MAFS 5250

Computational Methods for Pricing Structured Products

Final Examination – 2018 Spring

Time allowed: 2 hours **Course Instructor:** *Prof. Y. K. Kwok*

[points]

1. The temporal derivative $\dfrac{\partial V}{\partial \tau}$ in the Black-Scholes equation is approximated by the finite difference $\dfrac{V_j^{n+1} - V_j^n}{\Delta\tau}$, where it can be interpreted as the approximation at $(j\Delta x, n\Delta\tau)$ or $(j\Delta x, (n+1)\Delta\tau)$, where $x = \ln S$ and $\tau = T - t$.

   (a) Derive the corresponding two-level four-point explicit scheme and two-level four-point implicit scheme accordingly. Do these two schemes have the same order of accuracy? Explain your answer. Why we normally choose $\Delta\tau = \mathrm{O}(\Delta x^2)$? **[2]**

   (b) Explain why the information of the numerical boundary values can be taken up immediately by the numerical option values at the same time level in an implicit scheme while an explicit scheme falls short of this desirable feature. **[1]**

   (c) Apparently, the computational efforts of implementing the numerical calculations using the implicit scheme requires the solution of a tridiagonal system of equations. Explain why this does not pose computational hurdle when compared with the computational efforts of implementing the explicit scheme calculations. Briefly explain the fundamental logic in the Thomas algorithm in solving a tridiagonal system. You are not required to present the details of the Thomas algorithm. **[2]**

   (d) When one tries to use the implicit scheme to solve numerically an American option pricing model, explain why the direct dynamic programming procedure of getting the maximum among the numerical value obtained from solving the tridiagonal system of equations and the exercise payoff fails. **[1]**

2. This problem deals with the construction of the numerical boundary condition in the fully implied scheme used for solving the continuously monitored floating strike lookback call option. Let $c(S, m, \tau)$ denote the price of the floating strike lookback call option. Here, S is the asset price and m is the realized minimum asset price. The governing equation of $c(S, m, t)$ is given by

$$\frac{\partial c}{\partial \tau} = \frac{\sigma^2}{2} S^2 \frac{\partial^2 c}{\partial S^2} + (r - q) S \frac{\partial c}{\partial S} - rc,\ S \geq m,\ \tau > 0,\ \tau = T - t$$

   with auxiliary conditions:

$$\left.\frac{\partial c}{\partial m}\right|_{S=m} = 0 \text{ and } c(S, m, 0) = S - m.$$

   Here, σ is the volatility, q is the dividend yield and r is the interest rate.

   (a) Explain why the governing differential equation for $c(S, m, t)$ does not involve $\dfrac{\partial c}{\partial m}$. **[2]**

Hint: First, justify why $\left.\frac{\partial c}{\partial m}\right|_{S=m} = 0$, then explain why $\frac{\partial c}{\partial m}dm = 0$.

(b) Suppose we use the following transformation of variables:

$$x = \ln \frac{S}{m} \text{ and } V(x, \tau) = \frac{c(S, m, \tau)}{S} e^{q\tau},$$

the governing equation for $V(x, \tau)$ becomes

$$\frac{\partial V}{\partial \tau} = \frac{\sigma^2}{2} \frac{\partial^2 V}{\partial x^2} + \left(r - q + \frac{\sigma^2}{2}\right) \frac{\partial V}{\partial x}, \quad x > 0, \ \tau > 0.$$

Note that $S > m$ is equivalent to $x = 0$. Using the auxiliary condition:

$$\left.\frac{\partial c}{\partial m}\right|_{S=m} = 0,$$

show that

$$\frac{\partial V}{\partial x}(0, \tau) = 0.$$ **[2]**

(c) Suppose the fully implicit scheme is used to solve the lookback option model, find the coefficients in the following two-level four-point scheme in terms of r, q and $\frac{\sigma^2}{2}$:

$$V_j^n = a_{-1} V_{j-1}^{n+1} + a_0 V_j^{n+1} + a_1 V_{j+1}^{n+1}.$$ **[1]**

(d) Suppose the boundary $x = 0$ is placed at the grid $x = x_0$. Derive the reduced finite difference scheme that involves V_1^{n+1} and V_2^{n+1} using the numerical boundary condition that approximates the Neumann boundary condition: $\frac{\partial V}{\partial x}(0, \tau) = 0$. **[1]**

