MATH 4321 – Game Theory
Solution to Homework Three

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1. For a fixed value of y, $f(x, y)$ achieves its maximum at $x = 1$ when $y \leq 0$ and at $x = -1$ when $y > 0$. Therefore, we have

$$\max_{x \in C} f(x, y) = \begin{cases} (1-y)^2, & y \leq 0; \\ (-1-y)^2, & y > 0. \end{cases}$$

For $y \in [-1, 1]$, minimum of $\max_{x \in C} f(x, y)$ occurs at $y = 0$, so that $v^+ = \min_{y \in D} \max_{x \in C} f(x, y) = 1$.

On the other hand, $f(x, y) \geq 0$ and equals zero when $x = y$, so $\min_{y \in D} f(x, y) = 0$. We then have $v^- = \max_{x \in C} \min_{y \in D} f(x, y) = 0$.

2. (a) The payoff to each player is a quadratic function in its variable that the player can control while the control variable of the other player is held fixed. By invoking the corresponding first order conditions, we obtain

$$\frac{\partial u_1}{\partial q_1} = c + q_2 - 2q_1 = 0 \Rightarrow q_1 = BR_1(q_2) = \frac{c + q_2}{2},$$
$$\frac{\partial u_2}{\partial q_2} = c + q_1 - 2q_2 = 0 \Rightarrow q_2 = BR_2(q_1) = \frac{c + q_1}{2}.$$

(b) We solve simultaneously the above best response functions

$$q_1 = \frac{c + q_2}{2} \text{ and } q_2 = \frac{c + q_1}{2}$$

to obtain

$$q_1^* = q_2^* = c.$$

As a check, note that $u_1(q_1, q_2^*) = q_1(2c - q_1)$ has a local maxima at $q_1 = c$. Therefore, player 1 is worst off if he deviates from his part of the Nash pair.

3. (a) The payoff function for each player $i = 1, 2, \ldots, N$, is

$$u_i(r_1, \ldots, r_N) = b(r_i) - [f(r_i) + g(R - r_i)] = \sqrt{r_i} - 2r_i^2 - (R - r_i)^2,$$

where $R = r_1 + r_2 + \ldots + r_N$.

(b) Taking a partial derivative of u_i with respect to r_i gives

$$\frac{\partial u_i}{\partial r_i} = \frac{1}{2} \frac{1}{\sqrt{r_i}} - 4r_i = 0.$$

which implies $r_i = \frac{1}{4}$. Since $\frac{\partial^2 u_i}{\partial r_i^2} < 0$, we conclude that $(r_1, \ldots, r_N) = \left(\frac{1}{4}, \ldots, \frac{1}{4}\right)$ is the Nash equilibrium. The total amount of resources used by all the players is then $R = \frac{N}{4}$. When $N = 12$, the total resources used will be $R = 3$. The payoff to each player when $N = 12$ is

$$u_i\left(\frac{1}{4}, \cdots, \frac{1}{4}\right) = \frac{1}{2} - \frac{1}{8} - \left(\frac{11}{4}\right)^2 = -7.396.$$

(c) We set

$$F(R) = N\left(b\left(\frac{R}{N}\right) - f\left(\frac{R}{N}\right) - g\left(R - \frac{R}{N}\right)\right) = \sum_{i=1}^{N} u_i\left(\frac{R}{N}, \cdots, \frac{R}{N}\right).$$

To find the maximum of F, we take a derivative with respect to R and set to zero:

$$F'(R) = \left(4 - \frac{6}{N} - 2N\right) R + \frac{1}{2\sqrt{\frac{R}{N}}} = 0.$$

After some algebra, we get

$$R^s = \frac{N}{(2(4 + 2(N-1))^2)^{2/3}}.$$

Since $F''(R) = 4 - \frac{6}{N} - 2N - \frac{\sqrt{N}}{4R^{3/2}} < 0$ we know that R^s provides a maximum. When $N = 12$, we obtain $R^s = 0.192547$. The value of the maximum social welfare is $F(R^s) = 1.14004$.

