

# **MATH 392 — Topics in Mathematical Economics**

## **Topic 4 – Voting methods with more than 2 alternatives**

- 4.1 Social choice procedures
- 4.2 Analysis of voting methods
- 4.3 Arrow's Impossibility Theorem
- 4.4 Comparison voting methods

## 4.1 Social Choice Procedures

- A group of voters are collectively trying to choose among several alternatives, with the social choice (the “winner”) being the alternative receiving the most votes (based on a specified voting method).
- How to take in the information of some individual comparisons among the alternatives?
- What are the intuitive criteria to judge whether a social choice is “reasonably” acceptable? Is the choice the least unpopular, broadly acceptable, winning in all one-for-one contests, etc?

## Example

3 candidates are running for the Senate. By some means, we gather the information on the “preference order” of the voters.

22%	23%	15%	29%	7%	4%
<i>D</i>	<i>D</i>	<i>H</i>	<i>H</i>	<i>J</i>	<i>J</i>
<i>H</i>	<i>J</i>	<i>D</i>	<i>J</i>	<i>H</i>	<i>D</i>
<i>J</i>	<i>H</i>	<i>J</i>	<i>D</i>	<i>D</i>	<i>H</i>

First choice only

45% for *D*, 44% for *H* and 11% for *J*;  
*D* emerges as the “close” winner.

Contest between *H* and *D*

*H* scores  $(15 + 29 + 7)\% = 51\%$ .  
*D* scores  $(22 + 23 + 4)\% = 49\%$ ,

## General framework

A set  $A$  whose elements are called *alternatives* (or candidates);  $a, b, c$  etc.

A set  $P$  whose elements are called *people* (or voters);  $p_1, p_2, p_3$ , etc.

- Each person  $p$  in  $P$  has arranged the alternatives in a list according to preference.
- A social choice procedure is a fixed “receipt” for choosing an alternative based on the preference orderings of the individuals.
- Rational choice theory: Voters are assumed to make orderly choices that reflect their personal preferences and desires.

A “*social choice procedure*” is a function where a typical input is a sequence of individual preference rankings of the alternatives and an output is a single alternative, or a single set of alternatives if we allow ties.

- A sequence of individual preference lists is called a ‘profile’.
- The output is called the “social choice” or winner if there is no tie, or the “social choice set” or “those tied for winner” if there is a tie.

## Examples of social choice procedures

### 1. *Plurality voting*

Declare as the social choice(s) the alternative(s) with the largest number of first-place rankings in the individual preference lists.

*1980 US Presidential election:* Democrat Carter, Republican Reagan and Independent John Anderson

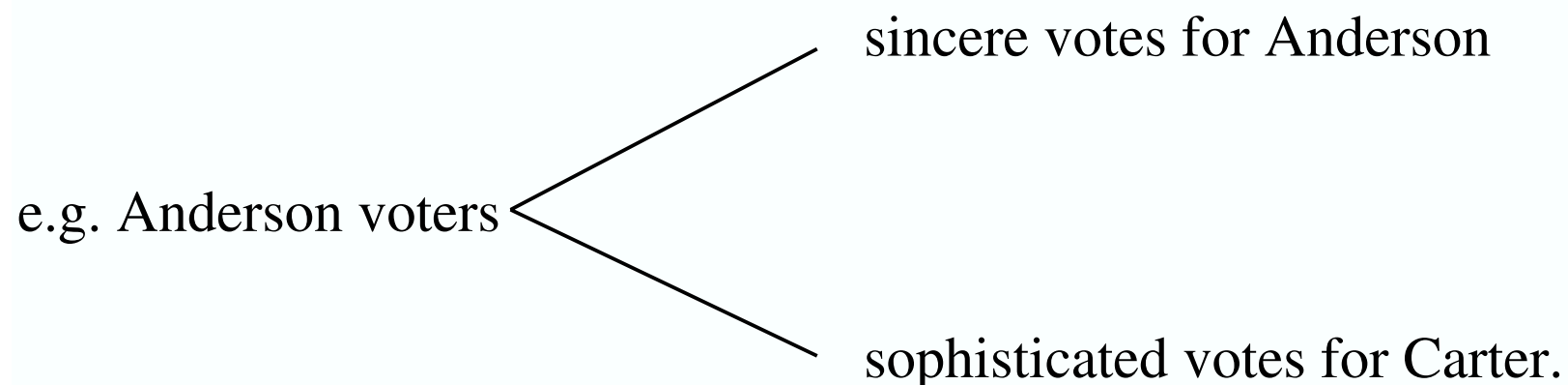
<i>Reagan voters (45%)</i>	<i>Anderson voters (20%)</i>	<i>Carter voters (35%)</i>
<i>R</i>	<i>A</i>	<i>C</i>
<i>A</i>	<i>C</i>	<i>A</i>
<i>C</i>	<i>R</i>	<i>R</i>

If voters can cast only one vote for their best choice, then Reagan would win with 45% of the vote.

- Reagan was perceived as much more conservative than Anderson who in turn was more conservative than Carter.

Since the chance of Anderson winning is slim, Anderson voters may cast their votes *strategically* to Carter so that their second choice could win.

- A voter's *sincere* strategy is to vote for her first choice.
- Reagan voters have a straightforward strategy: to vote sincerely.
- Following an admissible strategy that is not *sincere* is called *sophisticated* voting.



## Example

*3 voters*

*a*

*b*

*c*

*2 voters*

*b*

*a*

*c*

*4 voters*

*c*

*b*

*a*

"*c*" wins with first-choice votes;  
but 5-to-4 majority of  
voters rank *c* last.

Consider pairwise contests:-

*b* beats *a* by 6 to 3; *b* beats *c* by 5 to 4; *a* beats *c* by 5 to 4.

Note that *b* beats the other two but *b* is not the winner. Also, *c* loses to the other two but *c* is the winner. This is like Chen in 2000 Taiwan election.



## Plurality voting with run-off

Second-step election between the top two vote-getters in plurality election if no candidate receives a majority.

### Example

<i>6 voters</i>	<i>5 voters</i>	<i>4 voters</i>	<i>2 voters</i>
<i>a</i>	<i>c</i>	<i>b</i>	<i>b</i>
<i>b</i>	<i>a</i>	<i>c</i>	<i>a</i>
<i>c</i>	<i>b</i>	<i>a</i>	<i>c</i>

"a" beats "b" in the run-off

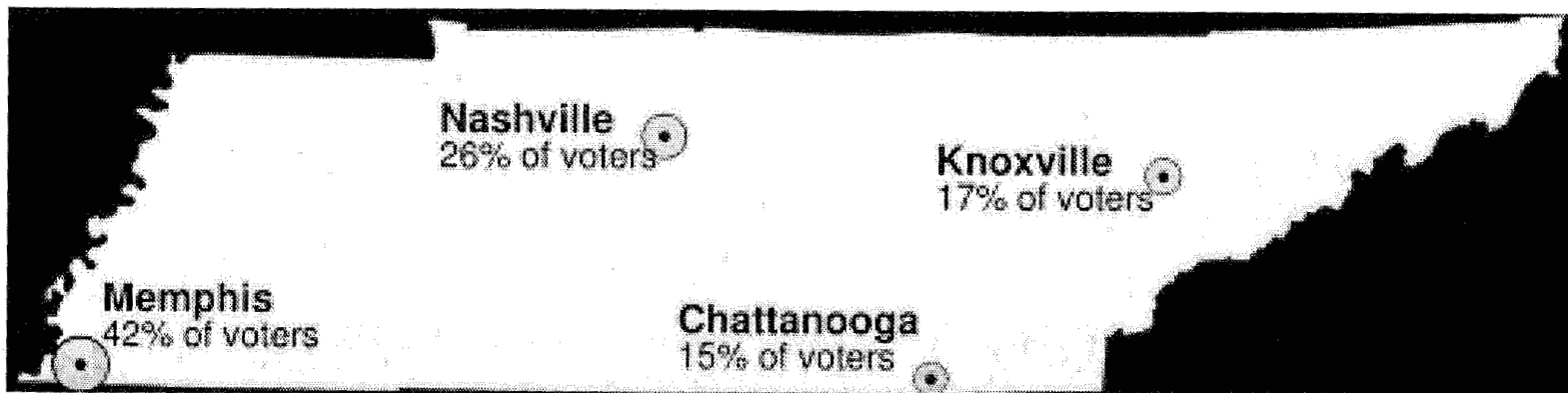
Now, suppose the last 2 voters change their preferences to  $abc$ , then "c" beats "a" in the run-off by a vote of 9 to 8. The moving up of "a" in the last 2 voters indeed hurts "a". (幫他變成害他)

## 2. *Borda count*

One uses each preference list to award “points” to each of  $n$  alternatives: bottom of the list gets zero, next to the bottom gets one point, the top alternative gets  $n - 1$  points.

The alternative(s) with the highest “scores” is the social choice.

- It sometimes elects broadly acceptable candidates, rather than those preferred by the majority, the Borda count is considered as a consensus-based electoral system, rather than a majoritarian one.



The candidates for the capital of the State of Tennessee are:

- Memphis, the state's largest city, with 42% of the voters, but located far from the other cities
- Nashville, with 26% of the voters, almost at the center of the state and close to Memphis
- Knoxville, with 17% of the voters
- Chattanooga, with 15% of the voters

<b>42% of voters (close to Memphis)</b>	<b>26% of voters (close to Nashville)</b>	<b>15% of voters (close to Chattanooga)</b>	<b>17% of voters (close to Knoxville)</b>
<b>1. Memphis</b> 2. Nashville 3. Chattanooga 4. Knoxville	<b>1. Nashville</b> 2. Chattanooga 3. Knoxville 4. Memphis	<b>1. Chattanooga</b> 2. Knoxville 3. Nashville 4. Memphis	<b>1. Knoxville</b> 2. Chattanooga 3. Nashville 4. Memphis

City	First	Second	Third	Fourth	Total points
Memphis	$42 \times 3$	0	0	$58 \times 0$	126
Nashville	$26 \times 3$	$42 \times 2$	$32 \times 1$	0	194
Chattanooga	$15 \times 3$	$43 \times 2$	$42 \times 1$	0	173
Knoxville	$17 \times 3$	$15 \times 2$	$26 \times 1$	$42 \times 0$	107

- The winner is Nashville with 194 points.

*Modification:* Voters can be permitted to rank only a subset of the total number of candidates with all unranked candidates being given zero points.

### 3. *Hare's procedure*

If no alternative is ranked first by a majority of the voters, the alternative(s) with the smallest number of first place votes is (are) crossed out from all reference orderings, and the first place votes are counted again.

#### **Example 1**

<i>5 voters</i>	<i>2 voters</i>	<i>3 voters</i>	<i>3 voters</i>	<i>4 voters</i>
<i>a</i>	<i>b</i>	<i>c</i>	<i>d</i>	<i>e</i>
<i>b</i>	<i>c</i>	<i>b</i>	<i>b</i>	<i>b</i>
<i>c</i>	<i>d</i>	<i>d</i>	<i>c</i>	<i>c</i>
<i>d</i>	<i>e</i>	<i>e</i>	<i>e</i>	<i>d</i>
<i>e</i>	<i>a</i>	<i>a</i>	<i>a</i>	<i>a</i>

“*b*” is eliminated first.

5 voters	2 voters	3 voters	3 voters	4 voters
<i>a</i>	<i>c</i>	<i>c</i>	<i>d</i>	<i>e</i>
<i>c</i>	<i>d</i>	<i>d</i>	<i>c</i>	<i>c</i>
<i>d</i>	<i>e</i>	<i>e</i>	<i>e</i>	<i>d</i>
<i>e</i>	<i>a</i>	<i>a</i>	<i>a</i>	<i>a</i>

Next, “*d*” is eliminated.

5 voters	2 voters	3 voters	3 voters	4 voters
<i>a</i>	<i>c</i>	<i>c</i>	<i>c</i>	<i>e</i>
<i>c</i>	<i>e</i>	<i>e</i>	<i>e</i>	<i>c</i>
<i>e</i>	<i>a</i>	<i>a</i>	<i>a</i>	<i>a</i>

There is still no majority winner, and “*e*” is crossed off. Lastly, “*c*” is then declared the winner.

\* Under plurality with run-off, *a* and *e* are the two top vote-getters, ending *e* as the social choice.

#### 4. *Coombs procedure*

Eliminate the alternative with the *largest* number of *last place* votes, until when one alternative commands the majority support.

Consider Example 1, the steps of elimination are

5 voters	2 voters	3 voters	3 voters	4 voters
<i>b</i>	<i>b</i>	<i>c</i>	<i>d</i>	<i>e</i>
<i>c</i>	<i>c</i>	<i>b</i>	<i>b</i>	<i>b</i>
<i>d</i>	<i>d</i>	<i>d</i>	<i>c</i>	<i>c</i>
<i>e</i>	<i>e</i>	<i>e</i>	<i>e</i>	<i>d</i>

“*e*” is eliminated, leaving

5 voters	2 voters	3 voters	3 voters	4 voters
<i>b</i>	<i>b</i>	<i>c</i>	<i>d</i>	<i>b</i>
<i>c</i>	<i>c</i>	<i>b</i>	<i>b</i>	<i>c</i>
<i>d</i>	<i>d</i>	<i>d</i>	<i>c</i>	<i>d</i>

“*b*”, with 11 first place votes, is now the winner.

## Example 2

5 voters	2 voters	4 voters	2 voters
<i>a</i>	<i>b</i>	<i>c</i>	<i>c</i>
<i>b</i>	<i>c</i>	<i>a</i>	<i>b</i>
<i>c</i>	<i>a</i>	<i>b</i>	<i>a</i>

- Coombs procedure eliminates “*c*” and chooses “*a*”.
- If the last two voters change to favor “*a*” over “*b*”, then “*b*” will be eliminated and “*c*” will win.

### 5. Dictatorship

Choose one of the voters and call her the dictator. The alternative on top of her list is the social choice.



## 6. *Sequential pairwise voting* (more than 2 alternatives)

- Two alternatives are voted on first; the majority winner is then paired against the third alternative, etc. The order in which alternatives are paired is called the *agenda* of the voting.

### *Example*

*A*: Reagan administration – supported bill to provide arms to the Contra rebels.

*H*: Democratic leadership bill to provide humanitarian aid but not arms.

