MATH 4321 – Game Theory
Solution to Homework One

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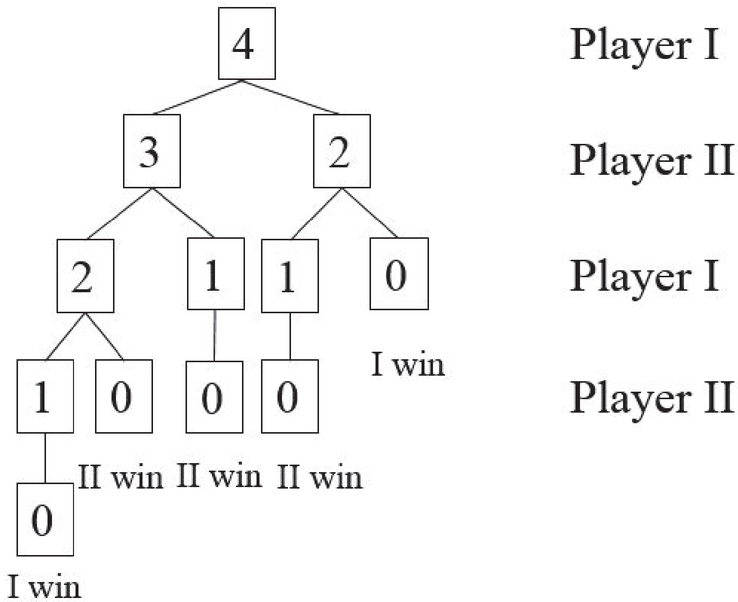
1. (a) With 4 pennies in single pile, the game tree is depicted as follows:



(b) Strategies for Player I (2 choices of strategy in later move when Player I plays 3 and no choice of strategy in later move when Player I plays 2)

(1) play $\boxed{3}$ then if at $\boxed{2}$ play $\boxed{1}$

(2) play $\boxed{3}$ then if at $\boxed{2}$ play $\boxed{0}$

(3) play $\boxed{2}$

Strategies for Player II (2 choices of strategy when Player I plays 3 and 2 choices of strategy when Player I plays 2)

(1) if at $\boxed{3}\rightarrow\boxed{2}$ if at $\boxed{2}\rightarrow\boxed{1}$

(2) if at $\boxed{3}\rightarrow\boxed{2}$ if at $\boxed{2}\rightarrow\boxed{0}$

(3) if at $\boxed{3}\rightarrow\boxed{1}$ if at $\boxed{2}\rightarrow\boxed{1}$

(4) if at $\boxed{3}\rightarrow\boxed{1}$ if at $\boxed{2}\rightarrow\boxed{0}$

I \ II	1	2	3	4	
1	1	1	-1	-1	min $= -1$
2	-1	-1	-1	-1	min $= -1$
3	-1	1	-1	1	min $= -1$
	max $= 1$	max $= 1$	max $= -1$	max $= 1$	

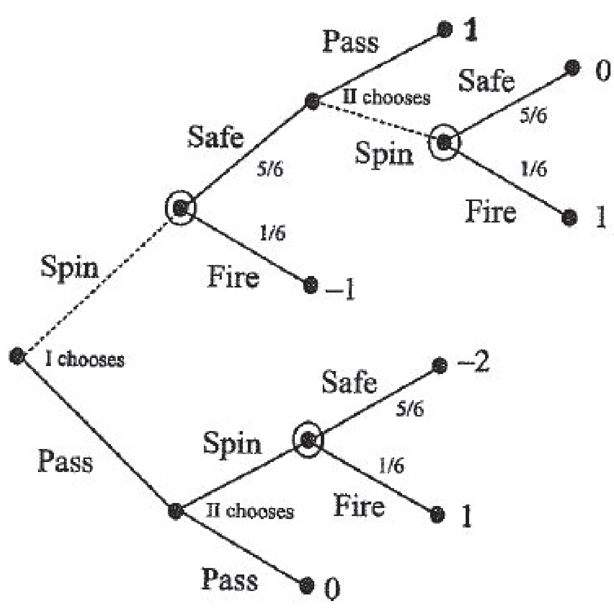
(c) We obtain $v^+ = -1$ and $v^- = -1$, so the value of the game is -1. All entries in the third column are saddle points. Once Player II players Strategy 3, Player I is indifferent to choose any of his 3 strategies. Player II always wins since Strategy 3 is a weakly dominant strategy for player II. This is obvious since one penny is always left behind for Player I under strategy 3 of Player II. Everyone would prefer to be the player who makes the second move (Player 2).

2. (a) The game matrix is given by

I \ II	1	2	3	4	5	
1	0	2	-1	-1	-1	min $= -1$
2	-2	0	2	-1	-1	min $= -2$
3	1	-2	0	2	-1	min $= -2$
4	1	1	-2	0	2	min $= -2$
5	1	1	1	-2	0	min $= -2$
	max $= 1$	max $= 2$	max $= 2$	max $= 2$	max $= 2$	

(b) We obtain $v^+ = 1$ and $v^- = -1$. Since $v^+ \neq v^-$, so there exists no saddle point in pure strategies.

