MATH 4321 – Game Theory

Homework One

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1. In a Nim game that starts with one pile holding 4 pennies, each player may take 1 or 2 pennies from the single pile. Suppose player I moves first. The game ends when there are no pennies left and the player who took the last penny pays 1 to the other player.

   (a) Draw the game tree for this 1×4 Nim as we did in 2×2 Nim.

   (b) Write down all the strategies for each player and then the game matrix.

   (c) Find the upper and lower values of the game, v^+, v^-. Would you rather be player I or player II?

2. Each of the two players must choose a number between 1 and 5. If a player's choice equals opposing player's choice + 1, she loses \$2; if a player's choice $\geq$ opposing player's choice + 2, she wins \$1. If both players choose the same number, the game is a draw.

   (a) Find the game matrix.

   (b) Find v^+ and v^- and determine whether a saddle point exists in pure strategies. If so, find it.

3. In the Russian roulette, suppose that if player I spins and survives and player II decides to pass, then the net gain to I is \$1 (the payoff rule is slightly different from the example in the lecture note). In this case, I gets all of the additional money that II had to put into the pot in order to pass. Draw the game tree and find the game matrix. What are the upper and lower values of the game? Find the saddle point in pure strategies.

4. Consider the following game matrix:

$$A = \begin{pmatrix} 3 & 6 \\ x & 0 \end{pmatrix}.$$

   Assuming that $x > 0$, find the value of x so that this game has a saddle point in pure strategies.

5. Let z be an unknown number and consider the matrices

$$A = \begin{pmatrix} 0 & z \\ 1 & 2 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 2 & 1 \\ z & 0 \end{pmatrix}.$$

   (a) Find $v(A)$ and $v(B)$ for any z.

(b) Now consider the game with matrix $A+B$. Find a value of z so that $v(A+B) < v(A)+v(B)$ and a value of z so that $v(A+B) > v(A)+v(B)$. Find the value of $A+B$ under the different choices of z.

This problem shows that the value of a zero-sum game is not a linear function of the matrix.

6. Find the saddle point in mixed strategies and value of the 2×2 zero-sum matrix game

$$A = \begin{pmatrix} 4 & -3 \\ -9 & 6 \end{pmatrix}$$

using the analytic formula.

7. Consider the zero-sum matrix game

$$A = \begin{pmatrix} 3 & 5 & 3 \\ 4 & -3 & 2 \\ 3 & 2 & 3 \end{pmatrix}.$$

Show that there is a saddle point in pure strategies at $(1,3)$ and find the value. Verify that $X^* = \left(\frac{1}{3}, 0, \frac{2}{3}\right)$, $Y^* = \left(\frac{1}{2}, 0, \frac{1}{2}\right)$ is a saddle point in mixed strategies.

Hint:
$$A^{-1} = \begin{pmatrix} \frac{13}{18} & \frac{1}{2} & -\frac{19}{18} \\ \frac{1}{3} & 0 & -\frac{1}{3} \\ -\frac{17}{18} & -\frac{1}{2} & \frac{29}{18} \end{pmatrix}.$$

8. Find the saddle point in mixed strategy for the zero-sum matrix game with matrix

$$A = \begin{pmatrix} 0 & 5 \\ 1 & 4 \\ 3 & 0 \\ 2 & 2 \end{pmatrix}.$$

Hint: Row 4 can be replicated by a convex combination of row 2 and row 3.

9. Four army divisions attack a town along two possible roads. The town has three divisions defending it. If there has been one defending division, then it needs 3 or 4 attacking divisions to capture the town. If there have been two defending divisions stationed, then the town is safe from attack. Even one division attacking an undefended road captures the town. Each commander must decide how many divisions to attack or defend each road. If the attacking commander captures a road to the town, then the town falls. Score 1 to the attacker if the town falls and -1 if it does not.

   (a) Find the payoff matrix.

   (b) Find the value of the game and the saddle point in mixed strategies.

10. Consider the matrix game $A = \begin{pmatrix} a_4 & a_5 & a_3 \\ a_1 & a_6 & a_5 \\ a_2 & a_4 & a_3 \end{pmatrix}$, where $a_1 < a_2 < \cdots < a_5 < a_6$. Use dominance to solve the game.