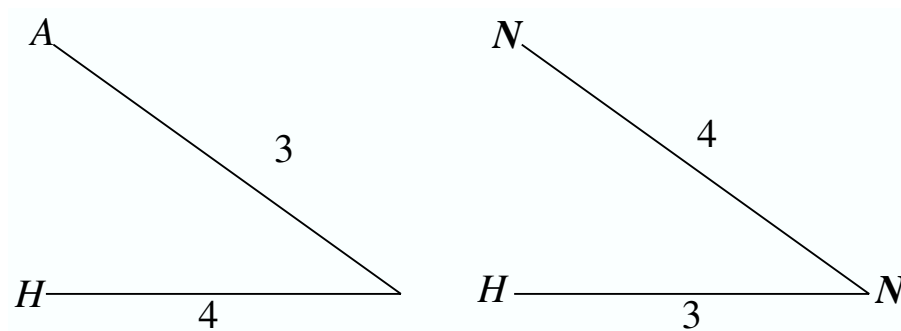
*N*: giving no aid to the rebels.

In the parliamentary agenda, the first vote was between *A* and *H*, with the winner to be paired against *N*. First, the form of aid is voted, then decide on whether aid or no aid.

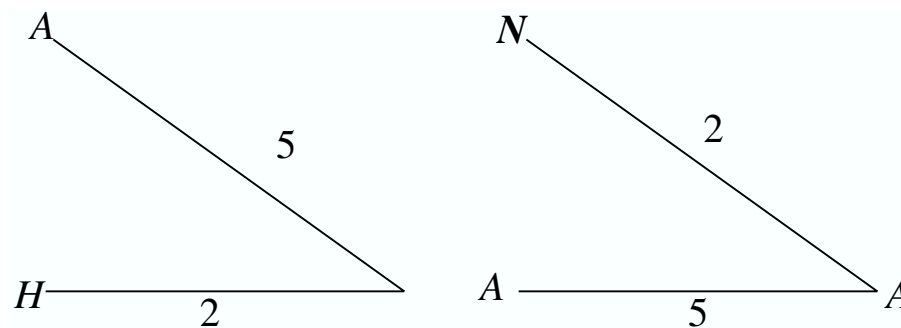
Suppose the preferences of the voters are:

<i>Conservative Republicans</i>	<i>Moderate Republicans</i>	<i>Moderate Democrats</i>	<i>Liberal Democrats</i>
<i>A</i>	<i>A</i>	<i>H</i>	<i>N</i>
<i>N</i>	<i>H</i>	<i>A</i>	<i>H</i>
<i>H</i>	<i>N</i>	<i>N</i>	<i>A</i>
$\begin{pmatrix} 2 \\ \text{voters} \end{pmatrix}$	$\begin{pmatrix} 1 \\ \text{voter} \end{pmatrix}$	$\begin{pmatrix} 2 \\ \text{voters} \end{pmatrix}$	$\begin{pmatrix} 2 \\ \text{voters} \end{pmatrix}$

- The Conservative Republicans may think that humanitarian aid is non-effective, either no arms or no aid at all. Moderate Republicans may think that some form of aid is at least useful.



Sincere voting



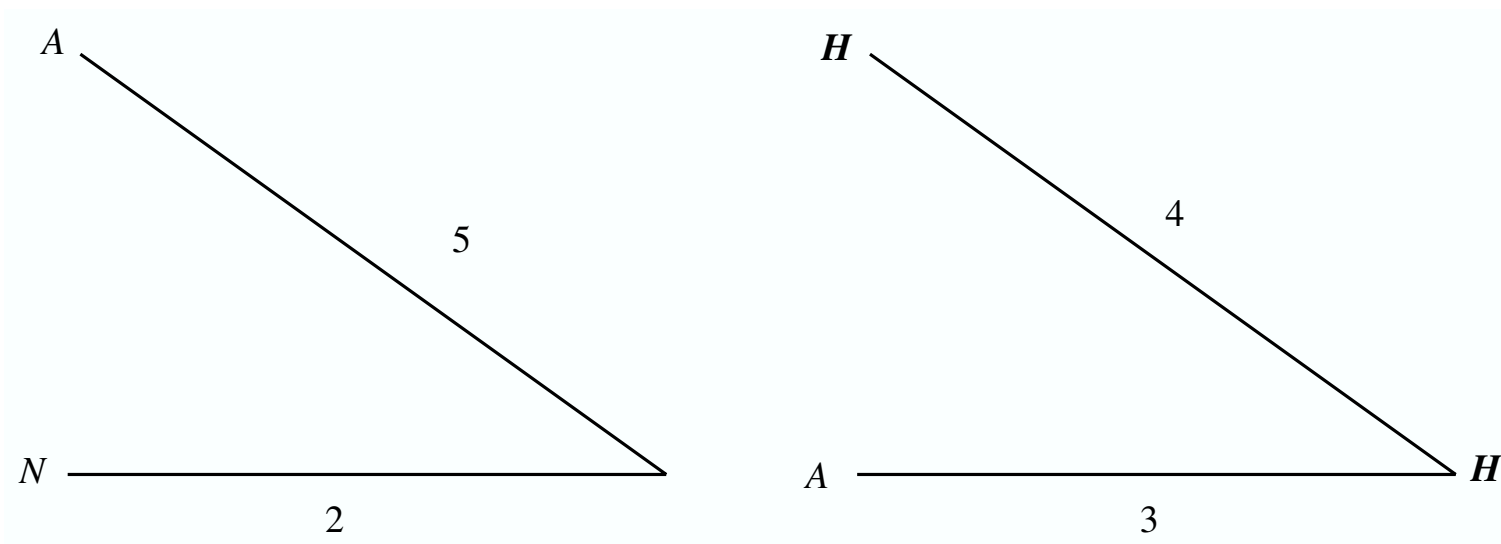
Sophisticated voting

By sophisticated voting, if voters can make  $A$  to win first, then  $A$  can beat  $N$  by 5 to 2.

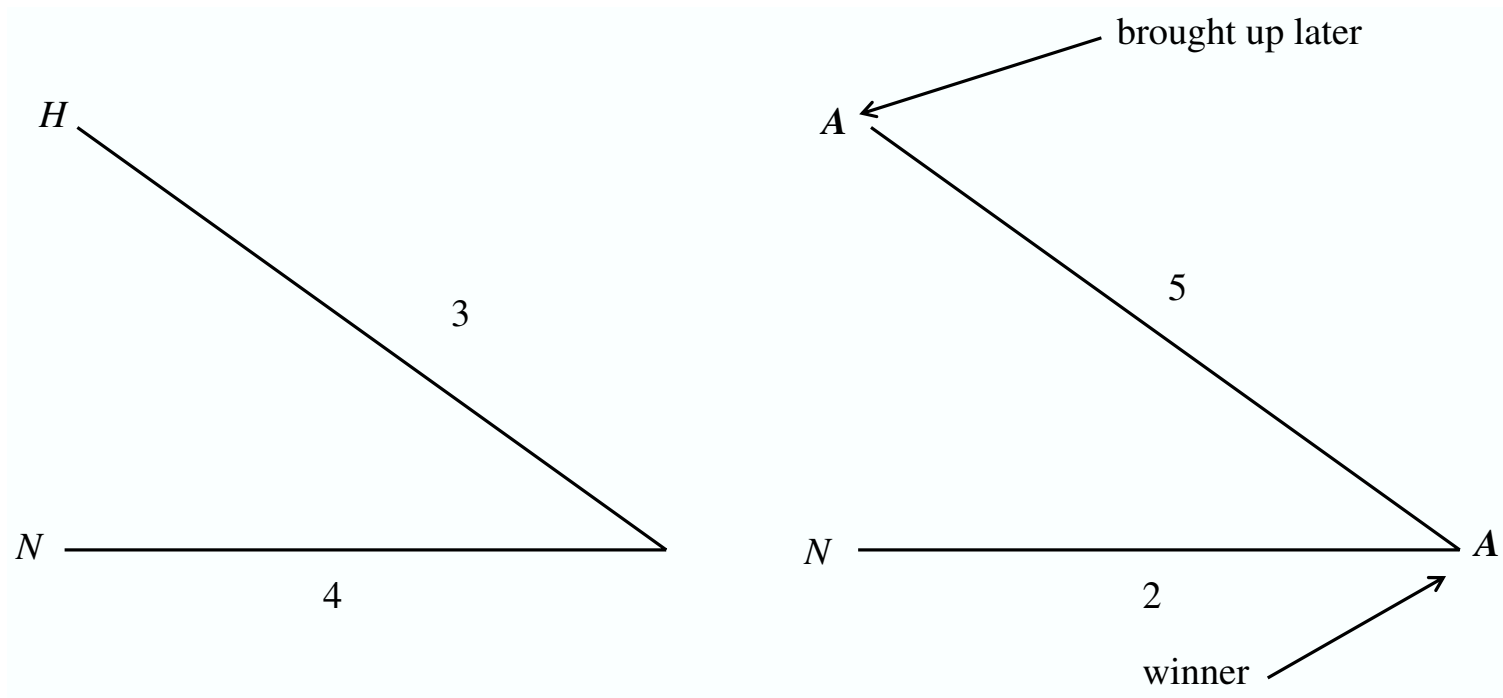
Republicans should vote sincerely for  $A$ , the liberal Democrats should vote sincerely for  $H$ , but the moderate Democrats should have voted sophisticatedly for  $A$  ( $N$  is the last choice for moderate Democrats).

## *Alternative agendas*

- produce *any one* of the alternatives as the winner under sincere voting:



Sincere voting



Sincere voting

Remark: The later you bring up your favored alternative, the better chance it has of winning.

## Example

Voters are unanimous  
in preferring  $b$  to  $d$ .

1 voter

$a$

$b$

$d$

$c$

1 voter

$c$

$a$

$b$

$d$

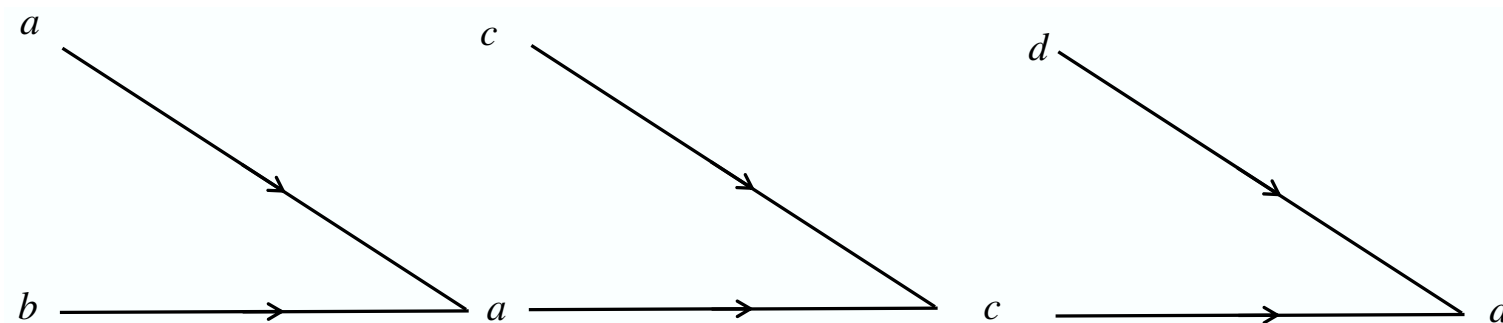
1 voter

$b$

$d$

$c$

$a$



## Example – voting paradox of Condorcet

Consider the following 3 preference listings of 3 alternatives

list #1	list #2	list #3
<i>a</i>	<i>c</i>	<i>b</i>
<i>b</i>	<i>a</i>	<i>c</i>
<i>c</i>	<i>b</i>	<i>a</i>

If *a* is the social choice, then #2 and #3 agree that *c* is better.

If *b* is the social choice, then #1 and #2 agree that *a* is better.

If *c* is the social choice, then #1 and #3 agree that *b* is better.

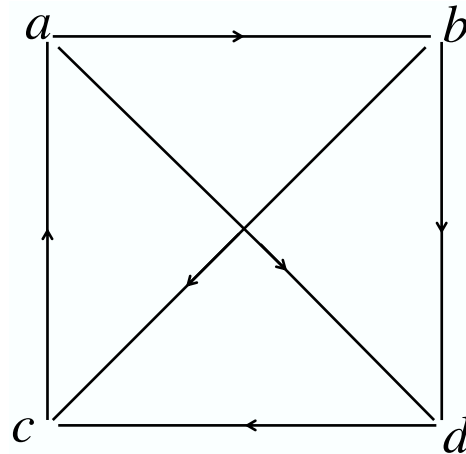
Two-thirds of the people are “constructively unhappy” in the sense of having a single alternative that all agree is superior to the proposed social choice.

Generalization to  $n$  alternatives and  $n$  people, involving unhappiness of  $\frac{n-1}{n}$  of the people:

## Loss of transitivity in pairwise contest

If  $a$  is preferred to  $b$  and  $b$  is preferred to  $c$ , then we expect  $a$  to be preferred to  $c$ .

1 voter	1 voter	1 voter
$a$	$c$	$b$
$b$	$a$	$d$
$d$	$b$	$c$
$c$	$d$	$a$



$a$  beats  $b$  in pairwise contest,  $b$  beats  $c$  in pairwise contest but  $a$  loses to  $c$  in pairwise contest.



## Chair's paradox

“Apparent power” needs not correspond to control over outcomes.

Consider the same example as in the voting paradox of Condorcet:

$A$	$B$	$C$
$a$	$b$	$c$
$b$	$c$	$a$
$c$	$a$	$b$

Here, the preference lists will not be regarded as inputs for the procedure, but only be used to “test” the extent to which each of  $A$ ,  $B$  and  $C$  should be happy with the social choice.

The social choice is determined by a standard voting procedure where voter  $A$  (Chair) also has a tie-breaking vote.

### *Definition*

Fix a player  $P$  and consider two strategies  $V(x)$  and  $V(y)$  for  $P$ . [ $V(x)$  as “vote for alternative  $x$ ”].  $V(x)$  is said to be weakly dominating for player  $P$  if

1. For every possible scenario (choice of alternatives for which to vote by the other players), the social choice resulting from  $V(x)$  is at least as good for player  $P$  as that resulting from  $V(y)$ .
2. There is at least one scenario in which the social choice resulting from  $V(x)$  is strictly better for player  $P$  than that resulting from  $V(y)$ .

A strategy is said to be weakly dominant for player  $P$  if it weakly dominates every other available strategy.