3. The game tree of the Russian roulette is depicted as follows:



Strategies for Player I:
I1 spin
I2 pass

Strategies for Player II:
II1 If I spins, then pass; if I passes, then spin (opposite play)
II2 If I spins, then pass; if I passes, then pass (always pass)
II3 If I spins, then spin; if I passes, then spin (always spin)
II4 If I spins, then spin; if I passes, then pass (same play)

The game matrix of the Russian roulette is given by

I \ II	II1	II2	II3	II4	
I1	$\frac{1}{6}(-1)+\frac{5}{6}(1)=\frac{2}{3}$	$\frac{2}{3}$ (same as I1, II1)	$\frac{1}{6}(-1)+\frac{5}{6}(1)[\frac{1}{6}(1)+\frac{5}{6}(0)]=-\frac{1}{36}$	$-\frac{1}{36}$ (same as I1, II3)	$\min=-\frac{1}{36}$
I2	$\frac{5}{6}(-2)+\frac{1}{6}(1)=-\frac{3}{2}$	0	$-\frac{3}{2}$ (same as I2, II1)	0 (same as I2, II2)	$\min=-\frac{3}{2}$
	$\max=\frac{2}{3}$	$\max=\frac{2}{3}$	$\max=-\frac{1}{36}$	$\max=0$	

Note that $v^+ = v^- = -\frac{1}{36}$. The saddle point under pure strategies is (I1, II3); that is, both players choose to spin under the solution concept of saddle point equilibrium. Note that even though the payoff under (I1, II4) is the same as $v^+ = v^- = -\frac{1}{36}$, it is not a maximum under the column strategy II4 (not observing the row-min and column-max property). Therefore, (I1, II4) is not a saddle point in pure strategies.

4. Consider the following 2 separate cases:

(i) If $0 < x \leq 3$, then

$$A = \begin{pmatrix} 3 & 6 \\ x & 0 \end{pmatrix} \begin{matrix} \min = 3 \\ \min = 0 \end{matrix}.$$
$$\max = 3 \quad \max = 6$$

We have $v^+ = v^- = 3$, so the game has $(1,1)$ as the saddle point in pure strategies.

(ii) If $x > 3$, then

$$A = \begin{pmatrix} 3 & 6 \\ x & 0 \end{pmatrix} \begin{matrix} \min = 3 \\ \min = 0 \end{matrix}.$$
$$\max = x \quad \max = 6$$

We have $v^+ = \min(x,6) > 3 = v^-$, so the game has no saddle point in pure strategies.

5. (a) Note that

$$v^-(A) = \max\big(1, \min(z,0)\big) = 1 \text{ and } v^+(A) = \min\big(1, \max(2,z)\big) = 1,$$

so the saddle point in pure strategies exists and it is at row 2 and column 1. Also, $v(A) = 1$.
On the other hand, we have

$$v^-(B) = \max\big(1, \min(z,0)\big) = 1 \text{ and } v^+(B) = \min\big(1, \max(2,z)\big) = 1,$$

so the saddle point in pure strategies exists and it is at row 1 and column 2. Also, $v(B) = 1$.

(b) (i) Suppose we pick $z = 3$ so that

$$A + B = \begin{pmatrix} 2 & 4 \\ 4 & 2 \end{pmatrix}.$$

Note that $E(X,\ 1) = 2x_1 + 4x_2$ and $E(X,\ 2) = 4x_1 + 2x_2$. Setting $E(X,\ 1) = E(X,\ 2)$ and imposing $x_1 + x_2 = 1$, we obtain

$$X^* = (\frac{1}{2}\ \frac{1}{2}).$$

In a similar manner, by solving $E(1, Y) = E(2, Y)$ and $y_1 + y_2 = 1$, we obtain $Y^* = (\frac{1}{2}\ \frac{1}{2})$. Also, we observe

$$v(A + B) = (\frac{1}{2}\ \frac{1}{2}) \begin{pmatrix} 2 & 4 \\ 4 & 2 \end{pmatrix} \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} = 3 > v(A) + v(B).$$

(ii) Next, we pick $z = -1$ so that

$$A + B = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}.$$

We obtain $X^* = Y^* = (\frac{1}{2}\ \frac{1}{2})$ and $v(A + B) = 1 < v(A) + v(B)$.

6. For the 2×2 game matrix $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, the probability vectors of mixed strategies are $X^* = (x^* \quad 1 - x^*)$, $x^* = \frac{a_{22} - a_{21}}{a_{11} - a_{12} - a_{21} + a_{22}}$, and $Y^* = (y^* \quad 1 - y^*)$, $y^* = \frac{a_{22} - a_{12}}{a_{11} - a_{12} - a_{21} + a_{22}}$;

$$v = \frac{1}{(1\ 1) A^{-1} \binom{1}{1}} = \frac{\det A}{(1\ 1) \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}}.$$

We obtain $X^* = (\frac{15}{22}, \frac{7}{22})$, $Y^* = (\frac{9}{22}, \frac{13}{22})$, $v = -\frac{3}{22}$.