11. Consider the zero-sum game with game matrix

$$\begin{pmatrix} 3 & -2 & 4 & 7 \\ -2 & 8 & 4 & 0 \end{pmatrix}.$$

   (a) Examine whether there is a saddle point in pure strategies. If not, determine the saddle point in mixed strategies.

   (b) Find the best response for player I to the strategy $Y = \left(\frac{1}{4}, \frac{1}{2}, \frac{1}{8}, \frac{1}{8}\right)$.

   (c) What is II's best response to I's best response?

12. You are in a bar and a stranger comes to you to play a matching coins game. The stranger proposes that you each call Heads or Tails. If you both call Heads, the stranger pays you \$3. If both call Tails, the stranger pays you \$1. If the calls do not match, then you pay the stranger \$2.

   (a) Formulate this as a two-person zero-sum game and solve it.

   (b) Suppose the stranger decides to play the strategy $\overline{Y} = \left(\frac{1}{3}, \frac{2}{3}\right)$. Find a best response and the expected payoff.

13. An entrepreneur, named Victor, outside Laguna beach can sell 500 umbrellas when it rains and 100 when the Sun is out along with 1000 pairs of sunglasses. Umbrellas cost him \$5 each and sell for \$10. Sunglasses wholesale for \$2 and sell for \$5. The vendor has \$2500 to buy the goods. Whatever he does not sell is lost as worthless at the end of day.

(a) Assume Victor's opponent is the weather, set up a payoff matrix with the elements of the matrix representing his net profit.

(b) Suppose Victor hears the weather forecast and there is a 30% chance of rain. What should he do?

14. For the zero-sum game with matrix

$$\begin{pmatrix} -1 & 0 & 3 & 3 \\ 1 & 1 & 0 & 2 \\ 2 & -2 & 0 & 1 \\ 2 & 3 & 3 & 0 \end{pmatrix},$$

it has been found that the saddle point in mixed strategies for Player II is $Y = \left(\frac{3}{7}, 0, \frac{1}{7}, \frac{3}{7}\right)$. Given that the value of the game is $\frac{9}{7}$, find the saddle point in mixed strategies for Player I.

15. Another method that can be used to solve a zero-sum matrix game uses calculus to find the interior saddle points. For example, consider

$$A = \begin{pmatrix} 4 & -3 & -2 \\ -3 & 4 & -2 \\ 0 & 0 & 1 \end{pmatrix}.$$

A strategy for each player is of the form $X = (x_1, x_2, 1-x_1-x_2)$, $Y = (y_1, y_2, 1-y_1-y_2)$, so we consider the function $f(x_1, x_2, y_1, y_2) = XAY^T$. Now solve the system of equations

$$\frac{\partial f}{\partial x_1} = \frac{\partial f}{\partial x_2} = \frac{\partial f}{\partial y_1} = \frac{\partial f}{\partial y_2} = 0$$

to get X^* and Y^*. If these are legitimate completely mixed strategies, then you can verify that they are optimal and then find $v(A)$. Carry out these calculations for A and verify that they give optimal strategies.

16. Each player displays either one or two fingers and simultaneously guesses how many fingers the opposing player will show. If both players guess correctly or both incorrectly, the game is a draw. If only one guesses correctly, that player wins an amount equal to the total number of fingers shown by both players. Each pure strategy has two components: the number of fingers to show and the number of fingers to guess. Find the game matrix and the saddle point strategies.

17. Two brothers, Curly and Shemp, inherit a car worth 8000 dollars. Since only one of them can actually have the car, they agree they will present sealed bids to buy the car from the other brother. The brother that puts in the highest sealed bid gets the car. They must bid in 1000 dollar units. If the bids happen to be the same, then they flip a coin to determine ownership and no money changes hands. Curly can bid only up to 5000 while Shemp can bid up to 8000. Find the payoff matrix with Curly as the row player and the expected net gain (since the car is worth 8000). Find v^-, v^+ and use dominance to solve the game.

18. Show that if

$$v^- = \max_X \min_Y E(X,Y) = \min_Y \max_X E(X,Y) = v^+,$$

then there exists a scalar v, $\hat{X}$ and $\hat{Y}$ such that

$$E(i,\hat{Y}) \le v \le E(\hat{X},j), \quad \text{for all } i \text{ and } j.$$