MAFS 5250 – Computational Methods for Pricing Structured Products

Mid-term Test, 2016

Time allowed: 90 minutes *Instructor: Prof. Y. K. Kwok*

[points]

1. Consider the construction of the binomial tree when the underlying asset pays single discrete dividend D between the $(k-1)^{\text{th}}$ and k^{th} time step. The asset price drops by the same amount as the dividend right after the dividend payment.

   (a) Explain why at the $(k+m)^{\text{th}}$ time steps, the number of nodes would be $(m+1)(k+1)$ rather than $k+m+1$ nodes as in the ususal reconnecting binomial tree. **[3]**

   (b) Describe the technique of splitting the asset price into the deterministic dividend component and risky component so that a reconnecting binomial tree can be constructed. Give details on the construction of the binomial tree where discrete dividend D is paid between the $(k-1)^{\text{th}}$ and k^{th} time step. **[4]**

2. Recall that the adjusted strike method $[K'(t_j) = \max(K, \overline{M}(t_j))]$ proposed by Cheuk and Vorst cannot be used to price an American fixed strike lookback call option. Explain why. **[4]**

   Hint Cheuk and Vorst argue that a guaranteed extra payment is received at maturity for a European fixed strike lookback call option whenever a newly updated maxima of the asset price is realized. However, in the American option counterpart, the date of receiving the guaranteed payment is uncertain due to potential early exercise of the American option. Explain why the dimension reduction technique of using the adjusted strike fails in the American option.

3. For a given percentile α, $0 \leq \alpha \leq 1$, the α-quantile of the asset price process $\{S_t\}_{t\in[0,T]}$ is defined by

$$B_{\text{inf}}(T;\alpha) = \inf\left\{B : \frac{1}{T}\int_0^T \mathbf{1}_{\{S_t \leq B\}}\, dt \geq \alpha\right\}.$$

   (a) Explain why $B_{\text{inf}}(T;1)$ gives the realized maximum asset price over $[0,T]$. **[1]**

   (b) The numerical approximate value of the continuously monitored European α-quantile call option is given by

$$V_\alpha(S,0) = e^{-rT}\sum_{j=-N}^{N} P[B_{\text{inf}} = S_j]\max(S_j - X, 0),$$

   where $S_j = S_0 u^j$ and X is the strike price.

   Define $V_{\text{cum}}^{\text{bin}}(\alpha, B)$ be the value of the binary option that pays \$1 at maturity if the cumulative time staying at or below the down-barrier B is less than α of the total life of the option, $0 \leq \alpha \leq 1$. Explain why

$$e^{-rT}P[B_{\text{inf}} = S_j] = V_{\text{cum}}^{\text{bin}}(\alpha, S_{j-1}) - V_{\text{cum}}^{\text{bin}}(\alpha, S_j). \quad \textbf{[3]}$$

   Hint $V_{\text{cum}}^{\text{bin}}(\alpha, S_j) = e^{-rT}P[B_{\text{inf}} > S_j]$.

(c) Describe the forward shooting grid algorithm that computes $V_{\text{cum}}^{\text{bin}}(\alpha, B)$. How to prescribe the terminal payoff of $V_{\text{cum}}^{\text{bin}}(\alpha, B)$? **[4]**

4. Recall that delivery of the accumulated shares in an accumulator is 3 trading days at the end of the accumulation period or the date of knock-out on which the stock price breaches the upper strike price.

   (a) Explain why the forward shooting grid algorithm is an effective approach to price an accumulator in response to the uncertainty in the knock-out date. **[2]**

   (b) Describe the backward induction procedure in the forward shooting grid algorithm based on the binomial tree and discuss the jump condition on the accumulator value across the ending date of an accumulation period. **[2]**

   (c) How to set the possible values of accumulated shares on the ending date of an accumulation period. **[1]**

5. In the Derman-Kani tree, the conditional probability

$$P_{i+1}^n = P[S((n+1)\Delta t) = S_{n+1}^{i+1} | S(n\Delta t) = S_n^i]$$

   is the risk neutral transition probability of making a transition from node (n, i) to node $(n+1, i+1)$. The forward price at level $n+1$ of S_n^i at level n is $F_n^i = e^{r\Delta t} S_n^i$.