7. There is a saddle point in pure strategies at $(1, 3)$. This is because the $(1, 3)$ entry corresponds to a minima of the 1st row and maxima of the 3th column. This gives the value of the game, $v = 3$. One can verify that

$$v = \frac{1}{J_n A^{-1} J_n^T} = \frac{1}{(1 \quad 1 \quad 1) \begin{pmatrix} \frac{13}{18} & \frac{1}{2} & -\frac{19}{18} \\ \frac{1}{3} & 0 & -\frac{1}{3} \\ -\frac{17}{18} & -\frac{1}{2} & \frac{29}{18} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}} = \frac{1}{\frac{1}{3}} = 3.$$

To search for the saddle point in mixed strategies, we attempt to use the formulas:

$$\begin{aligned} X^* &= vJ_3A^{-1} = 3(1\ 1\ 1) \begin{pmatrix} \frac{13}{18} & \frac{1}{2} & -\frac{19}{18} \\ \frac{1}{3} & 0 & -\frac{1}{3} \\ -\frac{17}{18} & -\frac{1}{2} & \frac{29}{18} \end{pmatrix} = (\frac{1}{3}\ 0\ \frac{2}{3}) \\ Y^{*T} &= vA^{-1}J_3^T = 3 \begin{pmatrix} \frac{13}{18} & \frac{1}{2} & -\frac{19}{18} \\ \frac{1}{3} & 0 & -\frac{1}{3} \\ -\frac{17}{18} & -\frac{1}{2} & \frac{29}{18} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \end{pmatrix}. \end{aligned}$$

Remark

A smart student may doubt about the applicability of the computational formulas:

$$X^* = vJ_3A^{-1} \text{ and } Y^* = vA^{-1}J_3^T,$$

when the saddle point mixed strategies X^* and Y^* are not completely mixed (not all components of X^* and Y^* are strictly positive). In this numerical example, we observe

$$\begin{aligned} E(2, Y^*) &= (4 \quad -3 \quad 2)\begin{pmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \end{pmatrix} = 3 = v; \\ E(X^*, 2) &= \begin{pmatrix} \frac{1}{3} & 0 & \frac{2}{3} \end{pmatrix}\begin{pmatrix} 5 \\ -3 \\ 2 \end{pmatrix} = \frac{5}{3} + \frac{4}{3} = 3 = v. \end{aligned}$$

In general, we have $E(2, Y^*) \leq v$ and $E(X^*, 2) \geq v$. In this example, though $x_2^* = 0$ and $y_2^* = 0$, it happens to have strict equality, where $E(2, Y^*) = v$ and $E(X^*, 2) = v$.

The computational formulas manage to give the saddle point in mixed strategies (though they are not completely mixed). On the other hand, the saddle point in pure strategies cannot be obtained using the computational formulas.

In summary, $x_i^* > 0$ and $y_i^* > 0$ for all i and j are the sufficient (but not necessary) conditions for the guaranteed success of finding X^* and Y^* using the computational formulas. When $x_i^* = 0$, it is still possible to have $E(i, Y^*) = v$; similarly for $y_j^* = 0$, we may still have $E(X^*, j) = v$. The computational formulas remain to be applicable, provided that $E(i, Y^*) = v$ and $E(X^*, j) = v$, for all i and j. We happen to have such a good luck in this example.

Alternatively, suppose we apply the two systems of equations to solve for X^*, Y^* and v, where

$$\begin{cases} v = E(1, Y) = 3y_1 + 5y_2 + 3y_3 \\ v = E(2, Y) = 4y_1 - 3y_2 + 2y_3 \\ v = E(3, Y) = 3y_1 + 2y_2 + 3y_3 \\ y_1 + y_2 + y_3 = 1 \end{cases};$$

and

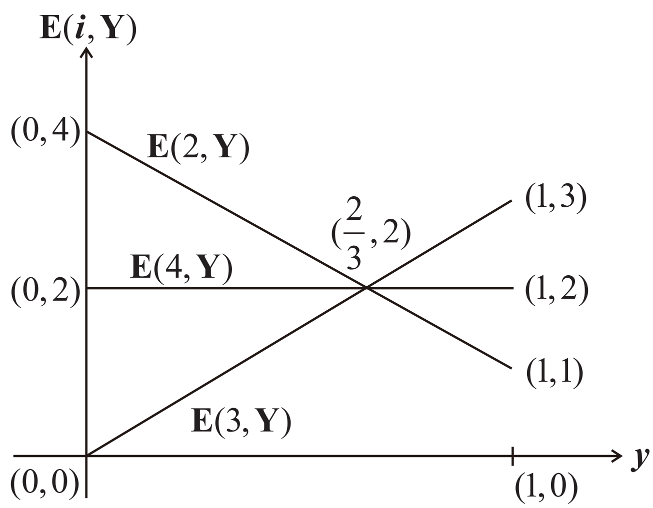
$$\begin{cases} v = E(X, 1) = 3x_1 + 4x_2 + 3x_3 \\ v = E(X, 2) = 5x_1 - 3x_2 + 2x_3 \\ v = E(X, 3) = 3x_1 + 2x_2 + 3x_3 \\ x_1 + x_2 + x_3 = 1 \end{cases}.$$

In the first system, performing subtraction of the third equation from the first equation, we obtain $3y_2 = 0$ giving $y_2 = 0$. Similarly, we obtain $x_2 = 0$ from the second system. We obtain $X^* = \begin{pmatrix} \frac{1}{3} & 0 & \frac{2}{3} \end{pmatrix}$, $Y^* = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}$ and $v = 3$. The two systems of equations give the same results for X^*, Y^* and v as well.