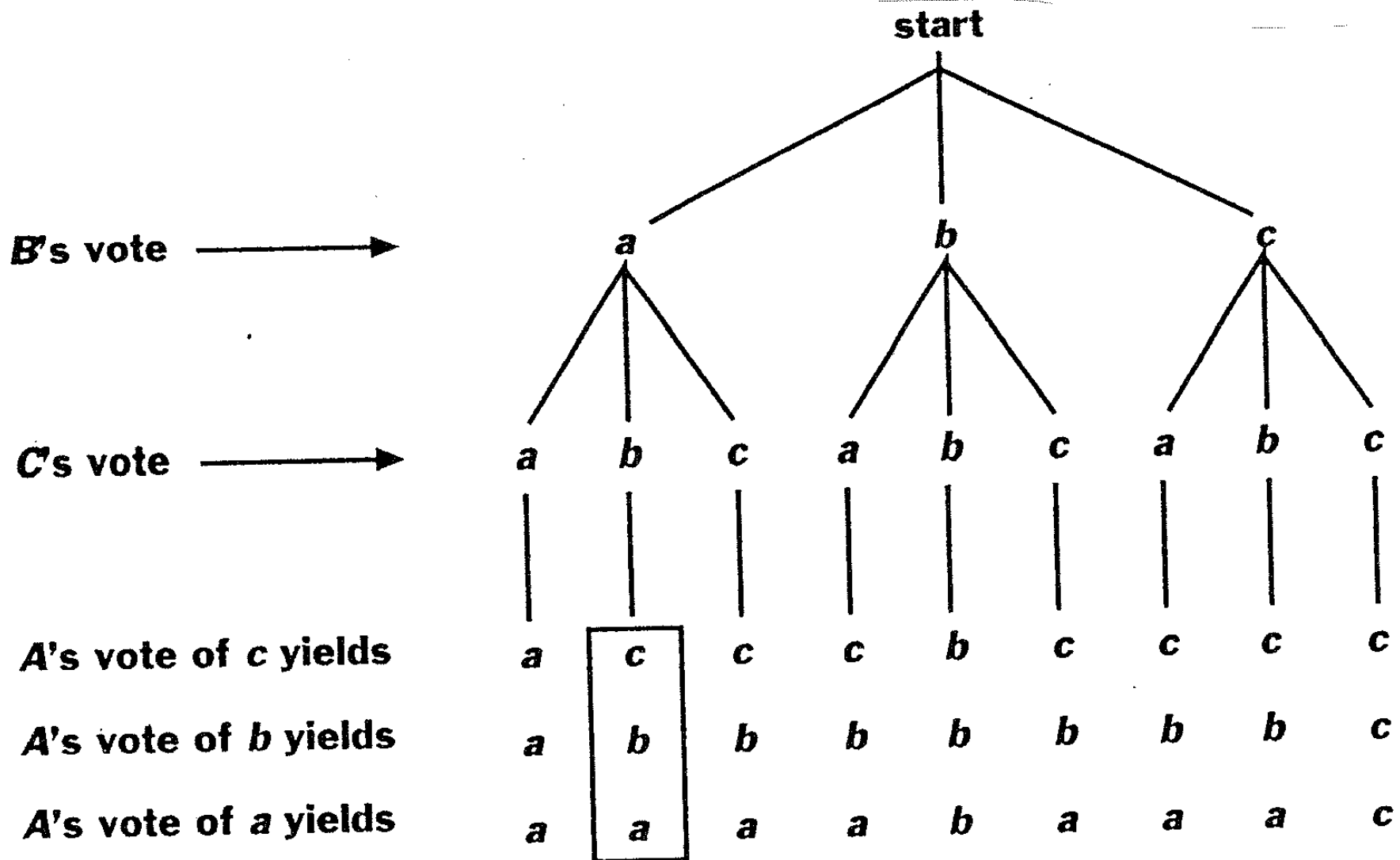
How to determine whether a strategy is a weakly dominant one? List all possible scenarios and compare the result achieved by using this strategy and all other strategies – use of a tree.

*Proposition*

“Vote for alternative  $a$ ” is a weakly dominant strategy for Chair.

*Proof* Consider the 9 possible scenarios in a tree.

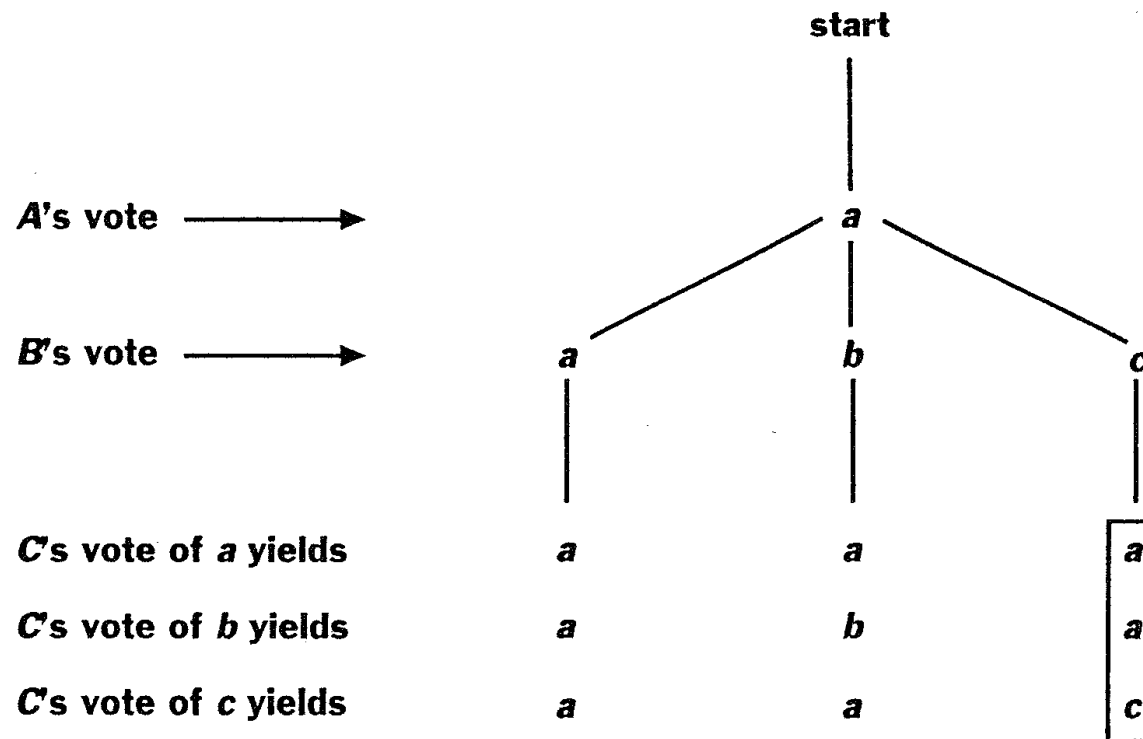
- Whenever there is a tie, Chair’s choice wins.



The outcome at the bottom of each column is never worse for  $A$  than either of the outcomes above it, and that it is strictly better than both in at least one case.

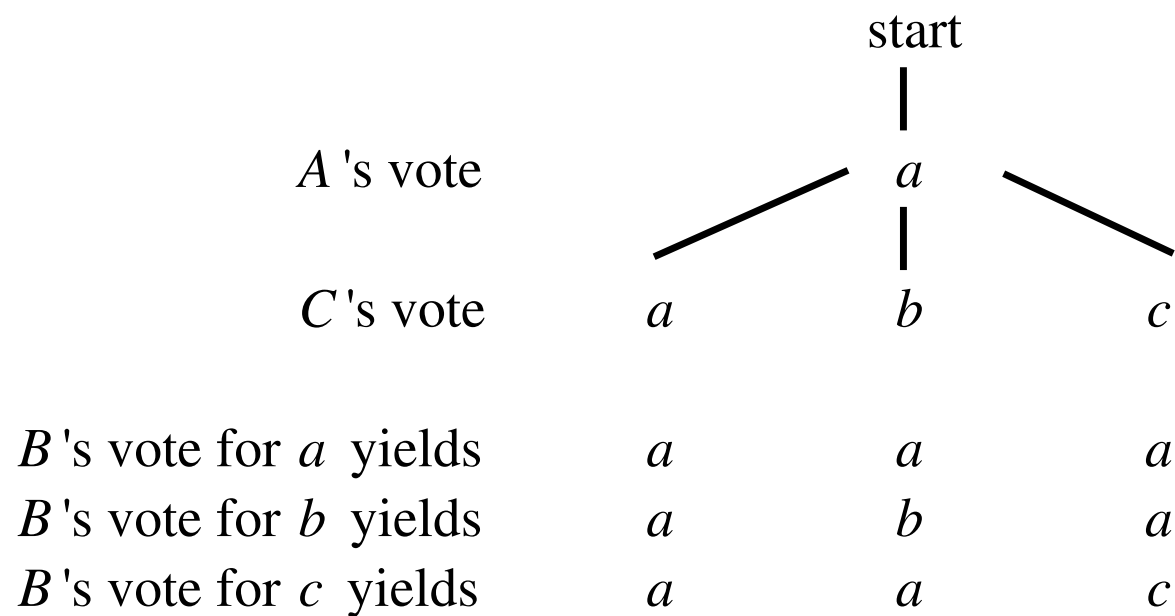
- Player  $A$  appears to have no rational justification for voting for anything except  $a$ .
- If we assume that  $A$  will definitely go with his weakly dominant strategy, then we analyze what rational self-interest will dictate for the other 2 players in the new game.

For player  $C$ : In the last column,  $C$ 's vote of  $b$  yields  $a$  since  $A$  is the Chair (tie-breaker).



"Vote for  $C$ " is a weakly dominant strategy for  $C$ .

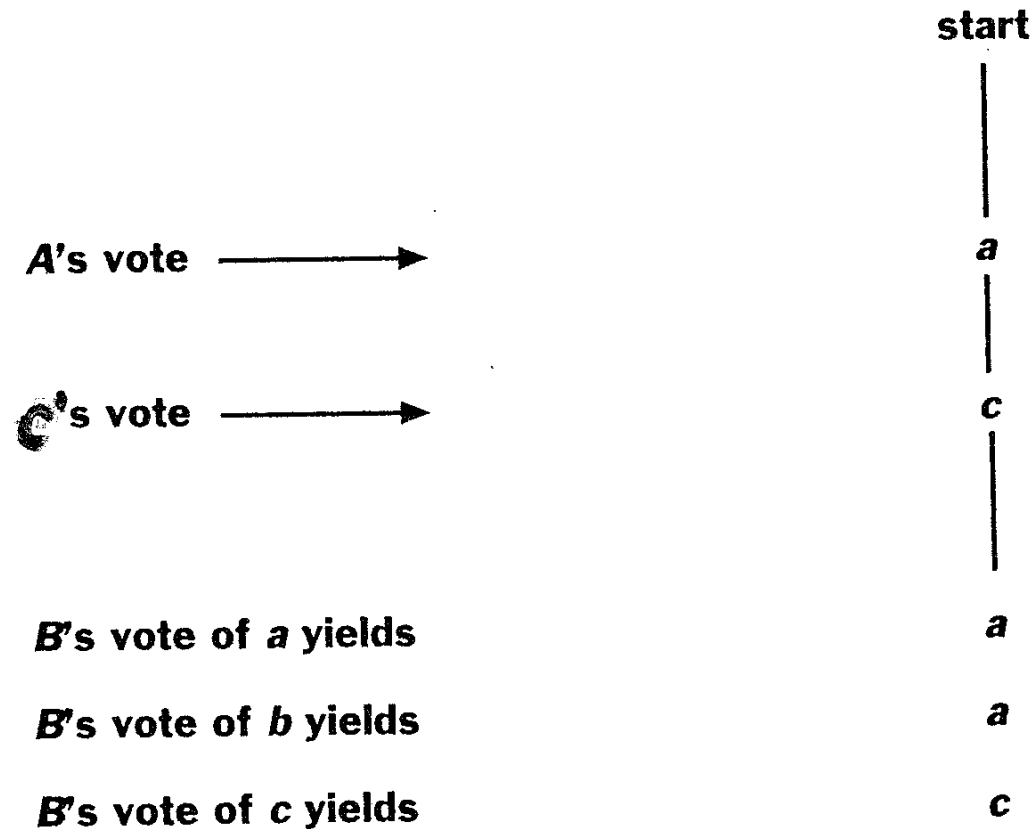
For player  $B$ :



$B$ 's preference:  $(b \quad c \quad a)$

“Vote for  $b$ ” is not a weakly dominant strategy for  $B$ .

In the new game where Player  $A$  definitely votes for  $a$  and Player  $C$  definitely votes for  $c$ , the strategy “vote for  $c$ ” is a weakly dominant strategy for Player  $B$ .



Sophisticated voting:  $A$  votes for  $a$ ,  $B$  votes for  $c$  and  $C$  votes for  $c$  yield  $c$ . Alternative  $c$  is  $A$ 's least preferred alternative even though  $A$  had the additional “tie-breaking” power.



## 4.2 Analysis of voting methods

Some properties that are, at least intuitively, desirable.

- If ties were not allowed, then we could have said “the” social choice instead of “a” social choice.

*Pareto condition*

If everyone prefers  $x$  to  $y$ , then  $y$  is not a social choice.

*Condorcet Winner Criterion* (Condorcet winner may not exist)

If there is an alternative  $x$  which could obtain a majority of votes in pairwise contests against *every* other alternative, a voting rule should choose  $x$  as the winner.

*Condorcet Loser criterion*

If an alternative  $y$  would lose in pairwise majority contests against every other alternative, a voting rule should *not* choose  $y$  as a winner.

## *Monotonicity Criterion* (幫他不會導致害他)

If  $x$  is a winner under a voting rule, and one or more voters change their preferences in a way favorable to  $x$  (without changing the order in which they prefer any other alternatives), then  $x$  should still be a winner.

## *Independence of irrelevant alternatives*

For any pair of alternatives  $x$  and  $y$ , if a preference list is changed but the relative positions of  $x$  and  $y$  to each other are not changed, then the new list can be described as arising from upward and downward shifts of alternatives other than  $x$  and  $y$ . Changing preferences toward these other alternatives should be irrelevant to the social preference of  $x$  to  $y$ .

- If we start with  $x$  a winner while  $y$  is a non-winner, people move some other alternative  $z$  around, then we cannot guarantee that  $x$  is still a winner. However, the independence of irrelevant alternatives says that  $y$  should remain a non-winner.

## Positive results

1. The plurality procedure satisfies the Pareto condition.

*Proof:* If everyone prefers  $x$  to  $y$ , then  $y$  is not on the top of any list (let alone a plurality), and thus  $y$  is certainly not a social choice.

2. The Borda count satisfies the Pareto condition.

*Proof:* If everyone prefers  $x$  to  $y$ , then  $x$  receives more points from each list than  $y$ . Thus,  $x$  receives a higher total than  $y$  and so  $y$  cannot be a winner.

3. The Hare system satisfies the Pareto condition.

*Proof:* If everyone prefers  $x$  to  $y$ , then  $y$  is not on the top of any list. Thus, either we have immediate winner and  $y$  is not among them or the procedure moves on and  $y$  is eliminated at the very next stage. Hence,  $y$  is not a winner.

4. Sequential pairwise voting satisfies the Condorcet winner criterion.

*Proof:* A Condorcet winner (if exists) always wins the kind of one-on-one contest that is used to produce the winner in sequential pairwise voting.

