MATH 4321 – Game Theory

Homework Three

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1. Let $f(x, y) = (x - y)^2$, the two intervals for x and y are $C = D = [-1,\ 1]$, respectively. Find $v^+ = \min_{y \in D} \max_{x \in C} f(x, y)$ and $v^- = \max_{x \in C} \min_{y \in D} f(x, y)$.

2. Two players decide on the amount of effort they each will exert on a project. The effort level of each player is $q_i \geq 0$, $i = 1, 2$. The payoff to each player is

$$u_1(q_1, q_2) = q_1(c + q_2 - q_1), \quad \text{and} \quad u_2(q_1, q_2) = q_2(c + q_1 - q_2),$$

where $c > 0$ is a constant. To model the synergistic effect between the two players, we observe that $u_1(q_1, q_2)$ is increased when the effort of player 2 is higher than that of player 1.

   (a) Find each player's best response function.

   (b) Find the Nash equilibrium.

3. There are N players each using r_i units of a resource whose total value is $R = \sum_{i=1}^{N} r_i$. The cost to player i for using r_i units of the resource is $f(r_i) + g(R - r_i)$, where we take $f(x) = 2x^2$, $g(x) = x^2$. This says that a player's cost is a function of the amount of the total used by that player and a function of the amount used by the other players. The revenue player i receives from using r_i units is $b(r_i)$, where $b(x) = \sqrt{x}$. Assume that the total resources used by all players is not unlimited so that $0 \leq R \leq R_0 < \infty$.

   (a) Find the payoff function for player $i = 1, 2, \ldots, N$.

   (b) Find the Nash equilibrium in general and when $N = 12$.

   (c) Now suppose that the total amount of resources is R and each player will use $\frac{R}{N}$ units of the total. Here, R is unknown and is chosen so that the *social welfare* is maximized, that is,

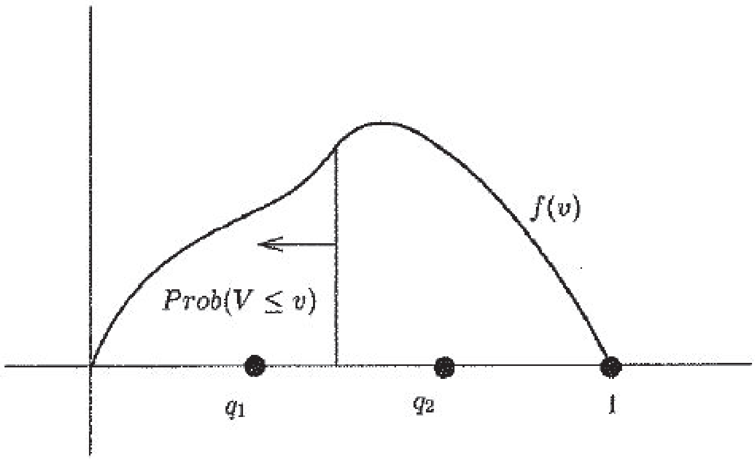
$$\text{Maximize } \sum_{i=1}^{N} u_i\left(\frac{R}{N}, \ldots, \frac{R}{N}\right),$$

   over $R \geq 0$. Practically, this means that the players do not act independently but work together so that the total payoffs to all players is maximized. Find the value of R that provides the maximum, R^s. Find it's value when $N = 12$.

4. Consider a variant of the voters' preference model in which voters' preferences are asymmetric. Specifically, suppose that each voter cares twice as much about policy differences to the left of her favorite position than about policy differences to the right of her favorite position. How does this affect the Nash equilibrium?

5. Suppose that voter preferences on an issue are distributed from $[0,1]$ according to a continuous probability density function $f(x) > 0$ and $\int_0^1 f(x)\ dx = 1$. The percentage of voters with position within $(x, x+dx)$ is given by $f(x)dx$. Recall $F(v) = P[V \le v] = \int_0^v f(x)\ dx$ and $P[V > v] = 1 - P[V \le v] = 1 - F(v)$.

   - $x = \frac{1}{2}$ - middle of the road
   - $x \in [0, \frac{1}{2})$ - leftist or liberal
   - $x \in (\frac{1}{2}, 1]$ - rightist or conservative



Area to the left of v is candidate I's percentage of the vote.

   We assume that voters always vote for the candidate nearest to their own positions. Write $\gamma = \frac{q_1+q_2}{2}$, and if $q_1 < q_2$, Player I receives $F(\gamma)$ percent of vote. The payoff functions for Players I and II are defined by the percentage of votes received so that

$$u_1(q_1, q_2) = \begin{cases} P[V \le \frac{q_1+q_2}{2}], & \text{if } q_1 < q_2; \\ \frac{1}{2}, & \text{if } q_1 = q_2; \\ P[V > \frac{q_1+q_2}{2}], & \text{if } q_1 > q_2. \end{cases}$$
$$u_2(q_1, q_2) = 1 - u_1(q_1, q_2).$$

   Given f, what position in $[0,1]$ should a politician take in order to maximize the votes.

