MATH 4321 – Game Theory

Homework Two

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1. Is every iterated dominance equilibrium made up of strategies that are not weakly dominated?

2. *Discoordination*
   Suppose that a man and a woman each chooses whether to go to a prize fight or a ballet. The man would rather go to the prize fight, and the woman to the ballet. What is more important to them, however, is that the man wants to show up to the same event as the woman, but the woman wants to avoid him.

   (a) Construct a game matrix to illustrate this game, choosing numbers to fit the preferences described verbally.

   (b) If the woman moves first, what will happen?

   (c) Does the game have a first-mover advantage?

   (d) Show that there is no pure strategy Nash equilibrium.

3. *Finding pure Nash equilibriums*
   Find all pure Nash equilibriums of the two games illustrated in the following tables. Can any of them be reached by iterated dominance?

   (a)

		Column		
		Left	*Middle*	*Right*
	Up	$10, 10$	$0, 0$	$-1, 15$
Row	*Sideways*	$-12, 1$	$8, 8$	$-1, -1$
	Down	$15, 1$	$8, -1$	$0, 0$

Payoffs to: (Row, Column).

   (b)

		Brydox	
		Flavor	*Texture*
Apex	*Flavor*	$-2, 0$	$0, 1$
	Texture	$-1, -1$	$0, -2$

Payoffs to: (Apex, Brydox).

4. *Three-by-Three equilibriums*
   Identify any dominated strategies and any pure strategy Nash equilibriums in the following game.

		Column		
		Left	*Middle*	*Right*
	Up	$1, 4$	$5, -1$	$-0, 1$
Row	*Sideways*	$-1, 0$	$-2, -2$	$-3, 4$
	Down	$0, 3$	$9, -1$	$5, 0$

Payoffs to: (Row, Column).

5. *Selfish and altruistic social behavior*
   Two people enter a bus. Two adjacent cramped seats are free. Each person must decide whether to sit or stand. Sitting alone is more comfortable than sitting next to the other person, which is more comfortable than standing.

   (a) Suppose that each person cares only about her own comfort. Model the situation as a strategic game. Is this game the same as the *Prisoner's Dilemma?* Find its Nash equilibrium (equilibriums?).

   (b) Suppose that each person is altruistic, ranking the outcomes according to the *other* person's comfort. However, out of politeness, one person prefers to stand than to sit if the other person stands. Model the situation as a strategic game. Is this game the *Prisoner's Dilemma?* Find its Nash equilibrium (equilibriums?).

   (c) Compare the people's comfort in the equilibriums of the two games.

6. Consider the game with the following bimatrix

	A	B	C
a	$1, 1$	$3, x$	$2, 0$
b	$2x, 3$	$2, 2$	$3, 1$
c	$2, 1$	$1, x$	$x^2, 4$

   (a) Find x so that the game has no pure Nash equilibrium.

   (b) Find x so that the game has (c, C) as pure Nash equilibrium.

7. *Extension of the Stag Hunt model*
   Extend the n-hunter *Stag Hunt* by giving each hunter K (a positive integer) units of effort, which she can allocate between pursuing the stag and catching hares. Denote the effort hunter i devotes to pursuing the stag by e_i, a nonnegative integer equal to at most K. The chance that the stag is caught depends on the smallest of all the hunters' efforts, denoted by $\min_j e_j$. ("A chain is as strong as its weakest link.") Hunter i's payoff to the action profile $(e_1, \ldots, e_n)$ is $2\min_j e_j - e_i$. This is because she is better off the more likely the stag is caught, the worse off the more effort she devotes to pursuing the stag, which means she catches fewer hares.

(a) Is the action profile $(e, \ldots, e)$, in which every hunter devotes the same effort to pursuing the stag, a Nash equilibrium for any value of e? What is a player's payoff to this profile? What is her payoff when she deviates to a lower or higher effort level?

(b) Is any action profile in which not all the players' effort levels are the same a Nash equilibrium? Consider a player whose effort exceeds the minimum effort level of all players. What happens to her payoff if she reduces her effort level to the minimum?

