MATH 4321 – Game Theory
Solution to Homework Two

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1. Strategy x is weakly dominated by strategy y only if y has a strictly higher payoff in some strategy profile and has a strictly lower payoff in no strategy profile. An iterated dominance equilibrium exists only if the iterative process results in a single strategy profile at the end.

   In order for x to be in the final surviving profile, it would have to weakly dominate the second-to-last surviving strategy for that player (call it x_2). Thus, it is strictly better than x_2 as a response to some profile of strategies of the other players: $\pi_i(x, s_{-i}) > \pi_i(x_2, s_{-i})$ for some particular set of strategies for the other players s_{-i} that has survived deletion so far. But for x_2 to have survived deletion so far means that x_2 must be at least as good as response to the profile s_{-i} as the third-to-last surviving strategy: $\pi_i(x_2, s_{-i}) \geq \pi_i(x_3, s_{-i})$, and in turn none of the earlier deleted x_i strategies could have done strictly better as a response to s_i or they would not have been weakly dominated. Thus x must be a strictly better response in at least one strategy profile than all the previously deleted strategies for that player and it cannot have been weakly dominated by any of them.

2. (a) The game matrix of the Battle of Sexes with unequal level of love is constructed as follows:

		Woman	
		Prize fight	Ballet
Man	Prize fight	(4,-2)	(-3,2)
	Ballet	(-4,1)	(3,-1)

   (b) Unlike the coordination game in Lecture Note, the woman wants to avoid the man. If the woman moves first, she is likely to choose ballet than prize fight though this is not a dominant strategy. Once the woman has chosen ballet, the man would choose ballet. Hence, (Ballet, Ballet) is the outcome.

   (c) The woman would prefer to move second so that she can avoid the man. Also the man wants to move second so that he can join the woman. Both face with the disadvantage of being the first mover. This is just the opposite to that of the coordination game where both have the first-mover advantage.

   (d) In all the outcomes, one of the two players can improve his or her payoff if he or she deviates from the strategy profile. Hence, there is no pure strategy Nash equilibrium. Intuitively, since the two players look for different types of companion where the man wants to be together while the woman wants to be alone, it is impossible to have a Nash equilibrium as one of them would choose to deviate unilaterally.

3. (a) (Down, Left) is a strong Nash equilibrium while (Sideways, Middle) is a Nash equilibrium. The Nash equilibrium (Down, Left) can be generated by the iterated dominance procedure through the order (Up, Right, Sideways, Middle).

   (b) (Texture, Flavor) is a strong Nash equilibrium while (Flavor, Texture) is a Nash equilibrium. The Nash equilibrium (Texture, Flavor) can be generated by the iterated dominance procedure through the order (Flavor, Texture).

4. Sideways and Middle are dominated strategies. (Up, Left) is a strong pure Nash equilibrium. It can be generated by the iterated dominance procedure through the order (Sideways, Middle, Right, Down).

5. (a) The payoffs of the two players can be prescribed as follows:

I \ II	Sit	Stand
Sit	(2,2)	(3,1)
Stand	(1,3)	(1,1)

   This game is not the same as the Prisoner's dilemma. Obviously, (Sit, Sit) is a pure Nash equilibrium.

   (b) Now, sitting alone is ranked the lowest while standing alone has the highest payoff. The new bi-matrix game is given below.

I \ II	Sit	Stand
Sit	(2,2)	(0,3)
Stand	(3,0)	(1,1)

   Like the Prisoner's dilemma, (Stand, Stand) is the pure Nash equilibrium though both players are better off if the profile is chosen to be (Sit, Sit).

   (c) Obviously, the pure Nash equilibrium (Sit, Sit) in the first game provides more comfort to the players. However, the comfort level is lowered by being altruistic. The pure Nash equilibrium is changed to (Stand, Stand).

6. (a) Consider the bimatrix

	A	B	C
a	$1, 1$	$3, x$	$2, 0$
b	$2x, 3$	$2, 2$	$3, 1$
c	$2, 1$	$1, x$	$x^2, 4$

   Potential pure Nash equilibrium may be (b, A) or (c, C) or (a, B). No pure Nash equilibrium exists if

$$x \leq 1 \text{ and } x^2 \leq 3.$$

   This is satisfied when $-\sqrt{3} \leq x \leq 1$.

(b) To have (c, C) as a pure Nash equilibrium, we need to have $x^2 \geq 3$ and $x \leq 4$, that is, $\sqrt{3} \leq x \leq 4$ or $x \leq -\sqrt{3}$.

7. (a) Consider the strategy profile $(e, ..., e)$, where e is a nonnegative integer from 1 to K. Suppose player i chooses $e_i < e$, while the other players do not change their choices, then his payoff becomes $2e_i - e_i = e_i < e$. On the other hand, if player i chooses $e_i > e$, then his payoff is $2e - e_i$, which is again less than e, Player i can never benefit from deviating the equilibrium strategies unilaterally. Hence, $(e, ..., e)$ is a pure Nash equilibrium.

   (b) Suppose $e_k > \min_j e_j$, player k can benefit from the deviation of his strategy since $2\min_j e_j - e_k$ would become larger when he chooses $e_k = \min_j e_j$ (reducing his effort level to the minimum). Hence, $(e_1, ..., e_n)$ with differing levels of effort is not a Nash equilibrium.