6. There are only 2 farmers and the farmers have a continuous number of sheep. We assume

$$u_1(x_1, x_2) = 100x_1 - x_1(x_1 + 2x_2) = x_1^2 + x_1[100 - 2(x_1 + x_2)],$$
$$u_2(x_1, x_2) = 100x_2 - x_1(x_1 + 2x_2) = x_2^2 + x_2[100 - 2(x_1 + x_2)].$$

   For $u_1(x_1, x_2)$, the gain is assumed to be linear in x_1 and loss is sum of quadratic terms x_1^2 and $2x_1x_2$. The proportional constant for gain is large compared to those of the two loss terms.

   Find the Nash equilibrium. What would happen when the two farmers reach an earlier agreement of grazing equal number of sheep? Do they both benefit from the agreement? Would the farmers have the incentive to cheat.

7. Two firms produce identical widgets. The price function is $P(q) = (150 - q)^+$ and the cost to produce q widgets is $C(q) = 120q - \frac{2}{3}q^2$ for each firm.

   (a) What are the profit functions for each firm?

   (b) Find the Nash equilibrium quantities of production.

   (c) Find the price of a widget at the Nash equilibrium level of production as well as the profits for each firm.

   (d) If firm 1 assumes that firm 2 will use its best response function, what is the best production level of firm 1 to maximize its profits assuming that firm 1 will then publicly announce the level. In other words, they will play sequentially, not simultaneously.

8. Suppose that the demand functions in the Bertrand model are given by

$$q_1 = D_1(p_1, p_2) = (\Gamma - p_1 + bp_2)^+ \quad \text{and} \quad q_2 = D_2(p_1, p_2) = (\Gamma - p_2 + bp_1)^+,$$

where $x^+ = \max(x, 0)$ and $1 \geq b > 0$. This says that the quantity of gadgets sold by a firm will increase if the price set by the opposing firm is too high. Assume that both firms have a cost of production: $c \leq \min(p_1, p_2)$. Since profit is revenue minus costs, the profit functions are given by

$$u_i(p_1, p_2) = D_i(p_1, p_2)(p_i - c), \ \ i = 1, 2.$$

   (a) Show that there is a unique Nash equilibrium at

$$p_1^* = p_2^* = \frac{\Gamma + c}{2 - b}.$$

   (b) Find the profit functions at equilibrium.

   (c) Suppose the firms have different costs and sensitivities so that

$$q_1 = D_1(p_1, p_2) = (\Gamma - p_1 + b_1 p_2)^+ \quad \text{and} \quad q_2 = D_2(p_1, p_2) = (\Gamma - p_2 + b_2 p_1)^+,$$

   and

$$u_i(p_1, p_2) = D_i(p_1, p_2)(p_i - c_i), \ \ i = 1, 2.$$

   Find a Nash equilibrium and the profits at equilibrium.

   (d) Find the equilibrium prices, production quantities, and profits if $\Gamma = 100$, $c_1 = 5$, $c_2 = 1$, $b_1 = \frac{1}{2}$, $b_2 = \frac{3}{4}$.

9. In the second-price sealed-bid auction, if two or more bidders are tied for high bid, a random device decides who will receive the object and the bidder receiving the object pays this tying bid. Show that the truth telling bidding strategy remains to be weakly dominating all other strategies.

10. Verify that in the first-price auction $(b_1, ..., b_N) = (v_1, ..., v_N)$, where $v_1 \geq v_2 \geq ... \geq v_N$, is a Nash equilibrium assuming $v_1 = v_2$; that is, the two highest valuations are equal in value.

11. A homeowner is selling her house by auction. Two bidders place the same value on the house at \$100,000, while the next bidder values the house at \$80,000. Should the homeowner use a first-price or second-price auction to sell the house, or does it matter? What if the highest valuation is \$100,000, the next is \$95,000 and the rest are not more than \$90,000?

12. In an *all-pay auction*, all the bidders must actually pay their bids to the seller, but only the high bidder gets the object up for sale. This type of auction is also called a *charity auction*. By following the procedure for a Dutch auction, show that the equilibrium bidding function for all players is $\beta(v) = \frac{N-1}{N} v^N$, assuming that bidders' valuations are uniformly distributed on the normalized interval $[0, 1]$. Find the expected total amount collected by the seller.