8. First, we compute

$$E(1,Y) = 5(1-y), \quad E(2,Y) = 4-3y, \quad E(3,Y) = 3y, \quad E(4,Y) = 2.$$

By plotting the lines $E(1,Y)$, $E(2,Y)$, $E(3,Y)$ and $E(4,Y)$ for $Y = (y,\ 1-y)$, $0 \le y \le 1$, we find the upper envelope. For $y \le \frac{1}{2}$, the line segment $E(1,Y)$ forms part of the upper envelope. However, when $y > \frac{1}{2}$, $E(1,Y) < E(2,Y)$. We observe that the lowest point of the upper envelope is given by the intersection of $E(2,Y)$ and $E(3,Y)$. We skip the line $E(1,Y)$ for simplicity of the plots since it is irrelevant in the subsequent calculation of Y^*. Also, the line $E(4,Y)$ also passes through the lowest point of the upper envelope.



We solve for $Y^* = (y^*,\ 1-y^*)$ by setting $E(2,Y^*) = E(3,Y^*)$. This leads to

$$y^* + 4(1-y^*) = 3y^*,$$

giving $y^* = \dfrac{2}{3}$. The saddle point mixed strategy of the column player is $Y^* = \left(\dfrac{2}{3},\ \dfrac{1}{3}\right)$. The value of the game is $E(3,Y^*) = 2$.

Also, note that $E(4,Y^*) = 2$ while $E(1,Y^*) = 5\left(\dfrac{1}{3}\right) < 2$. This indicates that row 1 is never used in the saddle point mixed strategy of the row player.

We observe that row 4 can be duplicated by the convex combination of $\dfrac{1}{2} \times$ row $2 + \dfrac{1}{2} \times$ row 3. First, we consider $X = (0,\ x,\ 1-x,\ 0)$ by leaving row 4 for a while and set $E(X,1) = E(X,2)$. This leads to

$$x + 3(1-x) = 4x$$

so that $x = \dfrac{1}{2}$. Accordingly, we have $E(X,1) = E(X,2) = 2 =$ value of the game. Since row $4 = \dfrac{1}{2} \times$ row $2 + \dfrac{1}{2} \times$ row 3, the most general saddle point mixed strategies of the row player is given by

$$X^* = \left(0,\ \frac{\alpha}{2},\ \frac{\alpha}{2},\ 1-\alpha\right), \quad \text{where} \quad 0 \le \alpha \le 1.$$

As a verification, we observe

$$E(X^*, 1) = \left(0,\ \frac{\alpha}{2},\ \frac{\alpha}{2},\ 1-\alpha\right)\begin{pmatrix} 0 \\ 1 \\ 3 \\ 2 \end{pmatrix} = 2$$

and

$$E(X^*, 2) = \left(0,\ \frac{\alpha}{2},\ \frac{\alpha}{2},\ 1-\alpha\right)\begin{pmatrix} 5 \\ 4 \\ 0 \\ 2 \end{pmatrix} = 2.$$

When $\alpha = 0$, we have the pure strategy $X^* = (0 \quad 0 \quad 0 \quad 1)$ for Player 1.

As a further verification, we observe

$$E(1, Y^*) = (0 \quad 5)\begin{pmatrix} \frac{2}{3} \\ \frac{1}{3} \end{pmatrix} = \frac{5}{3} < v; \quad E(2, Y^*) = (1 \quad 4)\begin{pmatrix} \frac{2}{3} \\ \frac{1}{3} \end{pmatrix} = 2 = v;$$
$$E(3, Y^*) = (3 \quad 0)\begin{pmatrix} \frac{2}{3} \\ \frac{1}{3} \end{pmatrix} = 2 = v; \quad E(4, Y^*) = (2 \quad 2)\begin{pmatrix} \frac{2}{3} \\ \frac{1}{3} \end{pmatrix} = 2 = v.$$

We deduce that x_1^* must be 0 since $E(1, Y^*) < 2$. However, x_2^*, x_3^* and x_4^* can be zero in a saddle point equilibrium, though $E(2, Y^*) = E(3, Y^*) = E(4, Y^*) = 2 = v$. Indeed, $x_2^* = x_3^* = 0$ when $\alpha = 0$ and $x_4^* = 0$ when $\alpha = 1$.