8. (a) The two pure Nash equilibriums are at: $X = (1, 0)$, $Y = (1, 0, 0)$ and $X = (0, 1)$, $Y = (0, 1, 0)$.

$$\begin{pmatrix} (5.2^*, 5.0^*) & (4.4, 4.4) & (4.4, 4.1) \\ (4.2, 4.2) & (4.6^*, 4.9^*) & (3.9, 4.3) \end{pmatrix}$$

x_1	x_2	y_1	y_2	y_3	$E_I(X^*, Y^*)$	$E_{II}(X^*, Y^*)$
1	0	1	0	0	5.2	5.0
0	1	0	1	0	4.6	4.9

The first pure Nash equilibrium gives higher payoff to both players, so it is likely that it will be played out by the players based on payoff dominant criterion.

   (b) The computation of the safety levels requires the maxmin strategy in zero sum games. For player I, we write

$$A = \begin{pmatrix} 5.2 & 4.4 & 4.4^* \\ 4.2 & 4.6 & 3.9 \end{pmatrix}.$$

The maxmin strategy for player I is seen to be

$$X^* = (1, 0),\ Y^* = (0, 0, 1) \text{ and } v(A) = 4.4.$$

Note that though the payoff at node $(1, 2)$ is also 4.4, it is not a saddle point. When player II chooses the second column, player I can be better of by playing the first row. For player II, it is necessary to consider

$$B^T = \begin{pmatrix} 5.0 & 4.2 \\ 4.4 & 4.9 \\ 4.1 & 4.3 \end{pmatrix}.$$

Using the graphical method, we obtain the maxmin strategy to be $X^* = (\frac{5}{13}, \frac{8}{13}, 0)$ and $Y^* = (\frac{7}{13}, \frac{6}{13})$. The value of the game $v(B^T) = (\frac{5}{13}, \frac{8}{13}, 0) \begin{pmatrix} 5.0 & 4.2 \\ 4.4 & 4.9 \\ 4.1 & 4.3 \end{pmatrix} \begin{pmatrix} \frac{7}{13} \\ \frac{6}{13} \end{pmatrix} = 4.63.$

   (c) To verify that the two pure Nash equilibriums are individually rational, it suffices to show $E_I(X^*, Y^*) \geq v(A)$ and $E_{II}(X^*, Y^*) \geq v(B^T)$. Since both 5.2 and 4.6 are larger than $v(A) = 4.4$, the result is verified for player I. Similarly, both 5.0 and 4.9 are larger than $v(B^T) = 4.63$, the result is also verified for player II.

9. (a) Let E denote immediate exit, T denote exit this quarter, and N denote exit next quarter. The game matrix is represented as below:

$$
[\mathrm{H}] \quad \begin{array}{c|ccc}
\mathrm{I} \backslash \mathrm{II} & E & T & N \\
\hline
E & (0,0) & (0,2) & (0,4) \\
T & (2,0) & (-1,-1) & (-1,1) \\
N & (4,0) & (1,-1) & (-2,-2)
\end{array}
$$

(b) There are no strictly dominated strategy but there is a weakly dominated one: T. To see this, we consider a convex combination of the payoffs for strategy E and N. When we have

$$2 \leq 0(\lambda) + 4(1-\lambda), \quad -1 \leq 0(\lambda) + 1(1-\lambda) \quad \text{and} \quad -1 \leq 0(\lambda) - 2(1-\lambda),$$

which is equivalent to $\frac{1}{2} \leq \lambda \leq \frac{1}{2}$, there is only one value of $\lambda = \frac{1}{2}$ satisfying the condition above. It corresponds to payoffs

$$0(\lambda) + 4(1-\lambda) = 2, \quad 0(\lambda) + 1(1-\lambda) = \frac{1}{2} > -1 \quad \text{and} \quad 0(\lambda) - 2(1-\lambda) = -1.$$

Therefore, strategy T is weakly dominated by a convex combination of E and N for both players.

(c) Since strategy T is weakly dominated, we cannot simply eliminate it from the game matrix.

Pure strategy Nash equilibriums
Consider the best response of Player I:

- When Player II chooses E, Player I will choose N and get the maximized payoff 4.
- When Player II chooses T, Player I chooses N to achieve the maximized payoff 1.
- When Player II chooses N, Player I chooses E and obtains zero payoff to avoid losing more.

Since the two players are identical, the best response for Player II is similar to Player I. From the above discussion, we conclude that the pure strategy Nash equilibria should be (E, N) and (N, E).

Mixed strategy Nash equilibriums
Let the optimal strategies of the row and column player be $X^* = (x_1, x_2, x_3)$ and $Y^* = (y_1, y_2, y_3)$, respectively.