4. In this case, the value function of a voter with the preferred policy x^* in response to the policy stand x of a candidate is given by

$$u(x; x^*) = \begin{cases} -(x - x^*) & \text{if } x > x^* \\ 0 & \text{if } x = x^* \\ -2(x^* - x) & \text{if } x < x^* \end{cases}.$$

That is, each voter cares twice as much about the deviation to the left of x^* as about the deviation to the right of x^*. In the lecture, the value function is chosen to be $u(x; x^*) = -|x - x^*|$.

First, we identify the citizen with preferred policy $\overline{x}$ who is indifferent between the two candidates. We then have

$$-2(\overline{x} - x_1) = -(x_2 - \overline{x}),$$

so

$$\overline{x} = \frac{2}{3}x_1 + \frac{1}{3}x_2 \text{ if } x_1 < x_2$$

and

$$\overline{x} = \frac{1}{3}x_1 + \frac{2}{3}x_2 \text{ if } x_1 \geq x_2.$$

We then proceed to derive the best response of each candidate as follows. Let m denote the median of the distribution of preferred policies of the voters, we obtain

$$BR_1(x_2) = \begin{cases} x_2 < x_1 < 3m - 2x_2 & \text{if } x_2 < m \\ m & \text{if } x_2 = m \\ \frac{3}{2}m - \frac{1}{2}x_2 < x_1 < x_2 & \text{if } x_2 > m \end{cases};$$

and

$$BR_2(x_1) = \begin{cases} x_1 < x_2 < 3m - 2x_1 & \text{if } x_1 < m \\ m & \text{if } x_1 = m \\ \frac{3}{2}m - \frac{1}{2}x_1 < x_2 < x_1 & \text{if } x_1 > m \end{cases}.$$

Since there is only one intersection point of the two best responses, the *only Nash equilibrium* is $(x_1^*, x_2^*) = (m, m)$.

5. Note that $F(\gamma)$ is an increasing function of γ. When $q_1 < q_2$, for a fixed value of q_2, $F(\gamma)$ is increasing with respect to an increase in q_1. Once q_1 increases beyond q_2, Player II starts to gain since $u_1(q_1, q_2) = 1 - F(\gamma)$ when $q_1 > q_2$.

   Let γ^* be the median of the random variable V, where $F(\gamma^*) = \frac{1}{2}$.

   Note that $u_i(\gamma^*, \gamma^*) = \frac{1}{2}$, $i = 1, 2$. We argue that (γ^*, γ^*) is a Nash equilibrium. This is because

$$\frac{1}{2} = u_1(\gamma^*, \gamma^*) \geq u_1(q_1, \gamma^*) = \begin{cases} F\left(\frac{\gamma^* + q_1}{2}\right) < F(\gamma^*) = \frac{1}{2} & \text{if } q_1 < \gamma^* \\ \frac{1}{2} & \text{if } q_1 = \gamma^* \\ 1 - F\left(\frac{\gamma^* + q_1}{2}\right) < F(\gamma^*) = \frac{1}{2} & \text{if } q_1 > \gamma^* \end{cases}.$$

   Similarly, we have

$$\frac{1}{2} = u_2(\gamma^*, \gamma^*) \geq u_2(\gamma^*, q_2).$$

6. Taking $\frac{\partial u_1}{\partial x_1} = 0$ and $\frac{\partial u_2}{\partial x_2} = 0$ to find the best response functions, we obtain

$$2x_1 + 100 - 4x_1 - 2x_2 = 0 \text{ and } 2x_2 + 100 - 4x_2 - 2x_1 = 0.$$

   Solving simultaneously to find the Nash equilibrium, we obtain $x_1 + x_2 = 50$. Due to symmetry of the payoff functions, each farmer should graze 25 sheep, yielding $u_i(25, 25) = 625$.

