MAFS 5250 – Computational Methods for Pricing Structured Products

Mid-term Test, 2019

Time allowed: 90 minutes *Instructor: Prof. Y. K. Kwok*

[points]

1. Consider the Cheuk-Vorst algorithm for pricing European fixed strike lookback put option whose terminal payoff is given by

$$\left(K - \min_{u \in [0,T]} S_u\right)^+,$$

where K is the fixed strike. In the binomial tree, let $\overline{m}(t_j)$ be the realized discrete minimum asset values up to time t_j, where

$$\overline{m}(t_j) = \min_{0 \leq i \leq j} S(t_i).$$

(a) Show that the terminal payoff under the discrete binomial tree can be decomposed into two terms:

$$\max\left(K - \overline{m}(t_j), 0\right) + \max(\min(\overline{m}(t_j), K) - \overline{m}(t_N; t_{j+1}), 0),$$

where

$$\overline{m}(t_N; t_{j+1}) = \min_{j+1 \leq i \leq N} S(t_i).$$ **[2]**

(b) Suppose we approximate K by $S_0 u^k$ for some integer k in the binomial tree, and there exists an integer m such that $\overline{m}(t_j) = S_0 u^m$. Take $M = \min(m, k)$, then $K'(t_j) = \min(K, \overline{m}(t_j)) = S_0 u^M$. We relate the adjusted strike $K'(t_j)$ with $S(t_j)$ in terms of an integer ℓ, where

$$\ell = \frac{\ln \frac{S(t_j)}{K'(t_j)}}{\ln u} \Leftrightarrow K'(t_j) = S(t_j) u^{-\ell}, \ \ell \geq 0.$$

Let $p_X(S(t_j), K'(t_j), t_j)$ denote the numerical lookback put price at t_j. Define the normalized lookback put price by

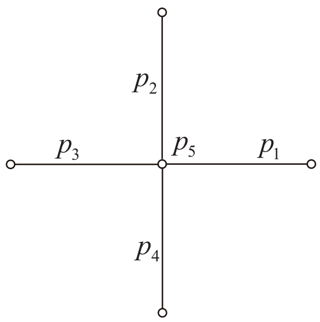
$$X(\ell, t_j) = \frac{p_X(S(t_j), K'(t_j), t_j)}{S(t_j)},$$

by virtue of the homogeneity property of p_X with respect to $S(t_j)$. Write down the binomial tree algorithm that relates nodal normalized put values at successive time points at t_j and t_{j+1}, in terms of discount factor $e^{-r\Delta t}$, proportional jump parameters u and d, and neutral risk probability of upward jump p. Make special care to distinguish (i) $\ell = 0$ and (ii) $\ell > 0$. Also, specify the numerical terminal condition at $t = t_N$ in the binomial calculations. **[5]**

2. In the construction of the two-dimensional lattice tree that approximates the dynamic joint evolution of

$$\frac{\ln S_i^{\Delta t}}{S_i} = \xi_i, \quad i = 1, 2,$$

where ξ_i is a normal random variable with mean $\left(r - \frac{\sigma_i^2}{2}\right)\Delta t$ and variance $\sigma_i^2 \Delta t$. Let the instantaneous correlation coefficient between ξ_1 and ξ_2 be ρ. Suppose we choose the 5-point node as follows:



ξ_1^a	ξ_2^a	probability
v_1	0	p_1
0	v_2	p_2
$-v_1$	0	p_3
0	$-v_2$	p_4
0	0	p_5

Here, $v_i = \sigma_i \sqrt{\Delta t}$, $i = 1, 2$. Recall that $p_1, p_2, \ldots, p_5$ are determined by equating two means, two variances, one covariance and sum of probability being one. Write down the six equations and show that the above choice of the 5-point node lead to inconsistency of equations so that such choice can never be adopted. **[3]**