(i) Suppose $x_1, x_2, x_3 > 0$, according to indifferent principle:

$$
\begin{aligned}
x_1 > 0 &\implies E_I(X^*, Y^*) = E(1, Y^*) = 0, \\
x_2 > 0 &\implies E_I(2, Y^*) = 2y_1 - y_2 - y_3 = 0, \\
x_3 > 0 &\implies E_I(3, Y^*) = 4y_1 + y_2 - 2y_3 = 0,
\end{aligned}
$$

which can be solved to be $y_1 = \frac{1}{3}$, $y_2 = 0$, $y_3 = \frac{2}{3}$. Correspondingly, we solve X^* by

$$
\begin{aligned}
y_1 > 0 &\implies E_{II}(X^*, Y^*) = E_{II}(X^*, 1) = 0, \\
y_2 = 0 \quad &\& \quad E_{II}(X^*, 2) = 2x_1 - x_2 - x_3 \leq 0, \\
y_3 > 0 &\implies E_{II}(X^*, 3) = 4x_1 + x_2 - 2x_3 = 0,
\end{aligned}
$$

which can be solved to be $x_1 = \alpha$, $x_2 = -2\alpha + \frac{2}{3}$, $x_3 = \alpha + \frac{1}{3}$, where $\alpha \in (0, \frac{1}{3})$.

(ii) Suppose $x_1 = 0, x_2, x_3 > 0$, then Column 2 is strictly dominated by Column 1 and is eliminated. We have the reduced matrix

$$\begin{matrix} (2,0) & (-1,1) \\ (4,0) & (-2,-2) \end{matrix}$$

When Player I is indifferent between the two rows, we have

$$2y_1 - y_3 = 4y_1 - 2y_3, \quad \text{and} \quad y_1 + y_3 = 1,$$

which yields $y_1 = \frac{1}{3}$, $y_2 = 0$, $y_3 = \frac{2}{3}$. Similarly, we find X^* by

$$0 = x_2 - 2x_3, \quad \text{and} \quad x_2 + x_3 = 1,$$

which yields $x_1 = 0$, $x_2 = \frac{2}{3}$, $x_3 = \frac{1}{3}$.

(iii) Suppose $x_2 = 0, x_1, x_3 > 0$, we have the reduced matrix

$$\begin{matrix} (0,0) & (0,2) & (0,4) \\ (4,0) & (1,-1) & (-2,-2) \end{matrix}$$

When Player I is indifferent between the two rows, we have

$$4y_1 + y_2 - 2y_3 = 0 \quad \text{and} \quad y_1 + y_2 + y_3 = 1,$$

which yields $y_1 = \beta$, $y_2 = -2\beta + \frac{2}{3}$, $y_3 = \beta + \frac{1}{3}$, where $\beta \in [0, \frac{1}{3}]$. When $\beta = 0$, $y_2, y_3 > 0$ and we have

$$2x_1 - x_3 = 4x_1 - 2x_3 \quad \text{and} \quad x_1 + x_3 = 1,$$

which yields $x_1 = \frac{1}{3}$, $x_2 = 0$, $x_3 = \frac{2}{3}$. When $\beta > 0$, $y_1, y_3 > 0$ and we have

$$4x_1 - 2x_3 = 0 \quad \text{and} \quad x_1 + x_3 = 1,$$

which also yields $x_1 = \frac{1}{3}$, $x_2 = 0$, $x_3 = \frac{2}{3}$.

(iv) Suppose $x_3 = 0, x_1, x_2 > 0$, then Column 2 is strictly dominated by Column 3 and is eliminated. We have the reduced matrix

$$\begin{matrix} (0,0) & (0,4) \\ (2,0) & (-1,1) \end{matrix}$$

By similar argument with (ii), we find that $y_1 = \frac{1}{3}$, $y_2 = 0$, $y_3 = \frac{2}{3}$. To find X^*, we write

$$0 = 4x_1 + x_2 \quad \text{and} \quad x_1 + x_2 = 1,$$

which could not happen when $x_1, x_2 \in [0, 1]$.

In conclusion, the mixed strategy Nash equilibriums are found to be $X^* = (\alpha, -2\alpha + \frac{2}{3}, \alpha + \frac{1}{3})$, $Y^* = (\frac{1}{3}, 0, \frac{2}{3})$ or $X^* = (\frac{1}{3}, 0, \frac{2}{3})$, $Y^* = (\beta, -2\beta + \frac{2}{3}, \beta + \frac{1}{3})$, where $\alpha, \beta \in [0, \frac{1}{3}]$.

10. There are two pure Nash equilibriums at (Stop, Go) and (Go, Stop). To find the mixed Nash equilibrium, we assume that the two players play pure mixed strategies. Let $X = (x, 1-x)$ and $Y = (y, 1-y)$, where $0 < x < 1$ and $0 < y < 1$. By applying the Equality Payoff theorem, we have

$$x + (1-\epsilon)(1-x) = 2x \text{ giving } x = \frac{1-\epsilon}{2-\epsilon};$$
$$y + (1-\epsilon)(1-y) = 2y \text{ giving } y = \frac{1-\epsilon}{2-\epsilon}.$$

The payoff to player I is $\frac{2(1-\epsilon)}{2-\epsilon}$ and the same for player II at the mixed Nash equilibrium. The probability that both players choose Go is $\left(\frac{1}{2-\epsilon}\right)^2 > \frac{1}{4}$ for $0 < \epsilon < 1$. Note that

$$\frac{\mathrm{d}}{\mathrm{d}\epsilon}\left(\frac{1}{2-\epsilon}\right)^2 = 2\left(\frac{1}{2-\epsilon}\right)^3 > 0 \text{ for } 0 < \epsilon < 1.$$

Therefore, the probability of both choosing Go is an increasing function of ϵ.