   (a) Relate P_{i+1}^n with F_n^i, S_{n+1}^i and S_{n+1}^{i+1}. **[2]**

   (b) The Arrow-Debreu price λ_n^i is defined by

$$\lambda_n^i = e^{-rn\Delta t} E[\mathbf{1}_{\{S(n\Delta t) = S_n^i\}} | S(0) = S_0].$$

   Explain why

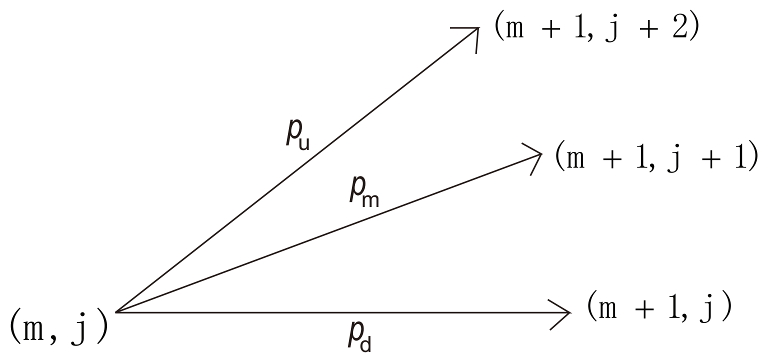
$$\lambda_{n+1}^{i+1} = e^{-r\Delta t}[\lambda_n^i P_{i+1}^n + (1 - P_{i+2}^n)\lambda_n^{i+1}], \ i = 0, 1, \dots, n-1.$$

   Find the corresponding formulas for λ_{n+1}^0 and λ_{n+1}^{n+1}. **[3]**

6. In the construction of the Hull-white implied interest rate tree, we start with the building of the tree for R^* that satisfies the following mean reversion process

$$dR^* = -\alpha R^* \, dt + \sigma \, dZ, \quad \alpha > 0.$$

   Consider the upward branching trinomial tree where



   The (m, j)-node at the m^{th} time step and j^{th} R^*-level moves to $(m+1, j+2)$, $(m+1, j+1)$ and $(m+1, j)$ nodes with respective probabilities p_u, p_m and p_d in the next time step.

(a) Explain why upper branching of the implied interest rate is required in the construction of the discrete tree of R^*. **[1]**

(b) By equating the mean and variance of the continuous process and discrete counterpart of R^*, find the governing equations for p_u, p_m and p_d. You are **NOT** required to solve them. **[3]**

(c) The key equation for the determination of α_m in the fitting of the discount bond prices is given by

$$P_{m+1} = \sum_{j=-n_m}^{n_m} Q_{m,j} \exp(-(\alpha_m + j\Delta R)\Delta t), \tag{A}$$

where n_m is the number of nodes on each side of the centered node at time $m\Delta t$, P_{m+1} is the discount bond price maturing at time $(m+1)\Delta t$ and $Q_{m,j}$ is the Arrow-Debreu price at the $(m,j)^{\text{th}}$ node. Derive eq.(A) based on the following definition of P_{m+1}, where

$$P_{m+1} = E[D(t_0, t_{m+1})|\mathcal{F}_{t_0}],$$

where $D(t_0, t_{m+1})$ is the discount factor from t_0 to t_{m+1}. **[4]**

Hint Use the tower rule in nested expectation and observe

$$Q_{m,j} = E\left[D(t_0, t_m)\mathbf{1}_{\{R(t_m)=\alpha_m+j\Delta R\}}|\mathcal{F}_{t_0}\right].$$

(d) Deduce the recursive relation that relates $Q_{m+1,j}$ and $Q_{m,k}$ in terms of $q(k,j)$, where $q(k,j)$ is the probability of moving from node (m,k) to node $(m+1,j)$. The summation is taken over all values of k for which $q(k,j)$ is nonzero. **[3]**

— *End* —