MAFS 5250 – Computational Methods for Pricing Structured Products

Mid-term Test, 2018

Time allowed: 90 minutes *Instructor: Prof. Y. K. Kwok*

[points]

1. The inherent difficulty to include discrete dividends in the binomial tree is that the number of nodes at maturity in the binomial tree increases as power of number of discrete dividends plus one. Suppose there is no discrete dividend, then the number of nodes at maturity is linear in n, where n is the number of time steps. Now, suppose a discrete dividend is paid between the $(k-1)^{\text{th}}$ and k^{th} time step. Explain why at the $(k+m)^{\text{th}}$ time step, the number of nodes would be $(m+1)(k+1)$ instead of $k+m+1$ nodes as in the usual recombining binomial tree. **[3]**

2. Consider the dynamic programming procedure for pricing a callable American call option, where K is the fixed call price and the exercise payoff is $S-X$. Recall that the continuation value at the (n,j) node is given by

$$(V_{\text{cont}})_j^n = \frac{pC_{j+1}^{n+1} + (1-p)C_{j-1}^{n+1}}{R},$$

where C_j^n is the American call value at the (n,j) node, R is the one-period discount factor and p is the probability of upward jump.

   (a) Explain why the callable American call value must be bounded between the call price K and the exercise payoff $S-X$. **[1]**

   (b) Explain why the most simplified dynamic programming procedure is given by

$$C_j^n = \min\left(\max\left((V_{\text{cont}})_j^n, S_j^n - X\right), K\right).$$

   Give your financial interpretation of the above procedure. **[3]**

   Hint In the lecture note, we derive the dynamic programming procedure based on the argument that the issuer chooses to call or restrain from calling so as to minimize the option value with reference to the two possible actions of the holder. Simplify this dynamic programming procedure using the result in part (a).

3. In the Cheuk-Vorst algorithm of pricing a European fixed strike lookback call option, we define the adjusted exercise price $K'(t_j)$, where

$$K'(t_j) = \max(\overline{M}(t_j), K).$$

Here, K is the strike price and $\overline{M}(t_j)$ is the realized maximum up to time t_j (known quantity at t_j). Show that the terminal payoff at t_N can be decomposed into

$$\max(\overline{M}(t_j) - K, 0) + \max(M(t_N; t_{j+1}) - K'(t_j), 0).$$

Here, $M(t_N; t_{j+1})$ is the future realized maximum between t_{j+1} and t_N. What is the interpretation of the decomposition, in particular, distinguish the two cases (i) $\overline{M}(t_j) \leq K$, and (ii) $\overline{M}(t_j) > K$? **[4]**

4. We design the forward shooting grid algorithm for pricing a call option with the strike reset feature. There are M reset dates, where on each of these reset dates t_i, $i = 1, 2, \ldots, M$, the call option's strike price is reset to the prevailing asset price S_{t_i} at t_i if the option is out-of-the-money at t_i.

   (a) Let X_0 denote the strike price set at initiation of the contract and X_i be the strike price set at t_i, $i = 1, 2, \ldots, M$. Explain how to incorporate this strike reset feature in the design of the forward shooting grid algorithm. Provide details on how to set X_0 as one of the nodal asset values in the trinomial tree and the design of the grid function for updating the strike price. **[3]**

   (b) Under certain condition, we can show that this strike reset call option resembles the discretely monitored floating strike lookback call option. Find the relation between these two call options. Give an explanation to your answer. **[2]**

5. For a given percentile α, $0 \leq \alpha \leq 1$, the α-quantile of $\{S_t\}_{t \in [0,T]}$ is defined by

$$B_{\inf}(T; \alpha) = \inf \left\{ B : \frac{1}{T} \int_0^T \mathbf{1}_{\{S_t \leq B\}} \mathrm{d}t \geq \alpha \right\}.$$

   Suppose a binary option that pays \$1 at maturity T if the cumulative time staying at or below the down-barrier B is less than α portion of the total life of the option, $0 \leq \alpha \leq 1$; otherwise the terminal payoff of the option is zero.

   Explain how to use the forward shooting grid algorithm to price this binary option. Specify the terminal payoff of this binary option in terms of α and the index in the grid function that counts the cumulative time. **[4]**

6. In the Derman-Kani algorithm for constructing the implied volatility tree, we define λ_n^i by

$$\lambda_n^i = e^{-rn\Delta t} E \left[\mathbf{1}_{\{S(n\Delta t) = S_n^i\}} \middle| S(0) = S_0 \right].$$

   (a) Starting with $\lambda_0^0 = 1$, explain why the successive iterates can be generated by

$$\begin{aligned} \lambda_{n+1}^0 &= e^{-r\Delta t} \left[\lambda_n^0 (1 - P_1^n) \right] \\ \lambda_{n+1}^{i+1} &= e^{-r\Delta t} \left[\lambda_n^i P_{i+1}^n + \lambda_n^{i+1} (1 - P_{i+2}^n) \right] \\ \lambda_{n+1}^{n+1} &= e^{-r\Delta t} \lambda_n^n P_{n+1}^n, \end{aligned}$$

   where P_{i+1}^n is the risk neutral transition probability of making the transition from node (n, i) to $(n+1, i+1)$. **[3]**

   (b) Let F_n^i be the forward price maturity at level $n+1$ of the nodal asset value S_n^i at the current level n. Find F_n^i in terms of S_n^i. Explain why F_n^i and P_{i+1}^n are related by

$$P_{i+1}^n = \frac{F_n^i - S_{n+1}^i}{S_{n+1}^{i+1} - S_{n+1}^i}.$$

   Explain why $F_n^i > S_{n+1}^{i+1}$ must be ruled out by arbitrage argument. **[2]**

7. In the Hull-White implied interest rate tree, the Δt-period rate R at the node (m, j) is $\alpha_m + j\Delta R$. Define Q_{ij} to be the discrete Arrow-Debreu price of the node (i, j). Let $D(t_0, t_{m+1})$ be the discount factor from t_0 to t_{m+1}.

   (a) Explain why

(i) $E\left[D(t_0,t_m)\mathbf{1}_{\{R(t_m)=\alpha_m+j\Delta R\}}\middle|\,\mathcal{F}_{t_0}\right] = Q_{m,j}$;

(ii) $E\left[D(t_m,t_{m+1}|\,R(t_m) = \alpha_m + j\Delta R\right] = \exp(-(\alpha_m + j\Delta R)\Delta t)$. **[2]**

(b) Recall the following relation:

$$\begin{aligned} P_{m+1} &= E[D(t_0,t_{m+1}|\mathcal{F}_{t_0})] \\ &= E[E[D(t_0,t_m)D(t_m,t_{m+1})|\mathcal{F}_{t_m}]|\mathcal{F}_{t_0}], \end{aligned}$$

where P_{m+1} is the price of a zero-coupon bond maturing at time $(m+1)\Delta t$. Find P_{m+1} in terms of $Q_{m,j}$ and $\alpha_m + j\Delta R$, $j = -n_m, -n_m+1, \ldots, n_m$, where n_m is the number of nodes on each side of the centered node at time $m\Delta t$. Show details of all the steps of derivation. **[3]**

— *End* —