11. (a) If both apply to the same firm, then the expected payoff is only half of the pay since there is only $50-50$ chance of getting the job. The bimatrix game is characterized by

I / II	apply firm 1	apply firm 2
apply firm 1	$\frac{p_1}{2},\ \frac{p_1}{2}$	$p_1,\ p_2$
apply firm 2	$p_2,\ p_1$	$\frac{p_2}{2},\ \frac{p_2}{2}$

Obviously, the pure Nash equilibriums are that the applicants apply to different firms.

(b) To search for the mixed Nash equilibriums, we apply the equality-payoff method, where

$$\frac{p_1 y_1}{2} + p_1 y_2 = p_2 y_1 + \frac{p_2 y_2}{2}$$

subject to $y_1 + y_2 = 1$. This gives

$$y_1 = \frac{2p_1 - p_2}{p_1 + p_2} \text{ and } y_2 = \frac{2p_2 - p_1}{p_1 + p_2},$$

so that $Y^* = (\frac{2p_1-p_2}{p_1+p_2}, \frac{2p_2-p_1}{p_1+p_2})$. Similarly, by symmetry, we obtain $X^* = (\frac{2p_1-p_2}{p_1+p_2}, \frac{2p_2-p_1}{p_1+p_2})$.

Under the mixed Nash equilibrium, each applicant has a higher probability to apply to firm 2 as it promises a higher pay. The expected payoff to each applicant is $\frac{3}{2}\frac{p_1 p_2}{p_1+p_2}$.

12. (a) Expected payoff of "Aggressive" and "Passive" played by Animal 1 are

$$0 \cdot q + 6 \cdot (1-q) = 6 - 6q \text{ and } 1 \cdot q + 3 \cdot (1-q) = 3 - 2q,$$

respectively. "Aggressive" has a higher expected payoff than "Passive" if and only if $q < \frac{3}{4}$. Hence, if $q < \frac{3}{4}$, Animal 1 should play "Aggressive" for sure, so $p = 1$. Otherwise, if $q > \frac{3}{4}$, Animal 1 should play "Passive" for sure, so $p = 0$. When $q = \frac{3}{4}$, Animal 1 is indifferent to any choice of p, where $p \in [0,1]$. Therefore, the best response function of Animal 1 is

$$B_1(q) = \begin{cases} p = 1 & \text{if } q < \frac{3}{4}, \\ p \in [0,1] & \text{if } q = \frac{3}{4}, \\ p = 0 & \text{if } q > \frac{3}{4}. \end{cases}$$

In a similar manner, expected payoff of "Aggressive" and "Passive" played by Animal 2 are

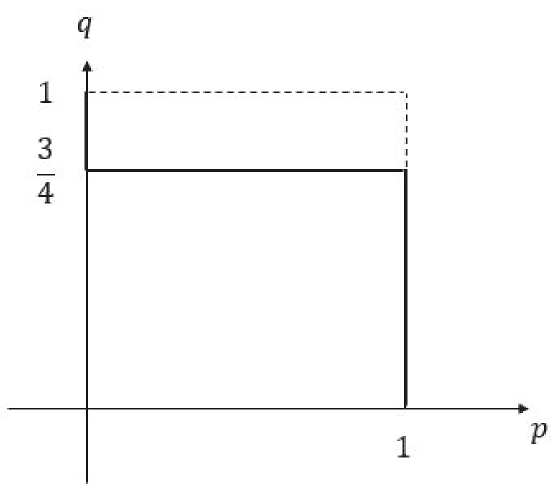
$$0 \cdot p + 6 \cdot (1-p) = 6 - 6p \text{ and } 1 \cdot p + 3 \cdot (1-p) = 3 - 2p,$$

respectively. The two expected payoffs are the same when $p = \frac{3}{4}$. Also, $6 - 6p > 3 - 2p$ when $p < \frac{3}{4}$. The best response function of Animal 2 is

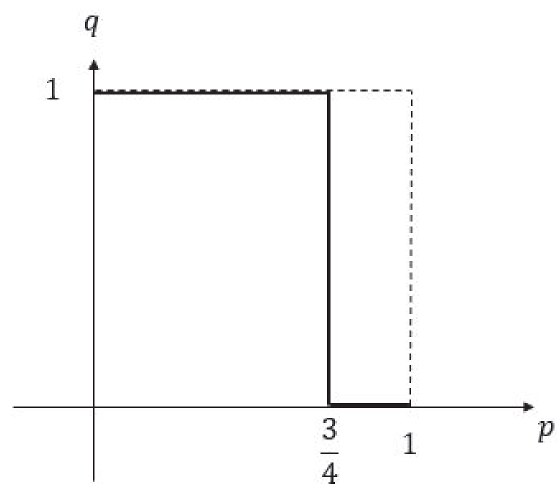
$$B_2(p) = \begin{cases} q = 1 & \text{if } p < \frac{3}{4}, \\ q \in [0,1] & \text{if } p = \frac{3}{4}, \\ q = 0 & \text{if } p > \frac{3}{4}. \end{cases}$$

Due to the symmetry in the payoff in the game matrix, the best response function $B_2(p)$ can be deduced from $B_1(q)$ by swapping the role of p and q.