(e) Recall the poor rate of convergence of the Cheuk-Vorst binomial scheme, where

$$V_j^n = \begin{cases} e^{-r\Delta t}[(1-p)dV_{j+1}^{n+1} + puV_{j-1}^{n+1}], & j \geq 1; \\ e^{-r\Delta t}[(1-p)dV_{j+1}^{n} + puV_{j}^{n+1}], & j = 0. \end{cases}$$

Here, d and u are proportional upward and downward jump in the binomial scheme, V_j^n is the numerical lookback option value normalized by the stock price. What is the finite difference approximation used by Cheuk and Vorst in approximating the Neumann boundary condition? Give your explanation of the poor rate of convergence of their scheme. **[2]**

3. Consider the numerical algorithm for pricing the participating policy, where the crediting mechanism of the policy value $P(t)$ across the sampling date t is governed by

$$P(t^+) = P(t^-) + \max(r_G P(t^-), \alpha[A(t) - P(t^-)] - \gamma P(t^-)).$$

Here, α and γ are the parameters in the bonus formula, r_G is the guaranteed minimum return, and $A(t)$ is the asset value process. Over the time interval $(t, t+1)$, we let $V_{t,k}^{i,j}$ denote the numerical option value of $V(t + 1 - k\Delta s, i\Delta A, j\Delta P)$, where Δs is the time step, ΔA and ΔP are the step width of $A(s)$ and $P(t)$, respectively. Note that $A(s)$ evolves according to the Geometric Brownian motion, $P(t)$ is updated only on sampling date.

(a) Suppose we use the fully implicit scheme, where

$$E^i V_{t,k+1}^{i-1,j} + H^i V_{t,k+1}^{i,j} + G^i V_{t,k+1}^{i+1,j} = V_{t,k}^{i,j},$$

$i = 1, 2, \ldots, I-1$. For the boundary nodal values, $V_{t,k+1}^{0,j}$ and $V_{t,k+1}^{I,j}$, we need to apply numerical boundary conditions to eliminate them so that the first equation and the last equation in the tridiagonal system of equations involve only two unknowns. From the following boundary conditions:
(i) $\dfrac{\partial V}{\partial s} - rV = 0$, (ii) $\dfrac{\partial^2 V}{\partial A^2} \to 0$ as $A \to \infty$;
explain why we can obtain

$$\begin{aligned} V_{t,k+1}^{0,j} &= (1 - r\Delta s) V_{t,k}^{0,j} \\ V_{t,k+1}^{I,j} &= 2V_{t,k+1}^{I-1,j} - V_{t,k+1}^{I-2,j}. \end{aligned}$$

use the above two relations to derive the modified finite difference scheme at $i = 1$ and $i = I - 1$. **[3]**

(b) Let $[P_0, P_{\max}]$ be the computational domain for P, where $P_0 = j_0 \Delta P$ and $P_{\max} = J_{\max} \Delta P$. Let $P(t^+) = \widetilde{j}\Delta P$ and $P(t^-) = j\Delta P$.

(i) Use the crediting mechanism formula to relate $\widetilde{j}$ and j. **[1]**

(ii) Let $\underline{j}$ denote the floor value of $\widetilde{j}$. Explain why when $\underline{j} + 1 \leq J_{\max}$, we have

$$V_{t-1,0}^{i,j} = V_{t,K}^{i,\widetilde{j}} \approx [1 - (\widetilde{j} - \underline{j})] V_{t,K}^{i,\underline{j}} + (\widetilde{j} - \underline{j}) V_{t,K}^{i,\underline{j}+1},$$

where K denotes the total number of time steps between t and $t+1$. Modify the above jump condition across the sampling date t when $\underline{j} + 1 > J_{\max}$. **[3]**

Hint: The first case corresponds to interpolation while the second case requires extrapolation.

4. This question addresses several issues in pricing formulation and numerical implementation of various features in defaultable convertible bonds.