5. The plurality procedure satisfies monotonicity.

*Proof:* If  $x$  is on the top of the most lists, then moving  $x$  up one spot on some list (and making no other changes) certainly preserves this.

6. The Borda count satisfies monotonicity

*Proof:* Swapping  $x$ 's position with the alternative above  $x$  on some list adds one point to  $x$ 's score and subtracts one point from that of the other alternative; the scores of all other alternatives remain the same.

7. Sequential pairwise voting satisfies monotonicity.

*Proof:* Moving  $x$  up on some list only improves  $x$ 's chances in one-on-one contests.

8. The dictatorship procedure satisfies the Pareto condition.

*Proof:* If everyone prefers  $x$  to  $y$ , then, in particular, the dictator does. Hence,  $y$  is not on top of the dictator's list and so is not a social choice.

9. A dictatorship satisfies monotonicity.

*Proof:* If  $x$  is the social choice then  $x$  is on top of the dictator's list. Hence, the exchange of  $x$  with some alternative immediately above  $x$  must be taking place on some list other than that of the dictator. Thus,  $x$  is still the social choice.

10. A dictatorship satisfies independence of irrelevant alternatives.

*Proof:* If  $x$  is the social choice and no one — including the dictator — changes his or her mind about  $x$ 's preference to  $y$ , then  $y$  cannot wind up on top of the dictator's list. Thus,  $y$  is not the social choice.

## Negative results

1. Sequential pairwise voting with a fixed agenda does not satisfy the Pareto condition.

*Proof:*

<b>Voter 1</b>	<b>Voter 2</b>	<b>Voter 3</b>
$a$	$c$	$b$
$b$	$a$	$d$
$d$	$b$	$c$
$c$	$d$	$a$

Everyone prefers  $b$  to  $d$ . But with the agenda  $a\ b\ c\ d$ ,  $a$  first defeats  $b$  by a score of 2 to 1, and then  $a$  loses to  $c$  by this same score. Alternative  $c$  now goes on to face  $d$ , but  $d$  defeats  $c$  again by a 2 to 1 score. Thus, alternative  $d$  is the social choice even though everyone prefers  $b$  to  $d$ . Alternative  $d$  has the advantage that it is bought up later.

2. The plurality procedure fails to satisfy the Condorcet winner criterion.

*Proof:* Consider the three alternatives  $a, b$ , and  $c$  and the following sequence of nine preference lists grouped into voting blocs of size four, three, and two.

Voters 1–4	Voters 5–7	Voters 8–9
$a$	$b$	$c$
$b$	$c$	$b$
$c$	$a$	$a$

- With the plurality procedure, alternative  $a$  is clearly the social choice since it has four first-place votes to three  $b$  and two for  $c$ .
- $b$  is a Condorcet winner,  $b$  would defeat  $a$  by a score of 5 to 4 in one-on-one competition, and  $b$  would defeat  $c$  by a score of 7 to 2 in one-on-one competition.



3. The Borda count does not satisfy the Condorcet winner criterion.
4. A dictatorship does not satisfy the Condorcet winner criterion.

*Proof:* Consider the three alternatives  $a, b$  and  $c$ , and the following three preference lists:

<b>Voter 1</b>	<b>Voter 2</b>	<b>Voter 3</b>
$a$	$c$	$c$
$b$	$b$	$b$
$c$	$a$	$a$

Assume that Voter 1 is the dictator. Then,  $a$  is the social choice, although  $c$  is clearly the Condorcet winner since it defeats both others by a score of 2 to 1.

5. The Hare procedure does not satisfy the Condorcet winner criterion.

*Proof:*

Voters 1–5	Voters 6–9	Voters 10–12	Voters 13–15	Voter 16–17
$a$	$e$	$d$	$c$	$b$
$b$	$b$	$b$	$b$	$c$
$c$	$c$	$c$	$d$	$d$
$d$	$d$	$e$	$e$	$e$
$e$	$a$	$a$	$a$	$a$

- $b$  is the Condorcet winner:  $b$  defeats  $a$  (12 to 5),  $b$  defeats  $c$  (14 to 3),  $b$  defeats  $d$  (14 to 3),  $b$  defeats  $e$  (13 to 4).
- On the other hand, the social choice according to the Hare procedure is definitely not  $b$ . That is, no alternative has the nine first place votes required for a majority, and so  $b$  is deleted from all the lists since it has only two first place votes.

6. The Hare procedure does not satisfy monotonicity.

*Proof*

<b>Voters 1–7</b>	<b>Voters 8–12</b>	<b>Voters 13–16</b>	<b>Voter 17</b>
<i>a</i>	<i>c</i>	<i>b</i>	<i>b</i>
<i>b</i>	<i>a</i>	<i>c</i>	<i>a</i>
<i>c</i>	<i>b</i>	<i>a</i>	<i>c</i>

Since no alternative has 9 or more of the 17 first place votes, we delete the alternatives with the fewest first place votes. In this case, that would be alternatives *c* and *b* with only five first place votes each as compared to seven for *a*. But now *a* is the only alternative left, and so it is obviously on top of a majority (in fact, all) of the lists. Thus, *a* is the social choice when the Hare procedure is used.

Favorable-to- $a$ -change yields the following sequence of preference lists:

<b>Voters 1–7</b>	<b>Voters 8–12</b>	<b>Voters 13–16</b>	<b>Voter 17</b>
$a$	$c$	$b$	$a$
$b$	$a$	$c$	$b$
$c$	$b$	$a$	$c$

If we apply the Hare procedure again, we find that no alternative has a majority of first place votes and so we delete the alternative with the fewest first place votes. In this case, that alternative is  $b$  with only four. But the reader can now easily check that with  $b$  so eliminated, alternative  $c$  is on top of 9 of the 17 lists. This is a majority and so  $c$  is the social choice.

7. The plurality procedure does not satisfy independence of irrelevant alternatives.

<b>Voter 1</b>	<b>Voter 2</b>	<b>Voter 3</b>	<b>Voter 4</b>
<i>a</i>	<i>a</i>	<i>b</i>	<i>c</i>
<i>b</i>	<i>b</i>	<i>c</i>	<i>b</i>
<i>c</i>	<i>c</i>	<i>a</i>	<i>a</i>

When the plurality procedure is used, *a* is a winner and *b* is a non-winner. Suppose that Voter 4 changes his or her list by moving the alternative *c* down between *b* and *a*. The lists then become:

<b>Voter 1</b>	<b>Voter 2</b>	<b>Voter 3</b>	<b>Voter 4</b>
$a$	$a$	$b$	$b$
$b$	$b$	$c$	$c$
$c$	$c$	$a$	$a$

Notice that we still have  $b$  over  $a$  in Voter 4's list. However, plurality voting now has  $a$  and  $b$  tied for the win with two first place votes each. Thus, although no one changed his or her mind about whether  $a$  is preferred to  $b$  or  $b$  to  $a$ , the alternative  $b$  went from being a non-winner to being a winner.

8. The Borda count does not satisfy independence of irrelevant alternatives.

*Proof:*

Voters 1–3	Voters 4 and 5
$a$	$c$
$b$	$b$
$c$	$a$

The Borda count yields  $a$  as the social choice since it gets 6 points ( $2 + 2 + 2 + 0 + 0$ ) to only five for  $b$  ( $1 + 1 + 1 + 1 + 1$ ) and four for  $c$  ( $0 + 0 + 0 + 2 + 2$ ).

Voter 1–3	Voter 4 and 5
$a$	$b$
$b$	$c$
$c$	$a$

The Borda count now yields  $b$  as the social choice with 7 points to only 6 for  $a$  and 2 for  $c$ .

9. The Hare procedure fails to satisfy independence of irrelevant alternatives.

*Proof:*

<b>Voter 1</b>	<b>Voter 2</b>	<b>Voter 3</b>	<b>Voter 4</b>
$a$	$a$	$b$	$c$
$b$	$b$	$c$	$b$
$c$	$c$	$a$	$a$

Alternative  $a$  is the social choice when the Hare procedure is used because it has at least half the first place votes,  $a$  is a winner and  $b$  is a non-winner.



<b>Voter 1</b>	<b>Voter 2</b>	<b>Voter 3</b>	<b>Voter 4</b>
<i>a</i>	<i>a</i>	<i>b</i>	<i>b</i>
<i>b</i>	<i>b</i>	<i>c</i>	<i>c</i>
<i>c</i>	<i>c</i>	<i>a</i>	<i>a</i>

Notice that we still have  $b$  over  $a$  in Voter 4's list. Under the Hare procedure, we now have  $a$  and  $b$  tied for the win, since each has half the first place votes. Thus, although no one changed his or her mind about whether  $a$  is preferred to  $b$  or  $b$  to  $a$ , the alternative  $b$  went from being a non-winner to being a winner.

10. Sequential pairwise voting with a fixed agenda fails to satisfy independence of irrelevant alternatives.

*Proof:*

Consider the alternative  $c, b$  and  $a$ , and assume this reverse alphabetical ordering is the agenda. Consider the following sequence of three preference lists:

<b>Voter 1</b>	<b>Voter 2</b>	<b>Voter 3</b>
$c$	$a$	$b$
$b$	$c$	$a$
$a$	$b$	$c$

In sequential pairwise voting,  $c$  would defeat  $b$  by the score of 2 to 1 and then lose to  $a$  by this same score. Thus,  $a$  would be the social choice (and thus  $a$  is a winner and  $b$  is a non-winner).

Suppose that Voter 1 moves  $c$  down between  $b$  and  $a$ , yielding the following lists:

<b>Voter 1</b>	<b>Voter 2</b>	<b>Voter 3</b>
$b$	$a$	$b$
$c$	$c$	$a$
$a$	$b$	$c$

Now,  $b$  first defeats  $c$  and then  $b$  goes on to defeat  $a$ . Hence, the new social choice is  $b$ . Thus, although no one changes his or her mind about whether  $a$  is preferred to  $b$  or  $b$  to  $a$ , the alternative  $b$  went from being a non-winner to being a winner. This shows that independence of irrelevant alternatives fails for sequential pairwise voting with a fixed agenda.

## Summary

	<b>Pareto</b>	<b>Condorcet Winner Criterion</b>	<b>Monotonicity</b>	<b>Independence of Irrelevant Alternatives</b>
Plurality	Yes	No	Yes	No
Borda	Yes	No	Yes	No
Hare	Yes	No	No	No
Seq pairs	No	Yes	Yes	No
Dictator	Yes	No	Yes	Yes

*Query:* The stated properties appear to be quite reasonable. Why haven't we presented a number of natural procedures that satisfy all of these properties and more?

## Condorcet voting methods

Recall that only the sequential pairwise voting satisfies the Condorcet winner criterion. However, Borda count does not satisfy the Condorcet winner criterion.

<i>3 voters</i>	<i>2 voters</i>	<i>Borda count:</i>
<i>a</i>	<i>b</i>	" <i>a</i> " is 6
<i>b</i>	<i>c</i>	" <i>b</i> " is 7
<i>c</i>	<i>a</i>	" <i>c</i> " is 2.

"*b*" is the Borda winner but "*a*" is the Condorcet winner. Worse, "*a*" has an absolute majority of first place votes. [Majority criterion: If a majority of voters have an alternative  $x$  as their first choice, a voting rule should choose  $x$ .]

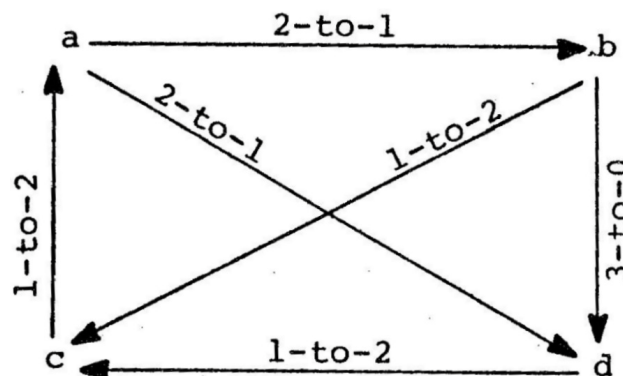
Why "*b*" wins in the Borda count? The presence of "*c*" enables the last 2 voters to weigh their votes for "*b*" over "*a*" more heavily than the first 3 voters' votes for "*a*" over "*b*". If "*c*" is removed, then "*a*" is chosen as the Borda winner. This is a violation of "Independence of Irrelevant Alternatives".

## *Black method*

Value the Condorcet criterion, but also believe that the Borda count has advantages.

- In cases where there is a Condorcet winner, choose it; otherwise, choose the Borda winner.

<i>voter A</i>	<i>voter B</i>	<i>Voter C</i>
<i>a</i>	<i>c</i>	<i>b</i>
<i>b</i>	<i>a</i>	<i>d</i>
<i>d</i>	<i>b</i>	<i>c</i>
<i>c</i>	<i>d</i>	<i>a</i>



- We check to see if one alternative beats all the other in pairwise contests. If so, that alternative wins. If not, we use the numbers to compute the Borda winner.
- Satisfies the Pareto, Condorcet loser, Condorcet winner and Monotonicity criteria. However, it does not satisfy

*Generalized Condorcet criterion:* If the alternatives can be partitioned into two sets  $A$  and  $B$  such that every alternative in  $A$  beats every alternative in  $B$  in pairwise contests, then a voting rule should **not** select an alternative in  $B$ .

The above criterion implies both the Condorcet winner and Condorcet loser criteria (take  $A$  to be the set which consists of only the Condorcet winner, or  $B$  to be the set which consists of only the Condorcet loser).

The following example shows that Black's rule violates this criterion:

<i>1 Voter</i>	<i>1 Voter</i>	<i>1 Voter</i>
<i>a</i>	<i>b</i>	<i>c</i>
<i>b</i>	<i>c</i>	<i>a</i>
<i>x</i>	<i>x</i>	<i>x</i>
<i>y</i>	<i>y</i>	<i>y</i>
<i>z</i>	<i>z</i>	<i>z</i>
<i>w</i>	<i>w</i>	<i>w</i>
<i>c</i>	<i>a</i>	<i>b</i>

- If we partition the alternatives as  $A = [a, b, c]$  and  $B = [x, y, z, w]$ , then every alternative in  $A$  beats every alternative in  $B$  by a 2-to-1 vote.
- Furthermore, there is no Condorcet winner, since alternatives  $a$  and  $b$  and  $c$  beat each other cyclically.
- When we compute Borda counts, we get:

<i>a</i>	<i>b</i>	<i>c</i>	<i>x</i>	<i>y</i>	<i>z</i>	<i>w</i>
11	11	11	12	9	6	3

By the Black rule,  $x$  is the winner.