8. Suppose that a married couple, both of whom have just finished medical school, now have choices regarding their residencies. One of the new doctors has three choices of programs, while the other has two choices. They value their prospects numerically on the basis of the program itself, the city, staying together, and other factors, and arrive at the bimatrix

$$\begin{pmatrix} (5.2, 5.0) & (4.4, 4.4) & (4.4, 4.1) \\ (4.2, 4.2) & (4.6, 4.9) & (3.9, 4.3) \end{pmatrix}.$$

(a) Find all the pure Nash equilibriums. Which one should be played?

(b) The safety value of Player I is the payoff that the player is guaranteed to receive using the maxmin strategy. This is the floor value which is given by $\max_{X \in S_n} \min_{Y \in S_m} XAY^T$. Find the safety levels for each player.

(c) A pure Nash equilibrium is said to be individually rational if the payoffs to all players at the pure Nash equilibrium are higher than their respective safety value. Verify that the pure Nash equilibriums are individually rational.

9. Consider two competing firms in a declining industry that cannot support both firms profitably. Each firm has three possible choices, as it must decide whether (i) to exit the industry immediately, (ii) at the end of this quarter, or (iii) the end of the next quarter. If a firm chooses to exit, then its payoff is 0. Each quarter that both firms operate yields each a loss equal to -1, and each quarter that a firm operates alone yields it a payoff of 2. For example, if firm 1 plans to exit at the end of this quarter while firm 2 plans to exit at the end of the next quarter, then the payoffs are $(-1, 1)$. This is because both firms lose -1 in the first quarter and firm 2 gains 2 in the second. Furthermore, since the payoff for each firm is the sum of its quarterly payoffs, the payoff to firm 2 is $2 - 1 = 1$. Also, suppose firm 1 chooses to exit immediately and firm 2 continues to operate for two quarters, the outcome is $(0, 4)$.

(a) Write down this game in matrix form.

(b) Are there any strictly dominated strategies? Are there any weakly dominated strategies? Find them if they exist.

(c) Find all possible Nash equilibriums, which may be pure or mixed or both.

10. Consider the Stop-Go game. Two drivers meet at an intersection at the same time. They have the options of stopping and waiting for the other driver to continue, or going. Here is the payoff matrix in which the player who stops while the other goes loses a bit less than if they both stopped.

I/II	Stop	Go
Stop	$1, 1$	$1-\epsilon, 2$
Go	$2, 1-\epsilon$	$0, 0$

Assume that $0 < \epsilon < 1$. Find all pure and mixed Nash equilibriums. Show that the probability of both Go is an increasing function of ϵ.

11. There are two companies each with exactly one job opening. Suppose firm 1 offers the pay p_1 and firm 2 offers pay p_2 with $p_1 < p_2 < 2p_1$. There are two prospective applicants each of whom can apply to only one of the two firms. They make simultaneous and independent decisions. If exactly one applicant applies to a company, that applicant gets the job. If both apply to the same company, the firm flips a fair coin to decide who is hired and the other is unemployed (payoff zero).

    (a) Find the game matrix.

    (b) Find all Nash equilibriums.

12. Consider the Hawk-Dove game with the following game matrix:

			Animal 2	
			q	$1-q$
			Aggressive	Passive
Animal 1	p	Aggressive	$0, 0$	$6, 1$
	$1-p$	Passive	$1, 6$	$3, 3$

    (a) Show that the best response functions of the two animals are

$$B_1(q) = \begin{cases} p = 0 & \text{if } q > \frac{3}{4} \\ p \in [0,1] & \text{if } q = \frac{3}{4} \\ p = 1 & \text{if } q < \frac{3}{4} \end{cases};$$

$$B_2(p) = \begin{cases} q = 0 & \text{if } p > \frac{3}{4} \\ q \in [0,1] & \text{if } p = \frac{3}{4} \\ q = 1 & \text{if } p < \frac{3}{4} \end{cases}.$$

    (b) Find all the mixed strategy Nash equilibriums.

13. In a symmetric game (A, B), we have $B = A^T$. For a 2×2 symmetric game, where

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

Suppose we change the matrix A to A', where

$$A' = \begin{pmatrix} a_{11} - a & a_{12} - b \\ a_{21} - a & a_{22} - b \end{pmatrix}, \quad B' = A'^T, \quad a \text{ and } b \text{ are any values.}$$

Show that the two symmetric games (A, B) and (A', B') have the same set of pure and mixed Nash equilibriums.