(a) Why we prefer to use stock price instead of firm value as the underlying state variable in the model formulation? **[1]**

(b) Why randomness in interest rate plays a secondary role in convertible bonds compared to their non-convertible counterparts? **[1]**

Hint: Bond floor can be visualized as strike.

(c) What is the role of the soft call requirement and how is it related to the Parisian feature? **[1]**

(d) How to price in the 30-day notice period requirement by modifying the call price? **[1]**

(e) Let h denote the hazard rate of arrival of default and R be the recovery rate upon default. To model default risk of the bond insurer, we modify the discount rate from the riskfree rate r to the risky discount rate $r + (1 - R)h$. Also, the modified drift rate becomes $r - q + h$, where h is visualized as negative dividend yield. Provide justification of these two modified terms in the pricing formulation. **[2]**

5. We consider the calculation of the delta of the Black-Scholes call price using the Monte Carlo simulation method.

(a) A straightforward approach to estimate delta Δ is to compute the call option price at two initial prices $S+h$ and $S-h$, and use the centered difference formula:

$$\Delta \approx \frac{c(S+h,t)-c(S-h,t)}{2h}.$$

Explain why numerical accuracy is highly degraded by looking at (i) finite precision arithmetics on computer calculations, (ii) magnification of standard errors in Monte Carlo simulation when h is small. **[1]**

(b) A better method is to use the same set of simulated normal random variables ϵ_j, $j=1,2,\ldots,N$, to generate simulated terminal stock price S_T, where $N\Delta t = T$. Based on two different initial stock prices S_0 and $S_0+\epsilon$, we obtain

$$\begin{aligned} S_T &= S_0 e^{(r-\frac{\sigma^2}{2})T+\sigma\sqrt{\Delta t}\sum_{j=1}^N \epsilon_j}, \\ S_T(\epsilon) &= (S_0+\epsilon)e^{(r-\frac{\sigma^2}{2})T+\sigma\sqrt{\Delta t}\sum_{j=1}^N \epsilon_j}. \end{aligned}$$

Let $\hat{c}(S_0+\epsilon)$ and $\hat{c}(S_0)$ be the respective simulated call price based on $S_0+\epsilon$ and S_0. Show that

$$|\hat{c}(S_0+\epsilon)-\hat{c}(S_0)| \le |S_T(\epsilon)-S_T| = \epsilon e^{(r-\frac{\sigma^2}{2})T+\sigma\sqrt{\Delta t}\sum_{j=1}^N \epsilon_j}.$$

Explain why

$$\mathrm{var}\left(\frac{\hat{c}(S_0+\epsilon)-\hat{c}(S_0)}{\epsilon}\right) = \mathrm{O}(1).$$

That is, the variance of the estimated delta $\widetilde{\Delta} = \dfrac{\hat{c}(S_0+\epsilon)-\hat{c}(S_0)}{\epsilon}$ remains bounded as $\epsilon \to 0$. **[4]**

Hint: Consider the various cases where the call option may expire in-the-money or out-of-the-money for differing values of $S_T(\epsilon)$ and S_T.

6. The control variate method attempts to reduce the variance of the estimate value $\widehat{V}_A$ of option A based on estimate value $\widehat{V}_B$ and known analytic value V_B of another similar option B. Suppose we use $V_B - \widehat{V}_B$ as control and define the control variate estimate to be

$$\widehat{V}_A^\beta = \widehat{V}_A + \beta(V_B - \widehat{V}_B).$$

(a) What is the rationale of this control variate method? **[1]**

(b) Determine the optimal control β^* that minimizes $\mathrm{var}(\widehat{V}_A^\beta)$. Explain why the control variate estimate using this optimal control β^* is guaranteed to decrease variance. **[3]**

7. (a) It had been commonly believed that the Monte Carlo simulation method cannot be used to price American options. Explain the nature of computational challenge that has to be overcomed. **[1]**

(b) Briefly explain the numerical procedure in the parametrization of the early exercise boundary in the Grant-Vora-Weeks algorithm when the Monte Carlo simulated paths have been generated. **[3]**

— *End* —