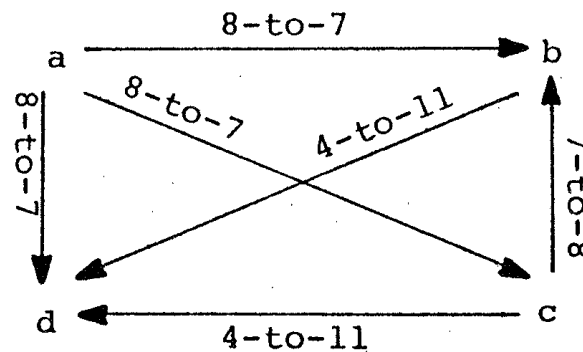


## Nanson method

- It is a Borda elimination scheme which sequentially eliminates the alternative with the lowest Borda count until only one alternative or a collection of tied alternatives remains.
- That this procedure will indeed always select the Condorcet winner, if there is one. Note that the Condorcet winner must gather more than half the votes in its pairwise contests with the other alternatives. There is no guarantee that the Condorcet winner wins in Borda count in each pairwise contest. However, by summing all  $n - 1$  pairwise contests with other alternatives, the Borda counts of the Condorcet winner must be higher than those of the sum of all other  $n - 1$  alternatives.
- Since it must always have a *higher than average* Borda count, it would never have the *lowest* Borda count and can never be eliminated in all steps.

3 Voters	4 Voters	4 Voters	4 Voters
<i>b</i>	<i>b</i>	<i>c</i>	<i>d</i>
<i>c</i>	<i>a</i>	<i>a</i>	<i>a</i>
<i>d</i>	<i>c</i>	<i>b</i>	<i>c</i>
<i>a</i>	<i>d</i>	<i>d</i>	<i>b</i>

The pairwise voting diagram is:



so that alternative *a* is the Condorcet winner. The Borda counts are *a* : 24, *b* : 25, *c* : 26 and *d* : 15. Hence, alternative *c* would be the Borda winner, and alternative *a* would come in next-to-last.

Under Nanson's procedure alternative  $d$  is eliminated and new Borda counts are computed:

3 Voters	4 Voters	4 Voters	4 Voters	
$b$	$b$	$c$	$a$	Borda $a : 16$
$c$	$a$	$a$	$c$	counts $b : 14$
$a$	$c$	$b$	$b$	$c : 15$

Alternative  $b$  is now eliminated, and in the final round alternative  $a$  beats  $c$  by 8-to-7.

- Since Nanson's procedure so cleverly reconciles the Borda count with the Condorcet criterion, it is a shame, but perhaps not surprising, to find that it shares the defect of other elimination schemes: it is not monotonic.

8 Voters	5 Voters	5 Voters	2 Voters
$a$	$c$	$b$	$c$
$b$	$a$	$c$	$b$
$c$	$b$	$a$	$a$

- The Borda counts are  $a : 21, b : 20$ , and  $c : 19$ . Hence  $c$  is eliminated, and then alternative  $a$  beats  $b$  by 13-to-7.
- If the last two voters change their minds in favor of alternative  $a$  over  $b$ , so that their preference ordering is  $cab$ , the new Borda counts will be  $a : 23, b : 18$  and  $c : 19$ . Hence  $b$  will be eliminated and then  $c$  beats  $a$  by 12-to-8. The change in alternative  $a$ 's favor has produced  $c$  as the winner.

*Query:* Suppose a Condorcet winner exists, will Nanson method observe monotonicity?

## *Copeland method*

- One looks at the results of pairwise contests between alternatives. For each alternative, compute the number of pairwise wins it has minus the number of pairwise losses it has. Choose the alternative(s) for which this difference is largest.
- It is clear that if there is a Condorcet winner, Copeland's rule will choose it: the Condorcet winner will be the only alternative with all pairwise wins and no pairwise losses. The Copeland rule also satisfies *all* of the other criteria we have considered.
- This method is more likely than other methods to produce ties. If its indecisiveness can be tolerated, it seems to be a very good voting rule indeed.
- It may come into spectacular conflict with the Borda count.

1 Voter	4 Voters	1 Voter	3 Voters
<i>a</i>	<i>c</i>	<i>e</i>	<i>e</i>
<i>b</i>	<i>d</i>	<i>a</i>	<i>a</i>
<i>c</i>	<i>b</i>	<i>d</i>	<i>b</i>
<i>d</i>	<i>e</i>	<i>b</i>	<i>d</i>
<i>e</i>	<i>a</i>	<i>c</i>	<i>c</i>
Copeland	<i>a</i> : 2	Borda	<i>a</i> : 16
scores:	<i>b</i> : 0	scores:	<i>b</i> : 18
	<i>c</i> : 0		<i>c</i> : 18
	<i>d</i> : 0		<i>d</i> : 18
	<i>e</i> : -2		<i>e</i> : 20

- Alternative *a* is the Copeland winner and *e* comes in last, but *e* is the Borda winner and *a* comes in last. The two methods produce diametrically opposite results.
- If we try to ask directly whether *a* or *e* is better, we notice that the Borda winner *e* is preferred to the Copeland winner, alternative *a*, by *eight* of the nine voters!

## Summary

- Sequential pairwise voting is bad because of the agenda effect and the possibility of choosing a Pareto dominated alternative.
- Plurality voting is bad because of the weak mandate (來自選民的授權) it may give – in particular, it may choose an alternative which would lose to *any* other alternative in a pairwise contest. This is a violation of the Condorcet Loser criterion.
- Plurality with run-off and the elimination schemes due to Hare, Coombs and Nanson all fail to be monotonic: changes in an alternative's favor can change it from a winner to a loser.
- Of these four elimination schemes, Coombs and Nanson are better than the others. They generally avoid disliked alternatives, the Nanson rule always detects a Condorcet winner when there is one, and the Coombs scheme almost always does.

- The Borda count takes positional information into full account and generally chooses a non-disliked alternative. Its major difficulty is that it can directly conflict with majority rule, choosing another alternative even when a majority of voters agree on what alternative is best. Thus, the Borda count would only be appropriate in situations where it is acceptable that an alternative preferred by a majority not be chosen if it is strongly disliked by a minority.
- The voting rules due to Copeland and Black appear to be quite strong. The Black rule directly combines the virtues of the Condorcet and Borda approaches to voting. The Copeland rule emphasizes the Condorcet approach. How it might be modified to avoid the most violent of conflicts with the Borda approach?

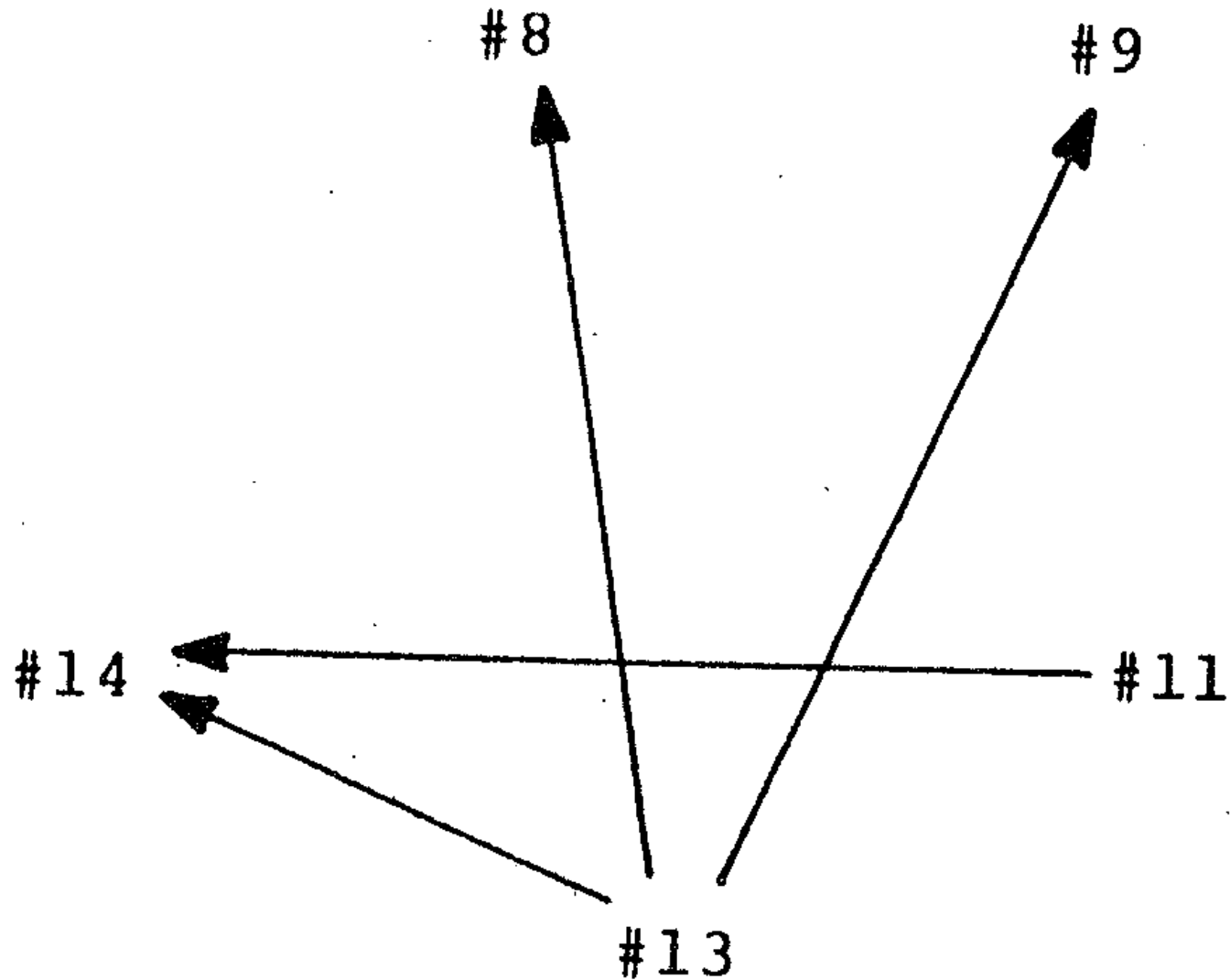


## Example

<i>B</i> 2 Votes	<i>P</i> 2 Votes	<i>C</i> 1 Vote	<i>W</i> 1 Vote
8	14	13	11
9	13	11-14	9-13
11	8-9-11	9	8
13		8	14
14			

The entries at the same level represent voter indifference among certain alternatives. We will assume that indifferent voters abstain in pairwise contest.

The pairwise voting diagram for this example is



#11 also beats #8 and #9 in pairwise contests (not shown).

1. Sequential pairwise voting. Alternative #11 and #13 will enter the voting at some stage, and will then be deadlocked.
2. Plurality voting. Alternative #8 and #14 will be tied. The adoption of run-off or Hare elimination will produce the same deadlock.
3. Coombs elimination. Alternative #14 will be the first eliminated since it has the most last-place votes.

<i>2 Votes</i>	<i>2 Votes</i>	<i>1 Vote</i>	<i>1 Vote</i>
8	13	13	11
9	8-9-11	11	9-13
11		9	8
13		8	

No alternative yet has a majority, and #8 (with 2.67 last-place votes) will be the next to go:

<i>2 Votes</i>	<i>2 Votes</i>	<i>1 Vote</i>	<i>1 Vote</i>
9	13	13	11
11	9-11	11	9-13
13		9	

Alternatives #9 and #13 (2.5 last place votes each) are then eliminated, and alternative #11 wins.

4. Borda count. In the Borda count with ties, tied options are awarded the *average* of the points which would normally be awarded to the places they hold. Thus in the preference list

14  
13  
8-9-11

alternatives #8, #9, and #11 occupy positions worth  $0 + 1 + 2 = 3$  points, so they are each awarded one point. The Borda counts are:

#8	#9	#11	#13	#14
11	11.5	12.5	14.5	10.5

and alternative #13 is the Borda winner.

5. Black. Since there is no Condorcet winner (#13 ties #11, but does not beat it), the Black rule awards the decision to the Borda winner, #13.

6. Nanson elimination. Under Nanson's elimination scheme, option #14, which has the lowest Borda count, is eliminated, and revised Borda counts are #8 : 8, #9 : 8.5, #11 : 9 and 13 : 10.5. Alternative #8 is then eliminated, yielding new Borda counts of #9 : 5.5, #11 : 6 and #13 : 6.5. Option #9 is then eliminated and then #11 and #13 tie.

7. The Copeland scores are

$$\begin{array}{lcl} \#8 : & 0 - 1 = & -1 \\ \#9 : & 0 - 1 = & -1 \\ \#11 : & 1 - 0 = & 1 \\ \#13 : & 3 - 0 = & 3 \\ \#14 : & 0 - 2 = & -2 \end{array}$$

with alternative #13 as the Copeland winner.

These results can be summarized as follows:

<i>Voting Method</i>	<i>Winner(s)</i>
Sequential pairwise	11, 13
Plurality	8, 14
Plurality with run-off	8, 14
Hare	8, 14
Coombs	11
Borda count	13
Black	13
Nanson	11, 13
Copeland	13

\$11 or #13 emerges when the emphasis is placed on the Condorcet winner criterion. The voting method used has a major effect upon the decision reached.

## Glimpse of Impossibility

- Rather than considering any particular social choice procedures, we would like to establish limitations on what kind of “better” procedures that can ever be found.
- Suppose we were asked to seek a social choice procedure that satisfies all four of desirable properties: Pareto, Condorcet Winner, Monotonicity and Independence of Irrelevant Alternatives. One possibility is to start with one of the four procedures that we looked at and to modify it in such a way that a property that was not satisfied by the original procedure would be satisfied by the new version.



## Theorem

There is no social choice procedure for three or more alternatives that satisfies both independence of irrelevant alternatives and the Condorcet winner criterion.

Our proof will be by contradiction: We will assume that we have a social choice procedure that satisfies both independence of irrelevant alternatives and the Condorcet winner criterion. We then show that if this procedure is applied to the profile that constitutes Condorcet's voting paradox, then it produces no winner.

## Proof

Assume that we have a social choice procedure that satisfies both independence of irrelevant alternatives and the Condorcet winner criterion. Consider the following profile from the voting paradox of Condorcet:

$a$	$c$	$b$
$b$	$a$	$c$
$c$	$b$	$a$

### *Claim 1*

The alternative  $a$  is a non-winner.

Consider the following profile (obtained by moving alternative  $b$  down in the third preference list from the voting paradox profile):

$a$	$c$	$c$
$b$	$a$	$b$
$c$	$b$	$a$

- Notice that  $c$  is a Condorcet winner (defeating both other alternatives by a margin of 2 to 1). Thus, our social choice procedure must produce  $c$  as the only winner. Thus,  $c$  is a winner and  $a$  is a non-winner for this profile.
- Suppose now that the third voter moves  $b$  up on his or her preference list. The profile then becomes that of the voting paradox. But no one changed his or her mind about whether  $c$  is preferred to  $a$  or  $a$  is preferred to  $c$ . By “independence of irrelevant alternatives”, and because we had  $c$  as a winner and  $a$  as a non-winner in the profile with which we began the proof of the claim, we can conclude that  $a$  is still a non-winner when the procedure is applied to the voting paradox profile.