(b) The two best response functions are plotted below:



$B_1(q)$: Best response function of Animal 1



$B_2(p)$: Best response function of Animal 2

The two best response functions intersect at 3 points in the p-q plane:

(i) $(0,1)$ that corresponds to the pure strategy Nash equilibrium (Passive, Aggressive);

(ii) $(1,0)$ that corresponds to the pure strategy Nash equilibrium (Aggressive, Passive);

(iii) $(\frac{3}{4}, \frac{3}{4})$ that corresponds to the mixed strategy Nash equilibrium with probability vectors: $\{(\frac{3}{4}, \frac{1}{4}), \frac{1}{4}, \frac{3}{4})\}$ for the mixed strategies played by the two animals.

13. For the symmetric game (A, B), the mixed strategy is given by

$$\begin{aligned} x^* &= \frac{b_{22} - b_{21}}{b_{11} - b_{12} - b_{21} + b_{22}} = \frac{a_{22} - a_{12}}{a_{11} - a_{21} - a_{12} + a_{22}}, \\ y^* &= \frac{a_{22} - a_{12}}{a_{11} - a_{12} - a_{21} + a_{22}}. \end{aligned}$$

For the new symmetric game (A', B'), where

$$(A', B') = \left(\begin{pmatrix} a_{11} - a & a_{12} - b \\ a_{21} - a & a_{22} - b \end{pmatrix}, \begin{pmatrix} a_{11} - a & a_{21} - b \\ a_{12} - a & a_{22} - b \end{pmatrix} \right),$$

the mixed strategy is given by

$$\begin{aligned} x'^* &= \frac{b'_{22} - b'_{21}}{b'_{11} - b'_{12} - b'_{21} + b'_{22}} = \frac{(a_{22} - b) - (a_{12} - b)}{(a_{11} - a) - (a_{21} - a) - (a_{12} - b) + (a_{22} - b)} = x^*, \\ y'^* &= \frac{a'_{22} - a'_{12}}{a'_{11} - a'_{12} - a'_{21} + a'_{22}} = \frac{(a_{22} - b) - (a_{12} - b)}{(a_{11} - a) - (a_{12} - a) - (a_{21} - b) + (a_{22} - b)} = y^*. \end{aligned}$$

Note that both symmetric games share the same best response functions. Therefore, they have the same set of pure and mixed Nash equilibriums.

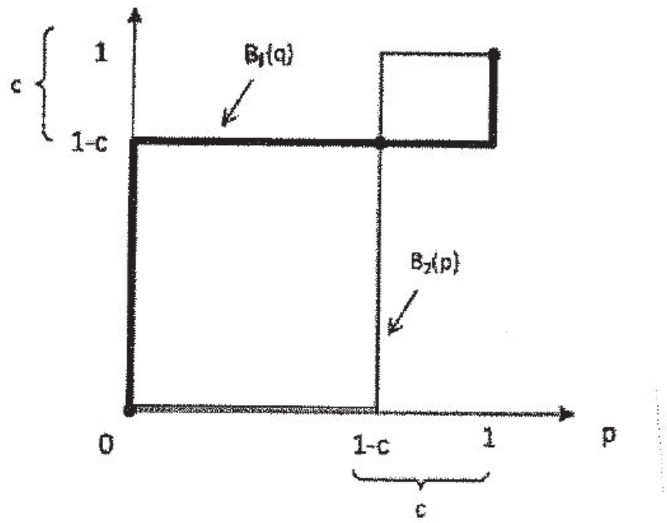
14. Let p be the probability that Player I chooses "no effort" in a mixed Nash equilibrium and q be the probability that Player II chooses "no effort". The expected payoffs are found to be

$$\begin{aligned} &\pi_1(\text{no effort}) = 0 \\ &\pi_1(\text{effort}) = q(-c) + (1-q)(1-c) = 1 - c - q \\ &\pi_2(\text{no effort}) = 0 \\ &\pi_2(\text{effort}) = p(-c) + (1-p)(1-c) = 1 - c - p. \end{aligned}$$

The best response functions are found to be

$$\begin{aligned} B_1(q) &= \begin{cases} p = 0 & \text{if } q < 1 - c \\ p \in [0,1] & \text{if } q = 1 - c \\ p = 1 & \text{if } q > 1 - c \end{cases}; \\ B_2(p) &= \begin{cases} q = 0 & \text{if } q < 1 - c \\ q \in [0,1] & \text{if } q = 1 - c \\ q = 1 & \text{if } q > 1 - c \end{cases}. \end{aligned}$$

The plots of $B_1(q)$ and $B_2(p)$ are shown below.



There is only one mixed Nash equilibrium: $(p, q) = (1-c, 1-c)$. As c increases, the equilibrium probabilities of "no effort" for both players decrease. There are two pure Nash equilibriums: $(p, q) = (0, 0)$ and $(p, q) = (1, 1)$.

15. There are two strategies for the expert.

   – honest: recommends a minor repair for a minor problem and a major repair for a major problem as recognized by himself (be aware of incompetence);
   – dishonest: recommends a major repair for any type of problem.