   Suppose the two farmers reach an earlier agreement with $x_1 = x_2$. Now, the payoff function of each player becomes

$$u_1(x, x) = x^2 + x(100 - 4x) = -3x^2 + 100x.$$

   The maximum occurs at $x = \frac{50}{3} < 25$. The payoff to each farmer if they follow the agreement is

$$u_1(\frac{50}{3}, \frac{50}{3}) = \frac{2500}{3} > 625.$$

   With an earlier agreement of equal production, both farmers can gain higher profit at a lower grazing level.

The farmers do have the incentive to cheat. To see that, suppose Player 1 assumes Player 2 to stick with the agreement of $\frac{50}{3}$, the new payoff is

$$u_1(x_1, \frac{50}{3}) = x_1^2 + x_1[100 - 2(x_1 + \frac{50}{3})],$$

which is maximized at $x_1 = \frac{100}{3}$, giving a higher payoff of $\frac{10,000}{9} = 1111.11$.

7. (a) The profit functions are

$$u_i(q_1, q_2) = q_i(150 - q_1 - q_2) - 120q_i + \frac{2}{3}q_i^2, \ i = 1, 2.$$

(b) Applying the first order condition, we obtain

$$q_1 = q_2 = 18.$$

(c) The price is $P(18 + 18) = 114$ and $u_1(18, 18) = u_2(18, 18) = 108$.

(d) The best response functions are

$$q_1(q_2) = \frac{3}{2}(30 - q_2) \text{ and } q_2(q_1) = \frac{3}{2}(30 - q_1), \ 0 \leq q_1, q_2 \leq 30.$$

Set $f(q_1) = u_1(q_1, q_2(q_1)) = q_1(\frac{7}{6}q_1 - 15)$, which is a parabola that is concave upward. The maximum of $f(q_1)$ occurs at the far right end point, where $q_1 = 30$. Correspondingly, $q_2(30) = 0$. We obtain $u_1(30, 0) = 600$.

8. (a) The maximization problem for firm i is defined by

$$\max_{p_i \geq 0} (\Gamma - p_i + bp_j)(p_i - c), \ i = 1, 2, \ i \neq j.$$

Applying the first order conditions, we obtain

$$\frac{\partial u_1}{\partial p_1} = \Gamma - 2p_1 + bp_2 + c = 0$$
$$\frac{\partial u_2}{\partial p_2} = \Gamma - 2p_2 + bp_1 + c = 0$$

giving

$$p_1^* = p_2^* = \frac{\Gamma + c}{2 - b}.$$

It is straightforward to check that

$$\frac{\partial^2 u_1}{\partial p_1^2} = -2 < 0 \text{ and } \frac{\partial^2 u_2}{\partial p_2^2} = -2 < 0.$$

(b) The profits of the two firms at the Nash equilibrium are

$$u_1(p_1^*, p_2^*) = u_2(p_1^*, p_2^*) = \left[\frac{\Gamma + c(b-1)}{2 - b}\right]^2.$$

The two firms have the same equilibrium profit.

(c) By using the same procedure as before, we obtain the equilibrium prices as follows:

$$p_1^* = \frac{2(\Gamma + c_1) + b_1(\Gamma + c_2)}{4 - b_1 b_2}$$
$$p_2^* = \frac{(\Gamma + c_1)b_2 + 2(\Gamma + c_2)}{b_1 b_2 - 4}.$$

The equilibrium profits of the two firms are

$$u_1(p_1^*, p_2^*) = \left[\frac{(\Gamma + c_1 b_2 + c_2)b_1 + 2\Gamma - 2c_1}{b_1 b_2 - 4}\right]^2,$$
$$u_2(p_1^*, p_2^*) = \left[\frac{(\Gamma + c_2 b_1 + c_1)b_2 + 2\Gamma - 2c_2}{b_1 b_2 - 4}\right]^2.$$