## *Claim 2*

The alternative  $b$  is a non-winner.

- Consider the following profile (obtained by moving alternative  $c$  down in the second preference list from the voting paradox profile):

$a$	$a$	$b$
$b$	$c$	$c$
$c$	$b$	$a$

Notice that  $a$  is a Condorcet winner (defeating both other alternatives by a margin of 2 to 1). Thus, our social choice procedure (which we are assuming satisfies the Condorcet winner criterion) must produce  $a$  as the only winner. Thus,  $a$  is a winner and  $b$  is a non-winner for this profile.

- Suppose now that the second voter moves  $c$  up on his or her preference list. The profile then becomes that of the voting paradox. But no one changed his or her mind about whether  $a$  is preferred to  $b$  or  $b$  is preferred to  $a$ . By “independence of irrelevant alternatives”, and because we had  $a$  as a winner and  $b$  as a non-winner in the profile with which we began the proof of the claim, we can conclude that  $b$  is still a non-winner when the procedure is applied to the voting paradox profile.

### *Claim 3*

It can be shown similarly that the alternative  $c$  is a non-winner.

- The above three claims show that when our procedure produces *no* winner. But a social choice procedure must always produce at least one winner. Thus, we have a contradiction and the proof is complete.

## 4.3 Arrow's Impossibility Theorem

### Social welfare function

1. accepts as input a sequence of individual preference lists of some set  $A$  (the set of alternatives), and,
  2. produces as output a listing (perhaps with ties) of the set  $A$ ; this list is called the *social preference list*.
- \* Allow ties in the output but not in the input.

Note that unlike a social choice procedure, the output is a “social preference listing” of the alternatives.

A social welfare function produces a listing of all alternatives, we can take alternative (or alternatives if tied) at the top of the list as the social choice.

### *Proposition*

Every social welfare function (obviously) gives rise to a social choice procedure (for that choice of voters and alternatives). Moreover (and less obviously), every social choice procedure gives rise to a social welfare function.

- We have a social choice procedure, how to use this procedure to produce a listing of all the alternatives in  $A$ .

### *Iteration procedure*

Simply delete from each of the individual preference lists those alternatives that we've already chosen to be on top of the social preference list. Now, input these new individual preference lists to the social choice procedure at hand. The new group of “winners” is precisely the collection of alternatives that we will choose to occupy the second place on the social preference list. Continuing this, we delete these “second-round winners” and run the social choice procedure again to obtain the alternatives that will occupy the third place in the social preference list, and so on until all alternatives have been taken care of.





A social welfare function aggregates individual preference lists into a social preference list.

## Definition

If  $A$  is a set (of alternatives) and  $P$  is a set (or people), then a *social welfare function* for  $A$  and  $P$  that it accepts as inputs only those sequences of individual preference listings of this particular set  $A$  that correspond to this particular set  $P$ .

- Assume for the moment that we have a fixed set  $A$  of three or more alternatives and a fixed finite set  $P$  of people. Our goal is to find a social welfare function for  $A$  and  $P$  that is “reasonable” in the sense of reflecting the will of the people.

## Social choice functions for two alternatives

- $n$  people and two alternatives:  $x$  and  $y$ .
- Simply voting for one of the alternatives instead of providing a preference list.
- Majority rule declares the social choice to be whichever alternative which has more than half the votes (possibility of a tie if the number of people is even).

### *Some examples of social welfare functions*

1. Designate one person as the dictator.
2. Alternative  $x$  is always the social choice.
3. The social choice is  $x$  when the number of votes for  $x$  is even.

## *Desirable properties of social welfare functions*

### 1. Anonymity

anonymous (不具名) if the social welfare function is invariant under permutation of the people

- Dictatorship does not satisfy anonymity

### 2. Neutrality

neutral if it is invariant under permutations of the alternatives

For example, if  $(H \ L \ H \ L \ L)$  yields  $L$ ; by swapping  $H$  for  $L$ ,

then  $(L \ H \ L \ H \ H)$  should yield  $H$ .

If  $\begin{pmatrix} a & b & c \\ c & a & b \\ b & c & a \end{pmatrix}$  produces  $\begin{pmatrix} c \\ b \\ a \end{pmatrix}$ , then  $\begin{pmatrix} c & b & a \\ a & c & b \\ b & a & c \end{pmatrix}$  produces  $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ . Note that we have swapped  $a$  for  $c$  and vice versa.

### 3. Monotonicity

If outcome is  $L$ , and one or more votes are changed from  $H$  to  $L$ , then the outcome is still  $L$ .

## Quota system

$n$  people and 2 alternatives; fix a number  $q$  that satisfies

$$\frac{n}{2} < q \leq n + 1.$$

Consider the procedure wherein the outcome is a tie unless one of the alternatives has at least  $q$  votes. If one of the alternatives has  $q$  or more votes, then it alone is the social choice.

1. If  $n$  is odd and  $q = \frac{n + 1}{2}$ , then the quota system is just majority vote.
2. If  $q = n + 1$  and there are only  $n$  people, then the outcome is always a tie. This corresponds to the procedure that declares the social choice to be a tie between the two alternatives regardless of how the people vote.

All quota systems satisfy anonymity, neutrality, and monotonicity. The first two properties are seen to be automatically satisfied by any quota system since the procedure does the direct votes counting.

## Theorem

Suppose we have a social welfare function for two alternatives that is anonymous, neutral, and monotone. Then that procedure is a quota system.

### *Proof*

It suffices to prove the following 2 conditions:

1. The alternative  $L$  alone is the social choice precisely when  $q$  or more people vote for  $L$ .
2.  $\frac{n}{2} < q \leq n + 1$ .

- The procedure is invariant under permutations of the people, so the outcome depends on the number of people who vote for, say,  $L$ .
- Let  $G$  denote the set of all numbers  $k$  such that  $L$  alone (獨贏) is the social choice when exactly  $k$  people vote for  $L$ .
  - (a) When  $G = \phi$ , this implies that  $L$  alone never wins. Also,  $H$  alone never wins by neutrality. In this case, the outcome is always a tie.
  - (b) If  $G$  is not empty, then we let  $q$  be the smallest number in  $G$ .

It is easily seen that Monotonicity  $\Rightarrow$  (1)

*Remark* Case (a) corresponds to  $q = n + 1$ .

- By neutrality, if  $k$  is in  $G$ , then  $n - k$  is definitely not in  $G$ . Otherwise, we would have  $H$  alone as the social choice when exactly  $n - k$  people voted for  $H$  (occurring automatically as  $k$  people voted for  $L$ ). This leads to a contradiction that  $L$  wins alone.



For example, take  $n = 11$  and  $q = 8$ . Now,  $k = 9$  is in  $G$  but  $n - k = 2$  cannot be in  $G$ . Otherwise, if 2 votes are sufficient for  $L$  to win, then 2 votes are also sufficient for  $H$  to win (neutrality property). However, when  $L$  receives 9 votes, then  $H$  receives 2 votes automatically. Both  $H$  and  $L$  win and this is contradicting to  $L$  wins alone when it receives 9 votes.

- By invoking monotonicity as a further step, if  $k$  is in  $G$ , then  $n - k$  cannot be as large as  $k$ . Thus,  $n - k < k$  or  $n < 2k$ . Hence,  $n/2 < k$  for any number that is in  $G$ .
- Lastly,  $q \leq n$  when  $G$  is non-empty and it suffices to take  $q$  to be  $n + 1$  when  $G = \phi$ . Thus,

$$n/2 < q \leq n + 1.$$

### *Remark*

When  $n$  is odd and we choose  $q > \frac{n+1}{2}$ , it is possible that the votes of both alternatives cannot achieve the quota. In this case, we have a tie. For example, we take  $n = 11$  and  $q = 7$ , suppose  $L$  has 6 votes and  $H$  has 5 votes, then a tie is resulted.

### **May Theorem**

If the number of people is odd and *ties are excluded*, then the only social welfare function for two alternatives that satisfies anonymity, neutrality and monotonicity is majority rule.

Note that at least one of the alternatives must receive number of votes to be  $\frac{n+1}{2}$  or above.

## Arrow's formalism

- Arrow (1950)'s desirable properties or axioms come into 3 types:
  - (i) axioms which restrict the inputs to a voting system — all individually rational preference orderings are allowed.

Say, voting system which works only when individuals are unanimous is not much of a voting system. This universal domain assumption say we cannot beg the question by assuming that all individuals have preferences of a certain type.

- (ii) axioms which restrict the outputs — the derived social preference ordering should be complete and transitive. The completeness axiom means that all questions of the form “Is  $X$  socially preferable to  $Y$ ” should have answers.

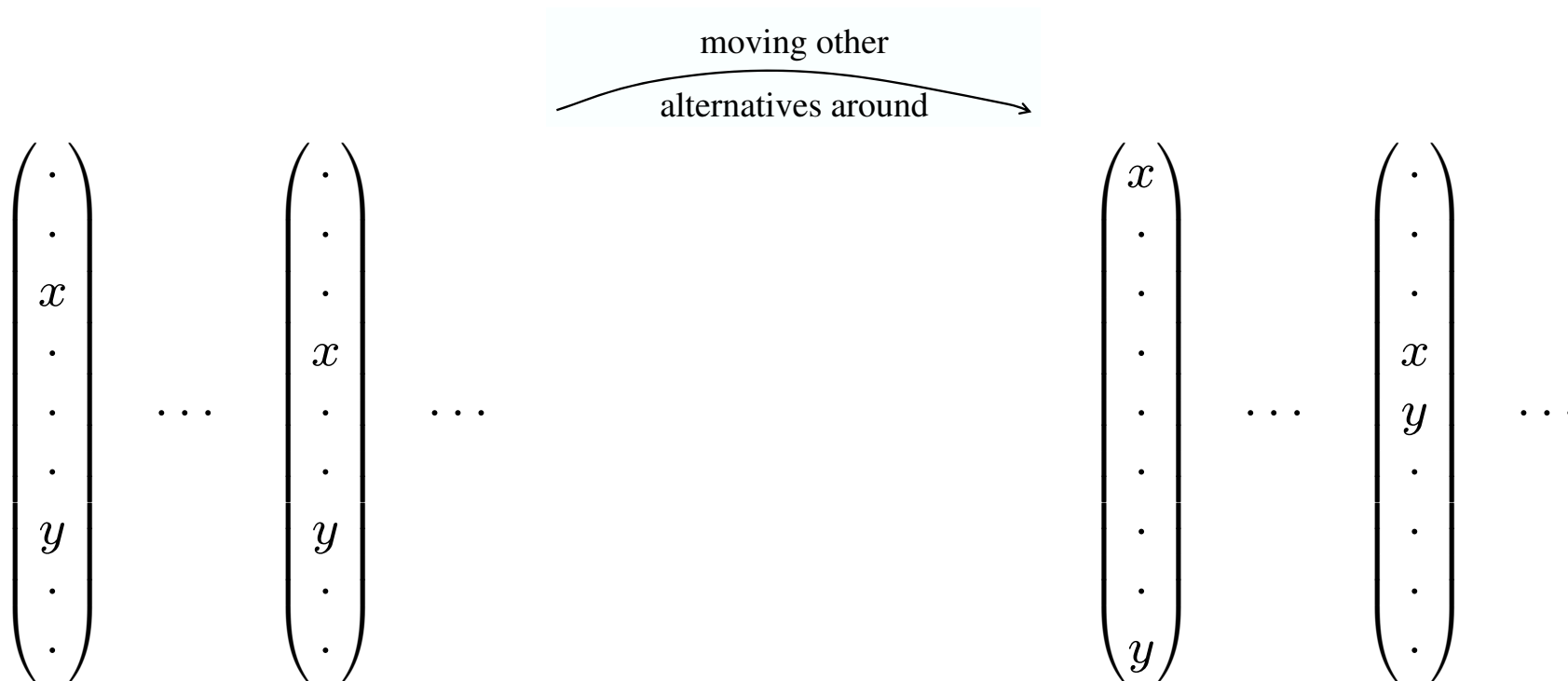
For example, the Pareto rule says that if some people prefer  $x$  to  $y$  and no one prefer  $y$  to  $x$  then  $x$  is socially preferable to  $y$ . However, the Pareto rule is incomplete since it cannot rank order  $x$  and  $y$  when some people prefer  $x$  to  $y$  and others prefer  $y$  to  $x$ .

(iii) axioms which put restrictions on how the voting system transform inputs into outputs.

A social welfare function (for  $A$  and  $P$ ) is called *weakly reasonable* if it satisfies the following three conditions:

1. Pareto: also called unanimity. Society put alternative  $x$  strictly above  $y$  whenever every individual puts  $x$  strictly above  $y$ . As a consequence, suppose the input consists of a sequence of identical lists, then this single list should also be the social preference list produced as output.
2. Independence of irrelevant alternatives (IIA): Suppose we have our fixed set  $A$  of alternatives and our fixed set  $P$  of people, but two different sequences of individual preference lists. Suppose also that exactly the same people have alternative  $x$  over alternative  $y$  in their list.

Suppose we have 6 voters, the 1<sup>st</sup> and the 4<sup>th</sup> voters place  $x$  above  $y$  while others place  $y$  above  $x$ . If we move other alternatives around to produce a new sequence, the social preference ordering between  $x$  and  $y$  remains unchanged.



**Interpretation of Independence of Irrelevant Alternatives**

Then we either get  $x$  over  $y$  in both social preference lists, or we get  $y$  over  $x$  in both social preference lists. The positioning of alternatives other than  $x$  and  $y$  in the individual preference lists is irrelevant to the question of whether  $x$  is socially preferred to  $y$  or  $y$  is socially preferred to  $x$ . In other words, the social relative ranking (higher or lower) of two alternatives  $x$  and  $y$  depends only on their relative ranking by every individual.

3. Monotonicity: If we get  $x$  over  $y$  in the social preference list, and someone who had  $y$  over  $x$  in his individual preference list interchanges the position of  $x$  and  $y$  in his list, then we still should get  $x$  over  $y$  in the social preference list.

## Non-dictatorship

There is no individual whose preference always prevails, that is, no individual's preference list is always the social preference list.

## Proposition

If  $A$  has at least three elements, then any social welfare function for  $A$  that satisfies both IIA and the Pareto condition will never produce ties in the output.

### *Proof*

- Assume, for contradiction, some sequence of individual preference lists result in a social preference list in which the alternatives  $a$  and  $b$  are tied, even though we are not allowing ties in any of the individual preference lists.
- Because of IIA, we know that  $a$  and  $b$  will remain tied as long as we don't change any individual preference list in a way that reverses that voter's ranking of  $a$  and  $b$ .