9. Let A_{ij} denote the Attacker's strategy of putting the i^{th} division on the first road and the j^{th} division on the second road. Let D_{ij} denote the Defender's strategy of putting the i^{th} division on the first road and the j^{th} division on the second road.

number of defending divisions	number of attacking divisions to win
0	1 or more
1	3 or more
2	–
3	–

(a) The payoff matrix is given by

$$\begin{array}{c} \\ A_{40} \\ A_{31} \\ A_{22} \\ A_{13} \\ A_{04} \end{array} \begin{array}{c} \begin{array}{cccc} D_{30} & D_{21} & D_{12} & D_{03} \end{array} \\ \begin{pmatrix} -1 & -1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 \end{pmatrix} \end{array}.$$

(b) The 1st and 3rd rows are dominated by the 2nd row; it is suboptimal to put 2 or 4 attacking divisions in a row. From intuition, it is inferior to put 2 or 4 attacking divisions in a road, since 2 attacking divisions serve the same role as 1 attacking division and 4 attacking divisions serve the same role as 3 attacking divisions.
In a similar manner, the 5th row is dominated by the 4th row.
For the column strategies, the 1st column is dominated by the 2nd column; the 4th column is dominated by the 3rd column.
The reduced payoff matrix becomes

$$\begin{array}{cc} & \begin{array}{cc} D_{21} & D_{12} \end{array} \\ \begin{array}{c} A_{31} \\ A_{13} \end{array} & \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \end{array}.$$

This payoff matrix resembles the Odds and Evens game. The value of this symmetric game is seen to be zero. The saddle point in mixed strategies is given by

$$X^* = \left(0,\ \frac{1}{2},\ 0,\ \frac{1}{2},\ 0\right) \quad \text{and} \quad Y^* = \left(0,\ \frac{1}{2},\ \frac{1}{2},\ 0\right).$$

10. Consider the matrix game

$$A = \begin{pmatrix} a_4 & a_5 & a_3 \\ a_1 & a_6 & a_5 \\ a_2 & a_4 & a_3 \end{pmatrix},\ a_1 < a_2 < a_3 < a_4 < a_5 < a_6.$$

We observe that:

column 2 is dominated by column 3

row 3 is dominated by row 1.

The reduced game matrix becomes

$$\begin{pmatrix} a_4 & a_3 \\ a_1 & a_5 \end{pmatrix}.$$

There is no saddle point in pure strategies, so we seek for a saddle point in mixed strategies. We compute

$$\begin{aligned} E(X,1) &= a_4 x_1 + a_1(1 - x_1) = (a_4 - a_1)x_1 + a_1 \\ E(X,2) &= a_3 x_1 + a_5(1 - x_1) = (a_3 - a_5)x_1 + a_5. \end{aligned}$$

By solving $E(X,1) = E(X,2)$, we obtain

$$x_1 = \frac{a_5 - a_1}{a_4 + a_5 - a_1 - a_3}.$$

The saddle point mixed strategy of Player I is

$$X^* = \left(\frac{a_5 - a_1}{a_4 + a_5 - a_1 - a_3},\quad \frac{a_4 - a_3}{a_4 + a_5 - a_1 - a_3},\quad 0\right).$$

Similarly, we can find the saddle point mixed strategy of Player II to be

$$Y^* = \left(\frac{a_5 - a_3}{a_4 + a_5 - a_1 - a_3}, \quad 0, \quad \frac{a_4 - a_1}{a_4 + a_5 - a_1 - a_3} \right).$$

The value of the game is given by

$$v(A) = E(1, Y^*) = (a_4 \quad a_3 \quad a_3) \begin{pmatrix} \frac{a_5 - a_3}{a_4 + a_5 - a_1 - a_3} \\ 0 \\ \frac{a_4 - a_1}{a_4 + a_5 - a_1 - a_3} \end{pmatrix} = \frac{a_4 a_5 - a_1 a_3}{a_4 + a_5 - a_1 - a_3}.$$

11. Given the zero-sum game matrix

$$A = \begin{pmatrix} 3 & -2 & 4 & 7 \\ -2 & 8 & 4 & 0 \end{pmatrix},$$

it is seen that column 3 and column 4 are dominated by column 1. After deleting these two dominated column strategies, the reduced game matrix becomes

$$\begin{pmatrix} 3 & -2 \\ -2 & 8 \end{pmatrix}.$$

There is no saddle point in pure strategies. We seek for saddle point in mixed strategies.

(a) Consider

$$\begin{aligned} E(X, 1) &= 3x_1 - 2(1 - x_1) = 5x_1 - 2 \\ E(X, 2) &= -2x_1 + 8(1 - x_1) = -10x_2 + 8. \end{aligned}$$

By equating $E(X, 1) = E(X, 2)$, we obtain $x_1 = \frac{2}{3}$. This gives

$$X^* = \left(\frac{2}{3}, \quad \frac{1}{3} \right) \quad \text{and} \quad v(A) = E(X^*, 1) = \left(\frac{2}{3}, \quad \frac{1}{3} \right) \begin{pmatrix} 3 \\ -2 \end{pmatrix} = 2 - \frac{2}{3} = \frac{4}{3}.$$

In a similar manner, we consider

$$\begin{aligned} E(1, Y) &= 3y_1 - 2(1 - y_1) = 5y_1 - 2 \\ E(2, Y) &= -2y_1 + 8(1 - y_1) = -10y_1 + 8. \end{aligned}$$

We obtain $Y^* = \left(\frac{2}{3}, \frac{1}{3}, 0, 0 \right)$.