(d) Substituting these values into the solution from (c), we have

$$p_1^* = 71.86,\ p_2^* = 77.45,\ u_1 = 4470.54 \text{ and } u_2 = 5844.34.$$

The optimal production quantities are found to be

$$q_1^* = \Gamma - p_1^* + b_1 p_2^* = 66.86 \text{ and } q_2^* = \Gamma - p_2^* + b_2 p_1^* = 76.45.$$

9. Let t_i be the truth telling bid of player i, which ties with another bid.

(i) Suppose the random device determines player i to be the winner, he pays t_i for the item. Consider that he uses $v_i > t_i$, then he pays the same amount t_i for the item. If he uses $v_i < t_i$, he loses and ends up with the same zero payoff.

(ii) Suppose player i is the loser, the use of $v_i > t_i$ would make him to be the winner but the net gain is zero. If he uses $v_i < t_i$, he remains to lose the auction anyway.

10. The payoff function of the i^{th} player is specified as follows:

$$u_i(b_1, ..., b_N) = \begin{cases} 0, & \text{if } b_i < M, \text{ she is not a high bidders;} \\ v_i - b_i, & \text{if } b_i = M, \text{ she is the sole high bidders;} \\ \frac{v_i - b_i}{k}, & \text{if } i \in \{k\}, \text{ she is one of } k \text{ high bidders;} \end{cases}$$

and recall that $v_1 \geq v_2 ... \geq v_N$. We have to show that $u_i(v_1, ..., v_N)$ gives a larger payoff to player i if player i makes any other bid $b_i \neq v_i$.

We assume that $v_1 = v_2$ so the two highest valuations are the same. Now for any player i if she bids less than $v_1 = v_2$ she does not win the object and her payoff is zero. If she bids $b_i > v_1$ she wins the object with payoff $v_i - b_i < v_1 = b_i < 0$. If she bids $b_i = v_1$, her payoff is $\frac{v_i - b_i}{3} = \frac{v_i - v_1}{3} < 0$. In all cases, she is worse off if she deviates from the bid $b_i = v_i$ as long as the other players stick with their valuation bids.

11. In the first case, it does not matter if she uses a first or second price auction. Either way she will sell it for $100,000. In the second case, the winning bid for player 1 is between $95,000 < b_1 < 100,000$ whether it is a first or second price auction. However, in the second price auction, the house will sell for $95,000. Thus, a first price auction is better for the seller.

12. The expected payoff of a bidder with valuation v who makes a bid of b is given by

$$u(b) = vP[b \text{ is high bid}] - b = v[F(\beta^{-1}(b))]^{N-1} - b = v[\beta^{-1}(b)]^{N-1} - b.$$

Let $y(b) = \beta^{-1}(b)$ so that $u(b) = vy(b)^{N-1} - b$. Differentiating $u(b)$ with respective to b (keeping v fixed) and setting it to be zero, the first order condition is given by

$$v(N-1)y(b)^{N-2}\frac{dy}{db} = 1, \quad y(0) = 0.$$

Set $v = y(b)$ so that the above differential equation becomes

$$(N-1)y(b)^{N-1}\frac{dy}{db} = 1, \quad y(0) = 0.$$

Separating the variables then integrating, and observing $y(0) = 0$, we obtain

$$\begin{aligned} (N-1)y^{N-1}dy &= db. \\ \frac{N-1}{N}y^N &= b. \end{aligned}$$

We finally obtain

$$\beta(v) = \frac{N-1}{N}v^N.$$

Since all bidders will actually pay their own bids and each bid is $\beta(v) = \frac{N-1}{N}v^N$, the expected payment from each bidder is

$$E[\beta(V)] = \frac{N-1}{N}\int_0^1 v^N \text{ d}v = \frac{N-1}{N(N+1)}.$$

Since there are N bidders, the total expected payment to the seller is $\frac{N-1}{N+1}$.