   Also, there are two strategies for the customer.

   – accept: buys whatever repair the expert recommends;
   – reject: buys a minor repair but seek some other remedy if a major repair is recommended.

   Assume that the players' preferences are represented by their expected monetary payoff. The players' payoffs are listed below.

(H,A): With probability r, the consumer's problem is major, so he pays E. With probability $1 - r$, it is minor. In this case, with probability s the expert correctly diagnoses it as minor. The consumer accepts his advice and pays I. With probability $1 - s$, the expert diagnoses it as major so he pays E. Thus his expected payoff is $-rE-(1-r)[sI+(1-s)E]$. The expert's profit is $r\pi + (1 - r)[s\pi + (1 - s)\pi']$. The gain to the incompetent expert when the customer accepts $= r\pi + (1-r)[\pi + (1-s)(\pi' - \pi)] - \pi = (1-r)(1-s)(\pi' - \pi)$.

(D,A): The customer's payoff is always E since he is always presented the problem as major. The true probability of minor is always $1-r$. Under which the expert receives π' as payoff (disregard the incompetence of the expert). Therefore, the expert's expected payoff is $r\pi + (1 - r)\pi'$.

(H,R): The expert earns the repair business only if the consumer's problem is minor and he diagnoses correctly. In this case, the expert's expected payoff is $(1 - r)s\pi$. The loss to this incompetent expert due to incorrect diagnosis $= (1-r)\pi-(1-r)s\pi = (1-r)(1-s)\pi$.

Similar explanation as before, the expected payoff of the customer is $-rE' - (1-r)[sI + (1-s)I']$.

(D,R): Same payoffs to both players as those without the incompetence issue. The payoff to the expert is always zero since he never earns the repair business. The expected customer's payoff is $-rE' - (1-r)I'$.

	Accept (q)	Reject $(1-q)$
Honest (p)	$r\pi + (1-r)[s\pi + (1-s)\pi']$, $-rE - (1-r)[sI + (1-s)E]$	$(1-r)s\pi, -rE' - (1-r)[sI + (1-s)I']$
Dishonest $(1-p)$	$r\pi + (1-r)\pi', -E$	$0, -rE' - (1-r)I'$

Expert's best response function

The expert is indifferent to "honest" or "dishonest" for a given q if and only if

$$q\{r\pi + (1-r)[s\pi + (1-s)\pi']\} + (1-q)(1-r)s\pi = q[r\pi + (1-r)\pi']$$

giving $q^* = \frac{\pi}{\pi'}$, which is the same result as that under $s = 1$ (full competence). Note that when $q^* = \frac{\pi}{\pi'}$, we observe $q^*(1-r)(1-s)(\pi' - \pi) = (1-q^*)(1-r)(1-s)\pi$. When the expert is honest, the gain to the incompetent expert when the customer accepts (gaining advantage by being incompetent) is the same as the loss to this incompetent expert when the customer rejects (incorrect diagnosis as a major problem leads to loss of business). This explains why q^* is independent of s.

When $q > q^*$, $p = 0$; that is, the expert should always be dishonest since the customer chooses "accept" with high probability. On the other hand, when $q < q^*$, the expert should always be honest with $p = 1$.

Customer's best response function

The customer is indifferent to "accept" or "reject" for a given p if and only if

$$\begin{aligned} & p\Big\{-rE - (1-r)[sI + (1-s)E]\Big\} + (1-p)(-E) \\ = \; & p\Big\{-rE - (1-r)[sI + (1-s)I']\Big\} + (1-p)[-rE' - (1-r)I'] \end{aligned}$$

giving $p^* = \frac{E - [rE' + (1-r)I]}{(1-r)s(E-I')}$.

Note that p^* becomes non-positive when $E \leq rE' + (1-r)I'$. In this case, we take $p^* = 0$ (see the argument in the lecture note).

We consider the more interesting case where $p^* > 0$; that is $E > rE' + (1-r)I'$. When $p > p^*$ (high probability of expert being honest), then the customer optimally chooses $q = 1$ (always accepts the advice). When $p < p^*$, then the customer's best response is $q = 0$.

Similar to the lecture note, the mixed Nash equilibrium is given by the intersection of the two best response functions, giving

$$(p^*, q^*) = \left(\frac{E - [rE' + (1-r)I']}{(1-r)s(E-I')}, \frac{\pi}{\pi'}\right).$$

When s becomes smaller, the expert's optimal mixed strategies is to choose a higher probability of being "honest".