(b) Given the chosen mixed strategy of Player II: $Y = \left(\frac{1}{4}, \frac{1}{2}, \frac{1}{8}, \frac{1}{8} \right)$, we have

$$\begin{aligned} E(X, Y) &= \left[3 \cdot \frac{1}{4} + (-2)\frac{1}{2} + 4 \cdot \frac{1}{8} + 7 \cdot \frac{1}{8} \right] x_1 \\ &\quad + \left[(-2) \cdot \frac{1}{4} + 8 \cdot \frac{1}{2} + 4 \cdot \frac{1}{8} + 0 \cdot \frac{1}{8} \right] (1 - x_1) \\ &= 4 - \frac{23}{8} x_1. \end{aligned}$$

Note that $\max\limits_{0\le x_1\le 1}\left\{4-\frac{23}{8}x_1\right\}=4$, which occurs at $x_1=0$. Hence, $X=(0,\quad 1)$ is the best response for player I to the strategy $Y=\left(\frac{1}{4},\ \frac{1}{2},\ \frac{1}{8},\ \frac{1}{8}\right)$. Alternatively, we observe

$$E(1,Y)=\begin{pmatrix}3 & -2 & 4 & 7\end{pmatrix}\begin{pmatrix}\frac{1}{4}\\ \frac{1}{2}\\ \frac{1}{8}\\ \frac{1}{8}\end{pmatrix}=\frac{4}{8}\quad\text{and}\quad E(2,Y)=\begin{pmatrix}-2 & 8 & 4 & 0\end{pmatrix}\begin{pmatrix}\frac{1}{4}\\ \frac{1}{2}\\ \frac{1}{8}\\ \frac{1}{8}\end{pmatrix}=4.$$

Since $E(2,Y)>E(1,Y)$, so the optimal strategy of Player I is to play the pure strategy of row 2. This shows an agreement with the earlier result.

(c) In a similar manner, it is easy to show that $Y=(1,\quad 0,\quad 0,\quad 0)$ is the best response for Player II when $X=(0,\quad 1)$.

12. (a) The game matrix is seen to be

$$\begin{array}{cc} & \begin{array}{cc} H & T \end{array} \\ \begin{array}{c} H \\ T \end{array} & \begin{pmatrix} 3 & -2 \\ -2 & 1 \end{pmatrix} \end{array}.$$

There is no saddle point in pure strategies, so we seek for a saddle point in mixed strategies. Note that

$$\begin{aligned} E(X,1) &= 3x_1-2(1-x_1)=5x_1-2 \\ E(X,2) &= -2x_1+(1-x_1)=-3x_1+1. \end{aligned}$$

By equating $E(X,1)=E(X,2)$, we obtain $X^*=\left(\frac{3}{8}\quad \frac{5}{8}\right)$. Also, the value of the game is $v(A)=E(X^*,1)=-\frac{1}{8}$. To solve for Y^*, we consider

$$E(1,Y)=v(A)\Leftrightarrow 3y_1-2(1-y_1)=-\frac{1}{8}$$

giving $5y_1=2-\frac{1}{8}=\frac{15}{8}$ or $y_1=\frac{3}{8}$. Therefore, $Y^*=\left(\frac{3}{8}\quad \frac{5}{8}\right)$.

(b) Note that

$$E(X,\overline{Y})=\left(3\cdot\frac{1}{3}-2\cdot\frac{2}{3}\right)x_1+\left[(-2)\frac{1}{3}+\frac{2}{3}\right](1-x_1)=-\frac{x_1}{3}.$$

We observe that $\max\limits_{0\le x_1\le 1}\left\{-\frac{x_1}{3}\right\}=0$, which occurs at $x_1=0$. The best response is the pure strategy $\overline{X}=(0\quad 1)$ and $E(\overline{X},\overline{Y})=0$.

13. (a) Victor's stocks for rain (strategy A): use \$2,500 to buy 500 umbrellas

Victor's stocks for sunny (strategy B): use \$2,000 to buy 1,000 sunglasses and \$500 to buy 100 umbrellas.

If it rains, then strategy A earns \$2,500 while strategy B loses \$2,000 − \$500 = \$1,500.

If it is sunny, then strategy A loses \$2,500 − \$1,000 = \$1,500 while strategy B earns \$3,000 + \$500 = \$3,500.

The payoff matrix is seen to be

Victor \ weather	rain	sunny
A	2,500	-1,500
B	-1,500	3,500

Let $(p, 1-p)$ be the mixed strategy of Victor. We find p by the payoff-equating method. Consider

$$\pi_{rain} = p(2{,}500) + (1-p)(-1{,}500)$$
$$\pi_{sunny} = p(-1{,}500) + (1-p)(3{,}500);$$

by equating $\pi_{rain} = \pi_{sunny}$, we obtain $p = \frac{5}{9}$.