Let  $c$  be any alternative that is distinct from  $a$  and  $b$ . Let  $X$  be the set of voters who have  $a$  over  $b$  in their individual preference lists, and let  $Y$  be the rest of the voters (who therefore have  $b$  over  $a$  in their lists).



yields

$ab$  (tied).



- Suppose we now insert  $c$  between  $a$  and  $b$  in the lists of the voters in  $X$ , and we insert  $c$  above  $a$  and  $b$  in the lists of the voters in  $Y$ . Then we will still get  $a$  and  $b$  tied in the social preference list (by independence of irrelevant alternatives), and we will get  $c$  over  $b$  by Pareto, since  $c$  is over  $b$  in every individual preference list. Thus, we have:



yields

$c$   
 $ab.$

- Independence of irrelevant alternatives guarantees us that, as for as  $a$  versus  $c$  goes, we can ignore  $b$ . Thus, we can conclude that if everyone in  $X$  has  $a$  over  $c$  and everyone in  $Y$  has  $c$  over  $a$ , then we get  $c$  over  $a$  in the social preference list.

- To get our desired contradiction, we will go back and insert  $c$  differently from what we did before. We insert  $c$  under  $a$  and  $b$  for the voters in  $X$ , and between  $a$  and  $b$  for the voters in  $Y$ . Using Pareto as before shows that we now get:



yields

$ab$   
 $c.$

- Independence of irrelevant alternatives guarantees us that, as far as  $a$  versus  $c$  goes, we can ignore  $b$ . Thus, we can now conclude that if everyone in  $X$  has  $a$  over  $c$  and everyone in  $Y$  has  $c$  over  $a$ , then we get  $a$  over  $c$  in the social preference list. This is the opposite of what we concluded above, and thus we have the desired contradiction.

## Question

Are there any weakly reasonable social welfare functions for  $A$  and  $P$ ?

Yes—appoint a dictator. Taking the dictator's entire individual preference listing of  $A$  and declaring it to be the social preference list. Why?

Dictatorship satisfies Pareto condition (if  $x$  is preferred over  $y$  by all, including the dictator, then  $x$  is socially preferred over  $y$ ), IIA (moving other alternatives would not change the social ranking of  $x$  and  $y$ ) and monotonicity (interchanging the relative order of  $x$  and  $y$  in lists other than that of the dictator is irrelevant).

**Theorem** (Arrow, 1950). If  $A$  has at least three elements and the set  $P$  of individuals is finite, then the only social welfare function for  $A$  and  $P$  satisfying the Pareto condition, independence of irrelevant alternatives, and monotonicity is a dictatorship.

## *Remark*

The reference to monotonicity is completely unnecessary. It is included simply because it makes the proof conceptually easier. Monotonicity can be removed by an additional lemma.

(Restatement of Arrow's Theorem). If  $A$  has at least three elements and the set  $P$  of individuals is finite, then it is impossible to find a social welfare function for  $A$  satisfying the Pareto condition, independence of irrelevant alternatives, and non-dictatorship.

## **Setup of the Proof**

Under the assumption of Pareto, IIA, and monotonicity, we would like to establish that there always exists a particular singleton voter where the social preference list is the same as the preference of this singleton voter – a dictator. (逃不過有一“獨裁者”的命運)

## Definition

$X$  is a set of people,  $a$  and  $b$  are alternatives. “ $X$  can force  $a$  over  $b$ ” means

“We get  $a$  over  $b$  in the social preference list whenever everyone in  $X$  places  $a$  over  $b$  in their individual preference lists.”

某組人能足夠保證把 $a$ 放在 $b$ 之上

- Our secret weapons are IIA and monotonicity. In order to show that  $X$  forces  $a$  over  $b$  it suffices to produce a *single sequence* of individual preference lists for which the following all hold.

1. Everyone in  $X$  has  $a$  over  $b$  in their lists.
  2. Everyone not in  $X$  has  $b$  over  $a$  in their lists.
  3. The resulting social preference list has  $a$  over  $b$ .
- IIA says that whether or not we get  $a$  over  $b$  in the social preference list does not depend in any way on the placement of other alternatives in the individual preference lists. Hence, in showing that  $X$  forces  $a$  over  $b$ , it suffices to consider a single sequence of individual preference lists with all other alternatives strategically placed (helping to get through our argument in a particular proof).
  - By virtue of monotonicity, it suffices to consider the “worst scenario” where those not in  $X$  place  $b$  above  $a$ .

## Definition “dictating set”

A set  $X$  is called a dictating set if  $X$  can force  $a$  over  $b$  whenever  $a$  and  $b$  are two distinctive alternatives in  $A$ .

1. If  $X$  is the set of all individuals, then  $X$  is a dictating set. This follows directly from the Pareto condition.
2. If  $p$  is one of the individuals and  $X$  is the set consisting of  $p$  alone, then  $X$  is a dictating set if and only if  $p$  is a dictator.

Dictatorship  $\Rightarrow$  “force  $a$  over  $b$ ” is obvious. On the other hand, if  $p$  can always force  $a$  over  $b$  for any pair of alternatives, the social preference list must coincide with his own preference list, then  $p$  is a dictator.

The strategy for passing from the very large dictating set  $P$  where we are starting to the very small dictating set  $\{p\}$  where we want to end up involves the following:

Show that if  $X$  is a dictating set, and if we split  $X$  into two sets  $Y$  and  $Z$  (so that everything in  $X$  is in exactly one of the two sets), then either  $Y$  is a dictating set or  $Z$  is a dictating set.

Under the assumption of Pareto, IIA and monotonicity, we would like to establish that there always exists a particular singleton voter where the social preference list is the same as the preference list of this singleton voter – a dictator. This is deduced from the result that there always exists a dictating set with only one element.



## Five Lemmas Yielding Arrow's Theorem

Notice that independence of irrelevant alternatives is directly appealed to only in Lemma 1.

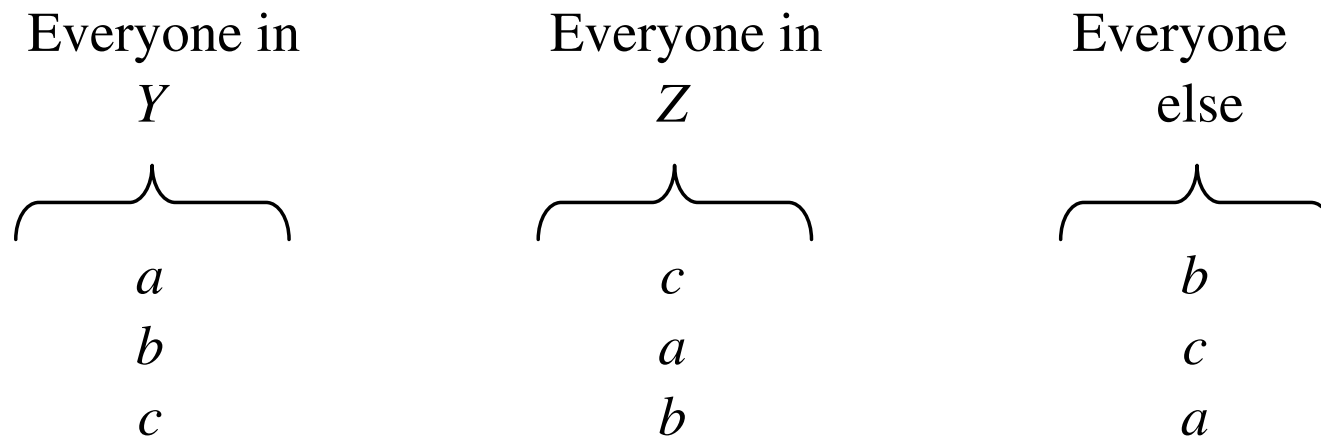
### Lemma 1

Suppose  $X$  forces  $a$  over  $b$  and  $c$  is an alternative distinct from  $a$  and  $b$ . Suppose now that  $X$  is split into two sets  $Y$  and  $Z$  (either of which may be the empty set) so that each element of  $X$  is in exactly one of the two sets. Then either  $Y$  forces  $a$  over  $c$  or  $Z$  forces  $c$  over  $b$ .

*Intuition:* If  $X$  has the power to force  $a$  high and  $b$  low, then either  $Y$  inherits the power to force  $a$  high or  $Z$  inherits the power to force  $b$  low.

## *Proof*

Consider what happens when the social welfare function under consideration is applied to the following sequence of individual preference lists:



- Alternatives other than  $a, b$ , and  $c$  can be placed arbitrarily in the individual preference lists. Notice that everyone in both  $Y$  and  $Z$  (and thus everyone in  $X$ ) has  $a$  over  $b$ .
- Since we are assuming that  $X$  forces  $a$  over  $b$ , this means that we get  $a$  over  $b$  in the social preference list.
- In particular, this means that we can't get both  $b$  over  $c$  and  $c$  over  $a$  in the social preference list, or else transitivity would give us  $b$  over  $a$  in the social preference list, contrary to what we just said.
- Thus, we must have either  $a$  over  $c$  or  $c$  over  $b$  in the social preference list.

(i) We Get  $a$  Over  $c$  in the Social Preference List

In this case, we have produced a *single sequence* of individual preference lists for which everyone in  $Y$  has  $a$  over  $c$  in their lists, everyone not in  $Y$  has  $c$  over  $a$  in their lists, and the resulting social preference list has  $a$  over  $c$ . This suffices to show that  $Y$  forces  $a$  over  $c$ .

(ii) We Get  $c$  Over  $b$  in the Social Preference List.

Proceed in a similar manner for  $Z$ .

## Lemma 2

Suppose  $X$  forces  $a$  over  $b$  and  $c$  is an alternative distinct from  $a$  and  $b$ . Then  $X$  forces  $a$  over  $c$  and  $X$  forces  $c$  over  $b$ .

Intuition: If  $X$  can force  $a$  over  $b$ , equivalently,  $X$  can force  $b$  under  $a$ , then  $X$  can force  $a$  over anything and  $X$  can force  $b$  under anything.

### *Proof*

- Using Lemma 1, set  $Y = X$  and  $Z = \phi$ . The conclusion is then that either  $X$  forces  $a$  over  $c$  (as desired) or the empty set forces  $c$  over  $b$  (which is ruled out by the Pareto condition.) Thus  $X$  forces  $a$  over  $c$ .
- In a completely analogous way, a consideration of the special case of Lemma 1 where  $Y$  is the empty set and  $Z$  is the whole set  $X$  shows that  $X$  forces  $c$  over  $b$ .

### Lemma 3

If  $X$  forces  $a$  over  $b$ , then  $X$  forces  $b$  over  $a$ .

Intuition: The forcing relation is symmetric.

*Proof*

Choose an alternative  $c$  distinct from  $a$  and  $b$ . (This is possible since we are assuming that we have at least three alternatives.) Assume that  $X$  forces  $a$  over  $b$ . Then, by Lemma 2,  $X$  forces  $a$  over anything. In particular,  $X$  forces  $a$  over  $c$ . But Lemma 2 now also guarantees that  $X$  forces  $c$  under anything — in particular,  $X$  forces  $c$  under  $b$ . This is the same as saying  $X$  forces  $b$  over  $c$ . Thus, by Lemma 2 one more time, we have that  $X$  forces  $b$  over anything, and so  $X$  forces  $b$  over  $a$  as desired. Briefly,

$$X \text{ forces } \frac{a}{b} \Rightarrow X \text{ forces } \frac{a}{c} \Rightarrow X \text{ forces } \frac{b}{c} \Rightarrow X \text{ forces } \frac{b}{a}.$$

## Lemma 4

Suppose there are two alternatives  $a$  and  $b$  so that  $X$  can force  $a$  over  $b$ . Then  $X$  is a dictating set.

Intuition: If  $X$  has a little local power, then  $X$  has complete global power.

### *Proof*

Assume  $X$  can force  $a$  over  $b$ , and assume  $x$  and  $y$  are two arbitrary alternatives. We must show that  $X$  can force  $x$  over  $y$ . Notice that Lemma 3 guarantees that  $X$  can also force  $b$  over  $a$ . Thus, Lemma 2 now lets us conclude that  $X$  can force  $a$  over or under anything and  $X$  can force  $b$  over or under anything.

(i)  $a = y$

Here, we want to show that  $X$  can force  $x$  over  $a$ . But since we know  $X$  can force  $a$  under anything, we have that  $X$  can force  $a$  under  $x$ . Equivalently,  $X$  can force  $x$  over  $a$ , as desired.

(ii)  $a \neq y$

Since  $X$  forces  $a$  over  $b$  and  $a \neq y$ , we know that  $X$  can force  $a$  over  $y$ . Equivalently,  $X$  can force  $y$  under  $a$ , and thus  $X$  can force  $y$  under anything. In particular,  $X$  can force  $y$  under  $x$ . Thus,  $X$  can force  $x$  over  $y$  as desired. Briefly,

$$X \text{ forces } \frac{a}{b} \Rightarrow X \text{ forces } \frac{a}{y} \Rightarrow X \text{ forces } \frac{x}{y}.$$



## Lemma 5

Suppose that  $X$  is a dictating set and suppose that  $X$  is split into two sets  $Y$  and  $Z$  so that each element of  $X$  is in exactly one of the two sets. Then either  $Y$  is a dictating set or  $Z$  is a dictating set.

### *Proof*

Choose three distinct alternatives  $a, b$ , and  $c$ . Since  $X$  is a dictating set, we have that  $X$  can force  $a$  over  $b$ . Lemma 1 now guarantees that either  $Y$  can force  $a$  over  $c$  (in which case  $Y$  is a dictating set by Lemma 4), or  $Z$  can force  $c$  over  $b$  (in which case  $Z$  is a dictating set by Lemma 4 again). This completes the proof of Arrow's impossibility theorem.

Lastly, we split a given dictating set (at least  $P$  is a dictating set) based on splitting a single element off the set at each step. We can always obtain a dictating set which is a singleton. The single element in that dictating set is a dictator.

## 4.4 Approval voting methods

- When more than 2 candidates are running, the plurality winner may fall short of a majority. For example, in the Presidential Race in Taiwan, 2000, Chen received less than 40% of the popular vote but he was declared the winner.
- The candidate preferred by the largest number of voter may be least preferred by even more voters.
- How to develop voting methods that satisfy the Condorcet criterion consistently and efficiently? The Condorcet criterion requires that the winner is the candidate who would defeat all other candidates in pairwise contests against them, provided that such a majority exists.

## Approval voting

- Voters cast one vote each for as many candidates as they wish, but do not cast more than one vote for any one candidate. The candidate receiving the most votes wins.