16. The payoffs of the auditing game are given by

		Suspects	
		Cheat (θ)	Obey ($1-\theta$)
IRS	Audit (γ)	$4-C, -F$	$4-C, -1$
	Trust ($1-\gamma$)	$0, 0$	$4, -1$

$$\pi_{IRS}(\text{audit}) = \theta(4-C) + (1-\theta)(4-C) = 4-C$$
$$\pi_{IRS}(\text{trust}) = \theta \cdot 0 + (1-\theta) \cdot 4 = 4-4\theta$$

The Equality of Payoff Theorem gives $\pi_{IRS}(\text{audit}) = \pi_{IRS}(\text{trust})$, we obtain

$$4-C = 4-4\theta$$

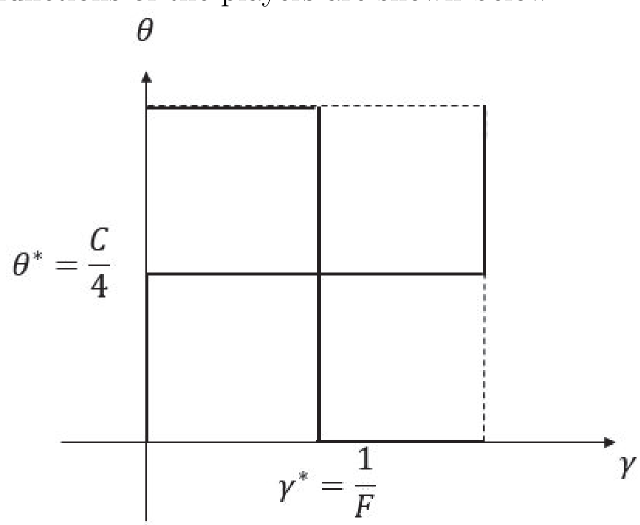
giving $\theta^* = \frac{C}{4}$. The corresponding payoff of IRS under the mixed Nash equilibrium is $\pi^*_{IRS} = 4-C$.

In a similar manner, we equate

$$-\gamma F = \pi^*_{suspect}(\text{cheat}) = \pi^*_{suspect}(\text{obey}) = -1.$$

We obtain $\gamma^* = \frac{1}{F}$ and $\pi^*_{suspect} = -1$.

The best response functions of the players are shown below



Since there is only one intersection point of the two best response functions, so there is only one mixed Nash equilibrium $(\gamma^*, \theta^*) = (\frac{1}{F}, \frac{C}{4})$.

Interestingly, the expected payoffs, π^*_{IRS} and $\pi^*_{suspect}$, are the same as those of (audit, obey).

17. (a) Victor's stocks for rain (strategy A): use \$2,500 to buy 500 umbrellas

Victor's stocks for sunny (strategy B): use \$2,000 to buy 1,000 sunglasses and \$500 to buy 100 umbrellas.

If it rains, then strategy A earns \$2,500 while strategy B loses \$2,000 − \$500 = \$1,500.

If it is sunny, then strategy A loses \$2,500 − \$1,000 = \$1,500 while strategy B earns \$3,000 + \$500 = \$3,500.

The payoff matrix is seen to be

Victor \ weather	rain	sunny
A	2,500	-1,500
B	-1,500	3,500

Let $(p, 1-p)$ be the mixed strategy of Victor. We find p by the payoff-equating method. Consider

$$\pi_{rain} = p(2{,}500) + (1-p)(-1{,}500)$$
$$\pi_{sunny} = p(-1{,}500) + (1-p)(3{,}500);$$

by equating $\pi_{rain} = \pi_{sunny}$, we obtain $p = \frac{5}{9}$.

(b) Consider

$$\begin{aligned} E(A,\ Y^0) &= (1\ 0)\begin{pmatrix} 2{,}500 & -1{,}500 \\ -1{,}500 & 3{,}500 \end{pmatrix}\begin{pmatrix} 0.3 \\ 0.7 \end{pmatrix} \\ &= 2{,}500 \times 0.3 - 1{,}500 \times 0.7 = -300 \\ E(B,\ Y^0) &= (0\ 1)\begin{pmatrix} 2{,}500 & -1{,}500 \\ -1{,}500 & 3{,}500 \end{pmatrix}\begin{pmatrix} 0.3 \\ 0.7 \end{pmatrix} \\ &= -1{,}500 \times 0.3 + 3{,}500 \times 0.7 = 2000; \end{aligned}$$

Victor should choose strategy B.

18. (a) Each merchant has 5 pure strategies to choose a location among 1, 2, 3, 4 and 5. The game matrix is construct as follows:

I \ II	1	2	3	4	5
1	(2.5, 2.5)	(1, 4)	(1.5, 3.5)	(2, 3)	(2.5, 2.5)
2	(4, 1)	(2.5, 2.5)	(2, 3)	(2.5, 2.5)	(3, 2)
3	(3.5, 1.5)	(3, 2)	(2.5, 2.5)	(3, 2)	(3.5, 1.5)
4	(3, 2)	(2.5, 2.5)	(2, 3)	(2.5, 2.5)	(4, 1)
5	(2.5, 2.5)	(2, 3)	(1.5, 3.5)	(1, 4)	(2.5, 2.5)

We find the Nash equilibrium using iterated elimination of strictly dominated strategies (IESDS). We observe that Column 5 is strictly dominated by Column 4 and Column 1 is strictly dominated by Column 2. Then Column 5 and Column 1 are eliminated. For the remaining bimatrix, Row 3 strictly dominates all the other rows, so we eliminate Row 1, 2, 4 and 5. For the remaining Row 3, Column 2 and Column 4 are strictly dominated by Column 3 and are eliminated. Finally, there is only one entry left, which is (3, 3). Then we find the Nash equilibrium where both merchants choose to take the location 3. It is also the unique Nash equilibrium since it is the unique survivor of the iterated elimination of strictly dominated strategies.

