$

Here, x is the logarithm of the stock price. **[2]**

(b) By observing the following boundary conditions: $\dfrac{\partial^2 V}{\partial x^2} \to 0$ as $x \to \infty$.

Derive the nodal equations at the right boundary nodes. Note that fictitious nodal values may be involved in the derivation procedure. **[2]**

(c) Quote the advantages of the fully implicit scheme over the fully explicit scheme and the Crank-Nicolson scheme. **[2]**

2. We consider the pricing of the continuously monitored European floating strike lookback call option. Let $c(S, m, t)$ denote the time-t value of the call option of stock price S and realized minimum stock price m.

(a) Explain *in details* why the boundary condition at $S = m$ is prescribed as

$$\frac{\partial c}{\partial m} = 0 \quad \text{at} \quad S = m.$$

[2]

(b) Suppose we choose the following set of similarity variables:

$$x = \ln\frac{S}{m} \quad \text{and} \quad V(x, \tau) = \frac{c(S, m, t)}{S}e^{-q\tau}, \ \tau = T - t.$$

The governing equation for the European floating strike lookback call option is given by

$$\frac{\partial V}{\partial \tau} = \frac{\sigma^2}{2}\frac{\partial^2 V}{\partial x^2} + \left(r - q + \frac{\sigma^2}{2}\right)\frac{\partial V}{\partial x}, \quad x > 0, \quad \tau > 0.$$

Note that you are **NOT** required to derive the above equation.

The “initial” condition at $\tau = 0$ is

$$V(x, 0) = 1 - e^{-x}, \quad x > 0;$$

while the Neumann boundary condition at $x = 0$ is

$$\frac{\partial V}{\partial x}(0, \tau) = 0, \quad \tau > 0.$$

Show that the explicit forward time finite difference scheme is given by

$$V_j^{n+1} = \frac{\alpha+\mu}{2}V_{j+1}^n + (1-\alpha)V_j^n + \frac{\alpha-\mu}{2}V_{j-1}^n$$

for some parameters α and μ. Express these two parameters in terms of r, q, σ^2, $\Delta\tau$ and Δx. **[2]**

(c) Deduce the numerical boundary condition at $x = 0$ under two scenarios:

(i) The boundary $x = 0$ is placed along a vertical layer of nodes. **[1]**

(ii) The boundary $x = 0$ is placed between two vertical layers of nodes. **[1]**

3. (a) Explain why we cannot apply the direct dynamic programming procedure for pricing an American option when an implicit finite difference scheme is used. **[2]**

(b) Suppose the implicit finite difference scheme takes the form

$$a_{-1}V_{j-1} + a_0V_j + a_1V_{j+1} = d_j, \quad j = 1, 2, \ldots, N.$$

Discuss the Projected Successive-Over-Relaxation (PSOR) procedure via the use of the Gauss-Seidel iterative scheme for pricing an American option. What is the role of the relaxation parameter ω in the PSOR procedure? **[4]**

4. (a) In the pricing model of a participating policy, explain why the value function of the participating policy has no jump while the policy account value faces discrete upward jump on the dates of bonus distribution. Is it paradoxical? Explain your answer. **[1]**

(b) Recall that the policy holder has the surrender right on the discrete dates of bonus distribution. What are the time instants at which it is optimal for the policyholder to exercise the surrender right? Give your explanation. **[1]**

5. It is claimed in the lecture note that the two-step procedure in pricing the discrete variance swap using finite difference calculations is equivalent to apply the tower rule in iterated expectation:

$$E_0^Q\left[\left(\frac{S_T - S_{T-\widetilde{\Delta}}}{S_{T-\widetilde{\Delta}}}\right)^2\right] = E_0^Q\left[E_{T-\widetilde{\Delta}}^Q\left(\frac{S_T - S_{T-\widetilde{\Delta}}}{S_{T-\widetilde{\Delta}}}\right)^2\right]. \qquad \text{(A)}$$

(a) Suppose we discretize the log-stock price dimension by $2\widetilde{M}$ grids in the finite difference scheme for solving the Black-Scholes equation. Explain why it is necessary to solve $2\widetilde{M}+1$ one-dimensional Black-Scholes equation over $(T-\widetilde{\Delta}, T]$. **[2]**

(b) Over the earlier time interval $[0, T-\widetilde{\Delta})$, we need to prescribe the terminal values of the contingent claim at $(T-\widetilde{\Delta})^-$ in order to perform the backward induction procedure starting at $(T-\widetilde{\Delta})^-$ and ending at time 0. How to extract the required terminal values from the solution of the $2\widetilde{M}+1$ one-dimensional problems over $(T-\Delta, T]$? Justify your answer. **[3]**

(c) How to interpret the two-step finite difference calculations as performing the evaluation of the iterated expectation formula (A) stated in the above? **[1]**

6. (a) Explain why the convertible bond value is less sensitive to fluctuation of the interest rate when compared to its non-convertible counterpart. **[1]**

   (b) Explain why we are interested to perform numerical valuation of the convertible bond value only in the domain of the stock price where the conversion value is less than the call price. **[2]**

   (c) Based on the observation in part(b), what would be the simplified version of the dynamic programming procedure at each node that takes care of the game between optimal holder's conversion and issuer's call? **[1]**

   (d) Would the issuer tend to delay call or choose to call earlier in the presence of the notice period requirement? Justify your answer. **[1]**

   (e) With discrete coupon payments received by the convertible bondholder, would the bondholder exercise the conversion privilege at time right before or right after a coupon payment date? **[1]**

7. (a) In the calculation of the delta of the Black-Scholes call price, explain why it is numerically unstable (highly susceptible to roundoff errors) to use the following forward difference formula:

$$\Delta \approx \frac{V(S+\epsilon, t) - V(S,t)}{\epsilon}, \quad \epsilon \approx 0.$$ **[1]**

   (b) Suppose we are interested in computing

$$\theta = E_f[h(X)],$$

   where X has a probability distribution function f. In the importance sampling procedure, we consider the expectation calculation of θ under another probability distribution g, where

$$\theta = E_g\left[h(X)\frac{f(X)}{g(X)}\right], \quad \text{where } h^*(x) = h(x)\frac{f(x)}{g(x)}.$$

   By considering the difference of the variances of the estimators:

$$\mathrm{var}_f(h(X)) - \mathrm{var}_g(h^*(X)),$$

   discuss the conditions on the choice of $g(x)$ such that a reduction in variance is achieved. **[3]**

   (c) (a) Explain the intrinsic difficulty in the use of Monte Carlo simulation method to price American options. **[1]**

   (b) Compare and contrast briefly the two effective methods for numerical valuation of American options using Monte Carlo simulation methods:

   (i) Method of parametrization of the early exercise boundary, and

   (ii) Linear regression method via basis functions. **[3]**

— *End* —