14. *Coordination game*

Two people can perform a task if and only if they both exert effort. They are both better off if they both exert effort and perform the task than if neither exerts effort (and nothing is accomplished); the worst outcome for each person is that she exerts effort and the other does not (in which case again nothing is accomplished). Let c be a positive number less than 1 that can be interpreted as the cost of exerting effort. Find all the mixed strategy Nash equilibriums of this game. How do the equilibriums change as c increases? Explain the reasons for the changes.

	No effort	Effort
No effort	$0, 0$	$0, -c$
Effort	$-c, 0$	$1-c, 1-c$

15. *Incompetent experts*

Consider a variant of the expert diagnosis model, in which the experts are not entirely competent. Assume that each expert always correctly recognizes a major problem but correctly recognizes a minor problem with probability $s < 1$; with probability $1 - s$ she mistakenly thinks that a minor problem is major, and, if the consumer accepts her advice, performs a major repair and obtains the profit π. Maintain the assumption that each consumer believes (correctly) that the probability her problem is major is r. As before, a consumer who does not give the job of fixing her problem to an expert bears the cost E' if it is major and I' if it is minor. Suppose, for example, that an expert is honest and a consumer rejects advice to obtain a major repair. With probability r the consumer's problem is major, so that the expert recommends a major repair, which the consumer rejects; the consumer bears the cost E'. With probability $1 - r$, the consumer's problem is minor. In this case with probability s the expert correctly diagnoses it as minor, and the consumer accepts her advice and pays I; with probability $1 - s$ the expert diagnoses it as major, and the consumer rejects her advice and bears the cost I'. The consumer's expected payoff in this case is $-rE' - (1-r)[sI + (1-s)I']$.

(a) Construct the payoffs for every pair of actions of the two players and find the mixed strategy equilibrium in the case $E > rE' + (1-r)I'$.

(b) Does incompetence breed dishonesty? How about more wary consumers?

16. *Auditing game*

The Internal Revenue Service (IRS) must decide whether to audit a certain class of suspect tax returns to discover whether they are accurate or not. The goal of the IRS is to either prevent or catch cheating at minimum cost. The suspects want to cheat only if they will not be caught. Let us assume that the benefit of preventing or catching cheating is 4, the cost of auditing is C, where $C < 4$, the cost to the suspects of obeying the law is 1, and the cost of being caught is the fine $F > 1$.

Auditing Game modeled as a 2×2 simultaneous move game

$$
\begin{array}{cccccc}
 & & & \textbf{Suspects} & & \\
 & & \text{Cheat } (\theta) & & \text{Obey } (1-\theta) & \\
 & \text{Audit } (\gamma) & 4-C, -F & \rightarrow & 4-C, -1 \\
\textbf{IRS} & & \uparrow & & \downarrow \\
 & \text{Trust } (1-\gamma) & 0, 0 & \leftarrow & 4, -1
\end{array}
$$

Payoffs to: (IRS, Suspects). Arrows show how a player can increase his payoff.

Find the best response functions of the two players and mixed Nash equilibrium.

17. An entrepreneur, named Victor, outside Laguna beach can sell 500 umbrellas when it rains and 100 when the Sun is out along with 1000 pairs of sunglasses. Umbrellas cost him \$5 each and sell for \$10. Sunglasses wholesale for \$2 and sell for \$5. The vendor has \$2500 to buy the goods. Whatever he does not sell is lost as worthless at the end of day.

    (a) Assume Victor's opponent is the weather, set up a payoff matrix with the elements of the matrix representing his net profit.

    (b) Suppose Victor hears the weather forecast and there is a 30% chance of rain. What should he do?

18. Suppose two merchants have to choose a location along the straight road. They may choose any point in $\{1, 2, \cdots, n\}$. Assume there is exactly one customer at each of these points and a customer will always go to the nearest merchant. If the two merchants are equidistant to a customer then they share that customer, that is, $\frac{1}{2}$ the customer goes to each store. For example, if $n = 11$ and if the player I chooses location 3 and player II chooses location 8, then the payoff to I is 5 and the payoff to II is 6.

    (a) Suppose $n = 5$. Find the bimatrix and find the Nash equilibrium.

    (b) Find the Nash equilibrium in general if $n = 2k + 1$, that is, n is an odd integer. Can we find a Nash equilibrium when n is even? Explain your answer.