MAFS 5250 – Computational Methods for Pricing Structured Products

Mid-term Test, 2017

Time allowed: 90 minutes *Instructor: Prof. Y. K. Kwok*

[points]

1. Consider the dynamic programming procedure applied to incorporate the game option between the issuer and holder in a callable American call option in a binomial tree calculation. Let X be the strike price and K be the call price.

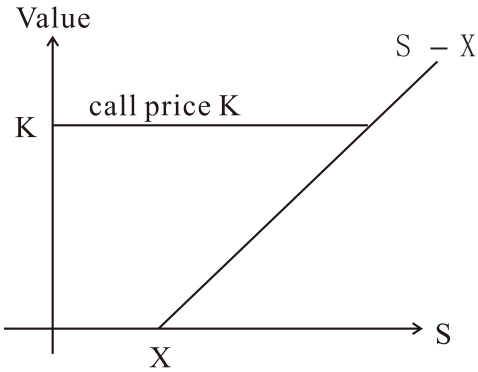
   (a) Give the financial interpretation of the following dynamic programming procedure:

   $$C_j^n = \max\left(\min\left(\frac{pC_{j+1}^{n+1} + (1-p)C_j^{n+1}}{R}, K\right),\ S_j^n - X\right),$$

   where p is the risk neutral probability of an upward move and R is the growth factor over one time step. **[2]**

   (b) Devise an alternative form of the above dynamic programming procedure and provide the corresponding financial interpretation of the alternative scheme. **[3]**

   (c) Explain why the price function of the callable American call option must be bounded between the call price K and the intrinsic value $S - X$.

   By observing this property, explain why the third and most simplified dynamic programming procedure is given by

   $$\min\left(\max\left(\frac{pC_{j+1}^{n+1} + (1-p)C_j^{n+1}}{R}, S_j^n - X\right),\ K\right).$$ **[4]**

2. (a) We use the stock price as the numeraire in the construction of the Cheuk-Vorst algorithm for pricing the floating strike lookback put option with terminal payoff: $\max_{t\in[0,T]} S_t - S_T$. We define $\hat{V}_t = \frac{V_t}{S_t}$ and construct the truncated binomial tree for the process $Y_t = \frac{S_t^{\max}}{S_t}$, $Y_t \geq 1$. Explain why the binomial algorithm for pricing the floating strike lookback put option is given by

   $$\widetilde{V}_j^n = e^{-r\Delta t}\left[p\widetilde{V}_{j-1}^{n+1}u + (1-p)\widetilde{V}_{j+1}^{n+1}d\right].$$

Be cautious with the stock price index and proportional constants u and d in the above binomial scheme. **[3]**

(b) Explain why the choice of stock price as numeraire cannot be directly extended to the fixed strike lookback call option, where the terminal payoff is given by $\max_{t \in [0,T]} S_t - X$. Here, X is the strike price. Outline the modification using the notion of adjusted strike price in order to achieve dimension reduction in the binomial pricing algorithm. In particular, discuss how to update the adjusted strike price and accumulate the terminal payoff when a newly realized maximum of the stock price occurs. The full details of the binomial pricing algorithm are not required. **[6]**

3. (a) Suppose the geometrical averaging of asset prices

$$G_n = (S_0 \; S_1 \; \cdots \; S_n)^{1/n+1}$$

is used in the payoff of an Asian option, find the correlation evolution function between G_{n+1}, G_n and S_{n+1}. **[3]**

(b) Since the number of possible realized averaging values of the asset price is 2^n, where n is the number of time steps in the binomial tree. Explain briefly how to reconcile this exponential growth of values of the path dependent function in the forward shooting grid algorithm. In particular, discuss the use of quantification of the averaging values and interpolation scheme in option value calculations. **[4]**