(b) The weather forecast dictates $Y^0 = (0.3, 0.7)$. We consider

$$\begin{aligned} E(A,\ Y^0) &= (1\ 0)\begin{pmatrix} 2{,}500 & -1{,}500 \\ -1{,}500 & 3{,}500 \end{pmatrix}\begin{pmatrix} 0.3 \\ 0.7 \end{pmatrix} \\ &= 2{,}500 \times 0.3 - 1{,}500 \times 0.7 = -300 \\ E(B,\ Y^0) &= (0\ 1)\begin{pmatrix} 2{,}500 & -1{,}500 \\ -1{,}500 & 3{,}500 \end{pmatrix}\begin{pmatrix} 0.3 \\ 0.7 \end{pmatrix} \\ &= -1{,}500 \times 0.3 + 3{,}500 \times 0.7 = 2000. \end{aligned}$$

Since $E(A, Y^0) < E(B, Y^0)$, Victor should choose the pure strategy B.

14. Let $X^* = (x_1\ x_2\ x_3\ x_4)$. Given that $Y^* = \left(\frac{3}{7}, 0, \frac{1}{7}, \frac{3}{7}\right)$ and $v = \frac{9}{7}$, we can check that $E(1, Y^*) = E(2, Y^*) = E(3, Y^*) = E(4, Y^*) = v$ are satisfied. As a remark, knowing that $y_2^* = 0$, we do not impose $E(X^*, 2) = v$ since $E(X^*, 2) \geq v$ in general. We determine X^*

by setting the following equations:

$$\begin{aligned}
E(X^*,\ 1) &= (x_1\ x_2\ x_3\ x_4)\begin{pmatrix}-1\\1\\2\\2\end{pmatrix}\\
&= -x_1 + x_2 + 2x_3 + 2x_4 = v = \frac{9}{7}\\
E(X^*,\ 3) &= 3x_1 + 3x_4 = \frac{9}{7}\\
E(X^*,\ 4) &= 3x_1 + 2x_2 + x_3 = \frac{9}{7}\\
x_1 + x_2 &+ x_3 + x_4 = 1.
\end{aligned}$$

Solving the 4 equations for the 4 unknowns, we obtain

$$X^* = (\frac{1}{12},\ \frac{13}{28},\ \frac{3}{28},\ \frac{29}{84}).$$

Interestingly, though $y_2 = 0$, we still observe

$$E(X^*, 2) = \begin{pmatrix}\frac{1}{12} & \frac{13}{28} & \frac{3}{28} & \frac{29}{84}\end{pmatrix}\begin{pmatrix}0\\1\\-2\\3\end{pmatrix} = \frac{9}{7} = v.$$

Remark

Though $y_j > 0 \Rightarrow E(X, j) = v$ or $E(X, j) > v \Rightarrow y_j = 0$, it is still plausible to have $y_j = 0$ while $E(X, j) = v$.

15. Consider the expected payoff

$$\begin{aligned}
f(x_1, x_2, y_1, y_2) &= XAY^T\\
&= 4x_1y_1 - 3x_1y_2 - 2x_1(1 - y_1 - y_2) - 3x_2y_1 + 4x_2y_2\\
&\quad - 2x_2(1 - y_1 - y_2) + (1 - x_1 - x_2)(1 - y_1 - y_2),
\end{aligned}$$

where $X = (x_1, x_2, 1 - x_1 - x_2)$ and $Y = (y_1, y_2, 1 - y_1 - y_2)$. We derive the system of equations by equating the first order partial derivatives of $f(x_1, x_2, y_1, y_2)$ to be zero:

$$\begin{aligned}
\frac{\partial f}{\partial x_1} &= 7y_1 - 3 = 0 \Rightarrow y_1 = \frac{3}{7}\\
\frac{\partial f}{\partial x_2} &= 7y_2 - 3 = 0 \Rightarrow y_2 = \frac{3}{7}\\
\frac{\partial f}{\partial y_1} &= 7x_1 - 1 = 0 \Rightarrow x_1 = \frac{1}{7}\\
\frac{\partial f}{\partial y_2} &= 7x_2 - 1 = 0 \Rightarrow x_2 = \frac{1}{7}.
\end{aligned}$$

Hence, we obtain the saddle point in mixed strategies (X^*, Y^*), where

$$X^* = (\frac{1}{7},\ \frac{1}{7},\ \frac{5}{7}) \text{ and } Y^* = (\frac{3}{7},\ \frac{3}{7},\ \frac{1}{7}).$$

Also, value of the game $= f(\frac{1}{7},\ \frac{1}{7},\ \frac{3}{7},\ \frac{3}{7}) = \frac{1}{7}$. As a check, we have

$$E(X^*,\ j) = \frac{1}{7},\ j = 1, 2, 3$$
$$E(i,\ Y^*) = \frac{1}{7},\ i = 1, 2, 3.$$

Hence, validity of these saddle point mixed strategies is verified.

16. The strategy can be represented by the ordered pair (a, b), where a is the number of fingers to show, $a = 1, 2$; b is the number of fingers to guess. There are 4 pure strategies for each player. The game matrix can be represented by

I \ II	(1,1)	(1,2)	(2,1)	(2,2)
(1,1)	0	2	-3	0
(1,2)	-2	0	0	3
(2,1)	3	0	0	-4
(2,2)	0	-3	4	0

The game matrix is skew symmetric, where $A = -A^T$, and this is a symmetric game. The value of the game is then equal to zero.