3. The key step in pricing the alpha quantile option is the observation of the relation:

$$e^{-rT} P\left[B_{\inf} > S_j\right] = V_{cum}^{bin}(\alpha, S_j), \tag{1}$$

where

$$B_{\inf}(T; \alpha) = \inf\left\{B : \frac{1}{T}\int_0^T \mathbf{1}_{\{S_t \le B\}}\, \mathrm{d}t \ge \alpha\right\}.$$

Here, $V_{cum}^{bin}(\alpha, S_j)$ denote the value of a binary option that pays \$1 at maturity T if the cumulative time staying at or below S_j is less than α of the total life of the option, $0 \le \alpha \le 1$.

(a) Give the justification of eq.(1). **[2]**

(b) For fixed value of B, explain how to compute $V_{cum}^{bin}(\alpha, B)$ numerically using the forward shooting grid approach. Specify the terminal payoff of $V_{cum}^{bin}(\alpha, B)$. **[4]**

4. In an accumulator, suppose the delivery of one or two units of stock on date t_i is determined by $S_{t_i} \ge K$ or otherwise, independent of whether knock-out occurs or not on t_i. There is a knock-out provision, where the accumulator will be knocked out on t_i if $S_{t_i} > H$, where $H > K$ and H is the upper knock-out level.

Suppose there are n observation dates and delivery of the stocks is made immediately on each observation date. Explain why the accumulator can be replicated by a portfolio of long and short position of up-and-out barrier call and put options. **[3]**

5. Let λ_n^i denote the Arrow-Debreu price of an option that pays \$1 if $S(n\Delta\tau)$ attains the value S_n^i, where

$$\lambda_n^i = e^{-rn\Delta t} E\left[\mathbf{1}_{\{S(n\Delta t) = S_n^i\}} \middle| S(0) = S_0\right].$$

(a) Explain why the price formula of the call option maturity at $(n+1)\Delta t$ and strike K under the discrete binomial tree can be expressed as

$$c((n+1)\Delta t; K) = \sum_{i=0}^{n+1} \lambda_{n+1}^i \max(S_{n+1}^i - K, 0).$$ **[1]**

(b) We set $K = S_n^i$. After some manipulation, it can be shown that

$$c((n+1)\Delta t; S_n^i) = \left[\lambda_n^i P_{i+1}^n (S_{n+1}^{i+1} - S_n^i) + \sum_{j=i+1}^{n} \lambda_n^j (F_n^j - S_n^i) \right] e^{-r\Delta t},$$

where

$$P_{i+1}^n = \frac{F_n^i - S_{n+1}^i}{S_{n+1}^{i+1} - S_{n+1}^i} \quad \text{and} \quad F_n^i = e^{r\Delta t} S_n^i.$$

Provide the financial interpretation of the above formula. **[4]**

6. The mean reversion feature of the short rate r with mean reversion rate a is captured by upward and downward branching of the binomial interest rate tree. Let $D(t_0, t_m)$ be the discount factor over (t_0, t_m) and the $(m, j)^{\text{th}}$ node in the trinomial interest rate tree is $R(t_m) = \alpha_m + j\Delta R$. The key step in the Hull-White algorithm is the determination of α_m in terms of discount bond price P_{m+1} and the discrete Arrow-Debreu price $Q_{i,j}$.

(a) Express $Q_{m,j}$ in terms of $D(t_0, t_m)$ and $R(t_m)$ via the discounted expectation formulation. **[2]**

(b) Show how to establish

$$E[D(t_0, t_{m+1}) | \mathcal{F}_{t_0}] = \sum_j Q_{m,j} \exp(-(\alpha_m + j\Delta R)\Delta t),$$

and use it to derive

$$\alpha_m = \frac{\ln \sum_{j=-n_m}^{n_m} Q_{m,j} e^{-j\Delta R \ \Delta t} - \ln P_{m+1}}{\Delta t}.$$ **[4]**

— *End* —