### *Example*

<i>2 voters</i>	<i>2 voters</i>	<i>1 voter</i>	<i>1 voter</i>	<i>Approval voters</i>
<i>a</i>	<i>d</i>	<i>a</i>	<i>a</i>	<i>a</i> : 4
<i>b</i>	<i>e</i>	<i>b</i>	<i>c</i>	<i>b</i> : 3
<i>d</i>		<i>d</i>	<i>e</i>	<i>c</i> : 1
<i>e</i>				<i>d</i> : 5
				<i>e</i> : 5

- It allows voters to express a choice (yes or no, approve or disapprove) for every candidate in the election. Unlike the Borda count method, voters do not need to rank their preference (simpler to implement).

## *Strategy-proof*

- Under plurality voting, suppose the candidate most preferred by some voters has no chance of winning, voters are impelled to vote “sophisticated” instead of “sincerely” — vote for a less preferred candidate who has a good chance of winning in order to oppose a more objectionable candidate. Voters, finding all the viable candidates distasteful, may not vote at all.
- Under approval voting, thus avoiding “wasting” their vote — encouraging a higher voter turnout. A candidate is better off with lukewarm support that is widespread than with enthusiastic support that is not. This system would be far more likely to elect Condorcet winners.

## Negative votes

- The introduction of negative votes would force the candidates to appear to the voters with positive programs, not just try to be the least unpopular candidate in the race. A threshold (say, at least 10% of the total votes cast) be set that a winning candidate's net vote could have to exceed for him to be elected.
- It can be shown that there is never any incentive for a voter to cast a negative vote in a two-candidate race since a voter can always bring about the same outcome with a positive vote as with a negative vote without a minimum threshold. However, the psychological impact of negative votes may be considerable when the net vote of the “winner” is negative.

- In three-candidate races, a voter can definitely do better, in some circumstances, by casting a negative vote against a candidate rather than a positive vote for either of the other two candidates.
- In races with more than 3 candidates, some of the advantages of negative voting break down. Whether the *idea* of negative voting can be incorporated in a voting scheme that allows for the fuller expression of voter preferences? A simple generalization of the idea of negative voting that allows a voter to cast “approval” votes for one *or more* candidates is equivalent to a negative voting strategy in specific cases.
- Approval voting opens up possible voting strategies for a voter that a single positive or a single negative vote does not permit.

## Negative Voting in Two-Candidate Contests

- Assume that  $X$  and  $Y$  are two candidates running for office, and every voter has a preference scale defined over these candidates.

$(X, Y)$ :  $X$  is preferred over  $Y$

$(Y, X)$ :  $Y$  is preferred over  $X$

$(X - Y)$ : indifferent between  $X$  and  $Y$

- Given negative voting, every voter has 4 strategies

(i)  $X$ : vote for  $X$

(ii)  $\overline{X}$ : vote against  $X$

(iii)  $Y$ : vote for  $Y$

(iv)  $\overline{Y}$ : vote against  $Y$

A voter is said to be rational if he chooses a strategy that is undominated: no other strategy that is as good in all contingencies and in at least one contingency, better.

- The state of nature is described by the ordered pair  $(x, y)$ , where  $x$  is the net vote cast for  $X$  and  $y$  is the net vote cast for  $Y$  by the  $n - 1$  other voters.
- To determine the undominated strategies of a voter, we have to concern ourselves with only a relatively few results — those in which the vote of a single voter can make a difference in the outcome.
- A voter's vote can *make a difference* if and only if it makes or breaks a tie.
- In a two-candidate race, a voter can break a tie (of the  $n - 1$  other voters) if the result is  $x = y$ , he can make a tie if  $x = y - 1$  or  $y = x - 1$ .
- For all other results (i.e., where  $|x - y| > 1$ ), a voter's vote would not change the *outcome*.



- If the voter's preference scale is  $(X, Y)$ , it follows that he will rank the tied outcome  $X/Y$  "in the middle" —  $(X, X/Y, Y)$ ; if his preference scale is  $(X - Y)$ , then  $(X - X/Y - Y)$  since indifference between  $X$  and  $Y$  implies indifference among  $X, X/Y$ , and  $Y$ .
- The outcomes associated with strategy  $\bar{X}$  are exactly the same as those associated with strategy  $Y$ , and the outcomes associated with strategy  $\bar{Y}$  are exactly the same as those associated with strategy  $X$ , for all relevant results. A negative vote is not uniquely advantageous in a two-candidate contest.

## Outcomes in Two-Candidate Contest with Negative Voting

---

Relevant results for $n - 1$ other voters			
Strategies of voter	$n - 1$ (even)	$n - 1$ (odd)	
	$x = y$	$x = y - 1$	$y = x - 1$
$X$	$X$	$X/Y$	$X$
$\overline{X}$	$Y$	$Y$	$X/Y$
$Y$	$Y$	$Y$	$X/Y$
$\overline{Y}$	$X$	$X/Y$	$X$

## Negative voting in 3-candidate contest without a runoff

We list 16 relevant results where one's vote can make a difference and 6 strategies of each voter.

- Consider the 3 first columns, which correspond to the 3 relevant results in which 2 candidates are tied with the same number of votes while the third candidate has one more vote.

	1	2	3
	$x$	$x$	$y$
Strategies	$= y$	$= z$	$= z$
of voter	$= z - 1$	$= y - 1$	$= x - 1$
$X$	$X/Z$	$X/Y$	$X$
$\overline{X}$	$Z$	$Y$	$X/Y/Z$
$Y$	$Y/Z$	$Y$	$X/Y$
$\overline{Y}$	$Z$	$X/Y/Z$	$X$
$Z$	$Z$	$Y/Z$	$X/Z$
$\overline{Z}$	$X/Y/Z$	$Y$	$X$

$(x, y, z)$ $= (0, 0, -2)$					
4	5	6	7	8	9
$x - 1$	$x - 1$	$y - 1$	$x - 1$	$x - 1$	$y - 1$
$= y - 1$	$= z - 1$	$= z - 1$	$= y - 1$	$= z - 1$	$= z - 1$
$= z$	$= y$	$= x$	$> z$	$> y$	$> x$
$X$	$X$	$X/Y/Z$	$X$	$X$	$Y/Z$
$Y$	$Z$	$Y/Z$	$Y$	$Z$	$Y/Z$
$Y$	$X/Y/Z$	$Y$	$Y$	$X/Z$	$Y$
$X$	$X/Z$	$Z$	$X$	$X/Z$	$Z$
$X/Y/Z$	$Z$	$Z$	$X/Y$	$Z$	$Z$
$X/Y/Z$	$X$	$Y$	$X/Y$	$X$	$Y$

$(x, y, z)$ $= (2, 1, 0)$						$(x, y, z)$ $= (0, 0, 0)$
10	11	12	13	14	15	16
$x - 1$	$x - 1$	$y - 1$	$y - 1$	$z - 1$	$z - 1$	$x$
$= y$	$= z$	$= x$	$= z$	$= x$	$= y$	$= y$
$> z$	$> y$	$> z$	$> x$	$> y$	$> x$	$= z$
$X$	$X$	$X/Y$	$Y$	$X/Z$	$Z$	$X$
$X/Y$	$X/Z$	$Y$	$Y$	$Z$	$Z$	$X/Z$
$X/Y$	$X$	$Y$	$Y$	$Z$	$Y/Z$	$Y$
$X$	$X$	$X/Y$	$Y/Z$	$Z$	$Z$	$X/Z$
$X$	$X/Z$	$Y$	$Y/Z$	$Z$	$Z$	$Z$
$X$	$X$	$Y$	$Y$	$X/Z$	$Y/Z$	$X/Y$

- It is assumed that the preference relation  $P$  and indifference relation  $I$  that underlie the scales of individual voters are transitive:

$$APB \quad \text{and} \quad BPC \Rightarrow APC.$$

Thus,  $(X, Y, Z) \Rightarrow (X, Z)$  and  $(X - Y - Z) \Rightarrow (X - Z)$ .

- There are 4 preference scales to be considered, indistinguishable except for order:

$$(X, Y, Z), (X, Y - Z), (X \cup Y, Z) \quad \text{and} \quad (X - Y - Z).$$

- For various preference scales, we seek for undominated strategies. Note that a unique undominated strategy is necessarily dominant — as good as, and in at least one contingency better than, any other strategy.

Preference scales	Undominated (dominant strategies)
$(X, Y, Z)$	$X, \bar{Z}$
$(X, Y - Z)$	$(X)$
$(X - Y, Z)$	$(\bar{Z})$
$(X - Y - Z)$	$X, \bar{X}, Y, \bar{Y}, Z, \bar{Z}$

- Consider the preference scale  $(X, Y - Z)$ , we note that the outcomes associated with strategy  $X$  are at least as good as the outcomes associated with his five other strategies. For example, if strategy  $X$  does not yield his first choice  $X$  as an outcome, neither do his other strategies.

- Is strategy  $X$  better than every other strategy in at least one contingency? For result 5, it yields  $X$ , which is a better outcome for our voter than strategies  $\bar{X}$ ,  $Y$ ,  $\bar{Y}$ , and  $Z$  — which all can lead to nonpreferred outcomes  $Y$  or  $Z$  — offer. But strategy  $X$  is also better than strategy  $\bar{Z}$ , because even though it leads to the same outcome as  $X$  for relevant result 5,  $X$  leads to a better outcome than  $\bar{Z}$  for relevant result 6 ( $X/Y/Z$  versus  $Y$ ). In this contingency, strategy  $X$  allows for the possibility that outcome  $X$  will be chosen, but strategy  $Z$  always ensures that nonpreferred outcome  $Y$  will be chosen.



- The fact that  $\overline{Z}$  is dominant when the preference scale of a voter is  $(X - Y, Z)$  demonstrates that a negative vote is uniquely advantageous in this situation. While it leads to no better an outcome than strategy  $X$  or strategy  $Y$  for 13 of the 16 relevant results, it leads to a definitely superior outcome for three results (better than  $X$  for results 6, 9, and 15; better than  $Y$  for results 5, 8, and 14).
- We say that  $\overline{Z}$  *dominates*  $X$  and  $Y$  (as well as all other strategies) for a voter with preference scale  $(X - Y, Z)$ .
- In words, the preference scale  $(X - Y, Z)$  of a voter says that he is indifferent between two candidates ( $X$  and  $Y$ ) but definitely prefers them to the third candidate ( $Z$ ). In such a situation, it is always rational for a voter to cast a negative vote against  $Z$  rather than a positive vote for either of the two candidates,  $X$  and  $Y$ , that he prefers.

## Negative voting in three-candidate contests with a runoff

What if the election is decided in a runoff between the top two vote-getters?

- For the purpose of comparing negative voting in a single plurality contest with voting in a plurality contest followed by a runoff, assume for now that a voter cannot cast a negative vote in the plurality contest.

Undominated (Dominated) Strategies for  
Four Preference Scales without Negative Voting

Preference scales	Undominated (dominant) strategies
$(X, Y, Z)$	$X, Y$
$(X, Y - Z)$	$(X)$
$(X - Y, Z)$	$X, Y$
$(X - Y - Z)$	$X, Y, Z$

- When a voter's strategy of voting against his last choice ( $\bar{Z}$ ) is unavailable, it may be advantageous for him to vote for his second choice [ $Y$  if his preference scale is  $(X, Y, Z)$ ], or either one of his first choices [ $X$  or  $Y$  if his preference scale is  $(X - Y, Z)$ ].
- Can a voter's choice of these apparently inferior strategies, when he is restricted to casting only a position vote, be rectified in a runoff? If we assume, as is common in many jurisdictions that allow for a runoff between the top two vote-getters, that the runoff occurs if and only if no candidate in the plurality election receives a majority of votes, the answer is "not necessarily,"
- The restriction of voters to a single positive vote may again force them to choose inferior strategies, compared with strategies available under negative voting, even when a runoff is permitted.

- Assume that there are five voters, whose preference scales are as follows (1)  $(X, Y - Z)$ ; (2)  $(X, Y - Z)$ ; (3)  $(Y, X - Z)$ ; (4)  $(Y, X - Z)$ ; (5)  $(X, Y, Z)$ .
- From the Table, we know that voters 1-2 and voters 3-4 have dominant strategies: vote for  $X$  and vote for  $Y$ , respectively. Now, if voters are restricted to a single positive vote in the plurality election, voter 5 has two undominated strategies (given that he has no information about the preference scales of the other voters and therefore cannot predict how they will vote): vote for  $X$  or vote for  $Y$ .
- Clearly, strategy  $X$  would be preferable if  $X$  and  $Y$  were tied and  $Z$  were out of the running; on the other hand, strategy  $Y$  would be preferable if  $Y$  and  $Z$  were tied and  $X$  were out of the running.

- Regardless of which undominated strategy voter 5 chooses, either  $X$  or  $Y$  will receive a majority of votes and there will be no runoff.
- If voter 5 chooses strategy  $Y$ , however, and  $Y$  thereby obtains a majority of votes, three of the five voters (1, 2, and 5) will be dissatisfied.
- For  $X$  is the so-called *Condorcet winner*, that is, the candidate preferred by a majority of voters in pairwise contests between each of the other candidates,  $Y$  and  $Z$ .
- In a pairwise contest between  $X$  and  $Y$ ,  $X$  would be preferred by voters 1, 2, and 5; similarly, these voters would also prefer  $X$  to  $Z$  in a pairwise contest between those two candidates. Yet, if all voters but voters 5 choose their dominant strategies, and voter 5 chooses his undominated strategy  $Y$ ,  $Y$  will defeat the Condorcet winner  $X$ .

- Would negative voting have prevented the choice of  $Y$  in a plurality election with a runoff if there were no majority winner (of net votes)? The answer is “yes”.
- Whichever of his undominated strategies voter 5 chose in the plurality election —  $X$  or  $\bar{Z}$  — he would have prevented the election of  $Y$ . If he had voted for  $X$ ,  $X$  would have defeated  $Y$  3 votes to 2, and there would have been no runoff. If he had voted against  $Z$ , the plurality contest would then have resulted in a 2-to-2 tie between  $X$  and  $Y$  (with  $Z$ 's net vote being  $-1$ ), and  $X$  would have won in the runoff since voter 5 would prefer him to  $Y$ .

- Allowing for a runoff in a plurality contest restricted to positive voting does not necessarily recapture the advantages of negative voting; specifically in our example, lead to the election of the Condorcet winner.
- Permitting negative voting, on the other hand, does lead to the election of the Condorcet winner, at least in our example, if there is a runoff between the top two vote-getters when neither finalist receives a majority of votes.
- In addition, if there is a Condorcet winner, negative voting is better able to find him.

- In summary, although negative voting offers no unique advantage to a voter in two-candidate contests, negative votes in a plurality contest among three candidates — with or without a runoff between the top two — may be uniquely advantageous for a voter, depending on his preference scale.
- In addition, negative voting can ensure the selection of a Condorcet winner when a runoff does not.
- The apparent advantages of negative voting do not carry through to plurality contests with more than three candidates.