(b) – When n is odd, we construct the game matrix as follows:

I \ II	1	2	$\cdots$	$k+1$	$\cdots$	$2k+1$
1	$(k+\frac{1}{2}, k+\frac{1}{2})$	$(1, 2k)$	$\cdots$	$(\frac{k+1}{2}, \frac{3k+1}{2})$	$\cdots$	$(k+\frac{1}{2}, k+\frac{1}{2})$
2	$(2k, 1)$	$(k+\frac{1}{2}, k+\frac{1}{2})$	$\cdots$	$(\frac{k+2}{2}, \frac{3k}{2})$	$\cdots$	$(k+1, k)$
$\cdots$						
$k+1$	$(\frac{3k+1}{2}, \frac{k+1}{2})$	$(\frac{3k}{2}, \frac{k+2}{2})$	$\cdots$	$(k+\frac{1}{2}, k+\frac{1}{2})$	$\cdots$	$(\frac{3k+1}{2}, \frac{k+1}{2})$
$\cdots$						
$2k+1$	$(k+\frac{1}{2}, k+\frac{1}{2})$	$(k, k+1)$	$\cdots$	$(\frac{k+1}{2}, \frac{3k+1}{2})$	$\cdots$	$(k+\frac{1}{2}, k+\frac{1}{2})$

The payoffs of Player I for outcome (Row i, Column j) can be represented by

$$\pi_I(i,j) = \begin{cases} \dfrac{i+j-1}{2}, & \text{if } i < j \\ k + \dfrac{1}{2}, & \text{if } i = j \\ 2k+1 - \dfrac{i+j-1}{2}, & \text{if } i > j \end{cases}$$

and the payoff of Player II is correspondingly $\pi_{II}(i,j) = 2k+1-\pi_I(i,j)$. We observe that the first and last columns are strictly dominated by Column $k+1$. By symmetry, it suffices to show that Column $k+1$ strictly dominates Column 1. This can be got by the following discussion:

1. When $i = 1$, we have $\frac{3k+1}{2} > k + \frac{1}{2}$.
2. When $i = 2, \cdots, k$, we have $2k+1-\frac{i+k}{2} \geq i+1 > \frac{i}{2}$.
3. When $i = k+1$, we have $k + \frac{1}{2} > \frac{k+1}{2}$.
4. When $i = k+2, \cdots, 2k+1$, we have $\frac{i+k}{2} > \frac{i}{2}$.

Then Column 1 and Column $2k+1$ are strictly dominated by Column $k+1$ and can be eliminated. Then location 1 and $2k+1$ will not be chosen. Since the players are identical, we also eliminate Row 1 and Row $2k+1$ for Player I.

The remaining matrix is $(2k-1)\times(2k-1)$. Since the discussion above is valid for any $k > 1$, we take $k' = k-1$ and have the new matrix to be $(2k'+1)\times(2k'+1)$. We can then use the similar method and find Row 2 and Row $2k$, Column 2 and Column $2k$ are strictly dominated by Row $k+1$ and Column $k+1$, respectively. We keep eliminating the strictly dominated rows and columns and finally get only one entry left, which is $(k+1,\ k+1)$. It uniquely survives the iterated elimination of strictly dominated strategies, so the unique Nash equilibrium for this case is (Row $k+1$, Column $k+1$). That is, both players choose the location $k+1$ at the middle.

– When n is even (i.e. $n = 2k$), we have the following bimatrix:

I \ II	1	$\cdots$	k	$k+1$	$\cdots$	$2k$
1	(k, k)	$\cdots$	$(\frac{k}{2}, \frac{3k}{2})$	$(\frac{k+1}{2}, \frac{3k-1}{2})$	$\cdots$	(k, k)
$\cdots$						
k	$(\frac{3k}{2}, \frac{k}{2})$	$\cdots$	(k, k)	(k, k)	$\cdots$	$(\frac{3k-1}{2}, \frac{k+1}{2})$
$k+1$	$(\frac{3k-1}{2}, \frac{k+1}{2})$	$\cdots$	(k, k)	(k, k)	$\cdots$	$(\frac{3k}{2}, \frac{k}{2})$
$\cdots$						
$2k$	(k, k)	$\cdots$	$(\frac{k+1}{2}, \frac{3k-1}{2})$	$(\frac{k}{2}, \frac{3k}{2})$	$\cdots$	(k, k)

We argue that (Row k, Column k) is a Nash equilibrium with payoff $\pi_I(k,k) = \pi_{II}(k,k) = k$. For Player I:

1. When $i = 1, \cdots, k-1$, $\pi_I(i,k) = \frac{i+k-1}{2} \leq k-1 < k$.
2. When $i = k+1$, $\pi_I(k+1,k) = k$.
3. When $i = k+2, \cdots, 2k$, $\pi_I(i,k) = 2k - \frac{i+k-1}{2} \leq k - \frac{1}{2} < k$.

Then Player I has no incentive to deviate from this equilibrium (i.e. $\pi_I(i,k) \leq \pi_I(k,k)$). Since the two players are identical, we have the same conclusion for Player II. This proves that (Row k, Column k) is a Nash equilibrium.