4. (a) The key step in the Hull-White interest rate tree construction is the determination of the parameter α_m using the following formula:

$$P_{m+1} = \sum_{j=-n_m}^{n_m} Q_{m,j} \exp(-(\alpha_m + j\Delta R)\Delta t),$$

where the short rate R at node (m, j) is $\alpha_m + j\Delta R$, $Q_{m,j}$ is the Arrow-Debreu price of the state at node (m, j), P_{m+1} is the price of a zero-coupon bond maturing at $(m+1)\Delta t$, and n_m is the number of nodes on each side of the central node at time $m\Delta t$. Prove the above formula, starting with the following relation:

$$P_{m+1} = E\left[D(t_0, t_{m+1}) \middle| \mathcal{F}_{t_0}\right] = E[E[D(t_0, t_m)D(t_m, t_{m+1})|\mathcal{F}_{t_m}]|\mathcal{F}_{t_0}],$$

where $D(t_0, t_{m+1})$ is the discount factor from t_0 to t_{m+1}. **[5]**

(b) In the Derman-Kani algorithm for constructing the implied binomial tree, it is necessary to relate the Arrow-Debreu price λ_i^n with the risk neutral probabilities and the observed call option prices with various implied binomial tree parameters. Recall that λ_i^n is defined by

$$\lambda_i^n = e^{-rn\Delta t} E_Q \left[\mathbf{1}_{\{S(n\Delta t) = S_i^n\}} \middle| S(0) = S_0 \right].$$

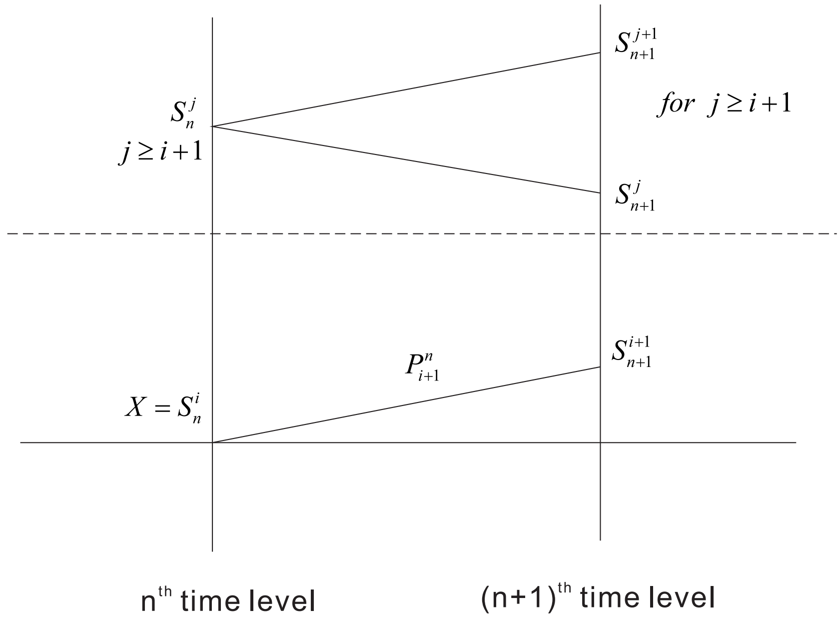
Using financial argument and nested expectation calculation, prove in details that the observed call option price with strike S_i^n and maturity date $(n+1)\Delta t$ is given by

$$c(S_i^n, (n+1)\Delta t) = e^{-r\Delta t} \left[\lambda_i^n p_{i+1}^n (S_{i+1}^{n+1} - S_i^n) + \sum_{j=i+1}^{n} \lambda_j^n (F_j^n - S_i^n) \right],$$

where

$$F_j^n = e^{r\Delta t} S_j^n \quad \text{and} \quad p_{j+1}^n = \frac{F_j^n - S_j^{n+1}}{S_{j+1}^{n+1} - S_j^{n+1}}.$$

Give justification of the occurrences of λ_j^n and $(F_j^n - S_i^n)$ in the above formula. **[5]**



5. (a) Give your comments on the advantages and disadvantages using explicit schemes and implicit schemes in finite difference calculations with regard to the following aspects:

   (i) incorporation of boundary conditions of the option pricing model,

   (ii) computational complexity,

   (iii) time step constraint. **[3]**

   (b) Show that the domain of dependence of the trinomial tree is $O(\sqrt{n})$, where n is the number of time steps. **[2]**

— *End* —