We solve for the saddle point mixed strategy of Player I by solving the following set of inequalities:

$$E(X, 1) \geq 0 \Leftrightarrow -2x_2 + 3x_3 \geq 0 \Leftrightarrow x_3 \geq \frac{2}{3}x_2 \tag{1}$$
$$E(X, 2) \geq 0 \Leftrightarrow 2x_1 - 3x_4 \geq 0 \Leftrightarrow x_1 \geq \frac{3}{2}x_4 \tag{2}$$
$$E(X, 3) \geq 0 \Leftrightarrow -3x_1 + 4x_4 \geq 0 \Leftrightarrow x_4 \geq \frac{3}{4}x_1 \tag{3}$$
$$E(X, 4) \geq 0 \Leftrightarrow 3x_2 - 4x_3 \geq 0 \Leftrightarrow x_2 \geq \frac{4}{3}x_3. \tag{4}$$

By inequalities (2) and (3), we obtain

$$x_1 \geq \frac{3}{2}x_4 \geq \frac{3}{2} \cdot \frac{3}{4}x_1 = \frac{9}{8}x_1 \Rightarrow x_1 = 0$$
$$x_4 \geq \frac{3}{4}x_1 \geq \frac{3}{4} \cdot \frac{3}{2}x_4 = \frac{9}{8}x_4 \Rightarrow x_4 = 0.$$

Given that $x_1 = 0$ and $x_4 = 0$, these dictate $x_2 + x_3 = 1$. Next, we consider $E(X, 1) \geq 0$ and $E(X, 4) \geq 0$, where $x_3 \geq \frac{2}{3}x_2$ and $x_2 \geq \frac{4}{3}x_3$, we obtain

$$x_3 \geq \frac{2}{3}(1 - x_3) \Leftrightarrow x_3 \geq \frac{2}{5}.$$

Similarly, we observe $x_2 \geq \frac{4}{3}(1 - x_2) \Leftrightarrow x_2 \geq \frac{4}{7}$. On the other hand, since $x_3 \geq \frac{2}{5}$ so that $x_2 \leq \frac{3}{5}$, together with $x_2 \geq \frac{4}{7}$, we have

$$\frac{3}{5} \geq x_2 \geq \frac{4}{7}.$$

Similarly, since $x_3 = 1 - x_2$, so

$$\frac{3}{7} \geq x_3 \geq \frac{2}{5}.$$

We conclude that the optimal mixed strategy of Player I is given by

$$X^* = (0,\ \alpha,\ 1-\alpha,\ 0),\ \text{where } \frac{3}{5} \geq \alpha \geq \frac{4}{7}.$$

By symmetry, we have the optimal mixed strategy $Y^* = (0, \alpha, 1-\alpha, 0)$, $\frac{3}{5} \geq \alpha \geq \frac{4}{7}$, for Player II.

17. This is a constant sum game of $4,000 (half of the worth of the car). We subtract $4,000 from the payoffs of the two brothers and end up with the following game matrix of a zero sum game that shows the payoff to Curly. In the table, we take $1,000 as one unit.

Curly \ Shemp	1	2	3	4	5	6	7	8
1	0	-2	-1	0	1	2	3	4
2	2	0	-1	0	1	2	3	4
3	1	1	0^{**}	0^{*}	1	2	3	4
4	0	0	0^{*}	0^{*}	1	2	3	4
5	-1	-1	-1	-1	0	2	3	4

It is obvious that the saddle points in pure strategies are $(3, 3)$, $(3, 4)$, $(4, 3)$ and $(4, 4)$, all give the value of the game to be 0. Therefore, the expected payoff is $4,000, same for both brothers. Note that strategy "3" dominates strategy "4" for both players. This dominance argument can be used to argue that $(3, 3)$ is the saddle point equilibrium strategies to be more likely to be played by both brothers.

18. Write $v = v^+ = v^-$. Let $\hat{X}$ be the corresponding mixed strategy of the row player and $\hat{Y}$ be the corresponding mixed strategy of the column player that give both the maximin and minimax. We deduce that $\min_Y E(X, Y)$ is maximized at $X = \hat{X}$ so that

$$\min_Y E(\hat{X}, Y) = \max_X \min_Y E(X, Y).$$

Similarly, note that $\max_X E(X, Y)$ is minimized at $Y = \hat{Y}$ so that

$$\max_X E(X, \hat{Y}) = \min_Y \max_X E(X, Y).$$

For any i and j, we deduce that

$$\begin{aligned} E(\hat{X}, j) &\geq \min_Y E(\hat{X}, Y) = \max_X \min_Y E(X, Y) = v \\ &= \min_Y \max_X E(X, Y) = \max_X E(X, \hat{Y}) \geq E(i, \hat{Y}). \end{aligned}$$

Remark:
The following statements are equivalent.

1. (X^*, Y^*) is a saddle point equilibrium pair such that

$$E(X, Y^*) \leq E(X^*, Y^*) = v \leq E(X^*, Y).$$

2. $v = \max_X \min_Y E(X, Y) = \min_Y \max_X E(X, Y)$.
3. $E(i, Y^*) \leq v \leq E(X^*, j)$ for all i and j.

Note that 2 $\implies$ 3 is established in the above, 1 $\implies$ 2 and 3 $\implies$ 1 are proven in the lecture notes.