MATH 4321 – Game Theory

Homework Four

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1. Two simple games are *equivalent* (or *isomorphic*) if the players can be labeled in such a way that the winning coalitions are the same in both games. Show that the following three-person games are all equivalent.

   (i) $N = \{A, B, C\}$. Approval is by majority vote, but A has a *veto*.

   (ii) $[3; 2, 1, 1]$.

   (iii) $[5; 3, 2, 2]$.

   (iv) $[17; 16, 1, 1]$.

2. Suppose the U.N. Security Council has eight nonpermanent members and three permanent members with passage of a bill requiring a total of seven votes subject to the veto power of each of the permanent members. How to express the yes-no system into a weighted voting system?

3. Which of the following properties about "winning coalition" are considered to be reasonable? Give your justification.

   (a) If X is a winning coalition and every voter in X is also in Y, then Y is also a winning coalition.

   (b) If X and Y are winning coalitions, then so is the coalition consisting of voters in both X and in Y.

   (c) If X and Y are disjoint (that is, have no voters in common), then at least one fails to be a winning coalition.

   (d) If X and Y are winning coalitions, then so is the coalition consisting of voters of $X \cup Y$.

   (e) If X is a winning coalition and X is split into two sets Y and Z so that every voter in X is in exactly one of Y and Z, then either Y is a winning coalition or Z is a winning coalition.

4. Calculate the Shapley-Shubik and Banzhaf indexes for the following weighted voting games:

   (a) $[5; 4, 2, 1, 1, 1]$;

   (b) $[9; 5, 4, 3, 2, 1]$.

5. A seven-person legislature has a three-person committee. Approval must be achieved by a majority of *both* the committee and the entire legislature. Denote the members by $AAAbbbb$. Compute the Shapley-Shubik and Banzhaf power indices. What is the ratio of power between a committee member and a noncommittee member?

6. The passage of a bill requires majority of votes. Calculate the Shapley-Shubik and Banzhaf indices for the large stockholder L with 40% of the shares if the remaining shares are split evenly among

(a) five other stockholders;

(b) seven other stockholders.

Give the intuition why the large stockholder is less powerful in (b) than he is in (a)?

7. Consider a voting system consisting of 3 big states and 6 small states, passage of a bill requires "yes" vote from all the big states and at least 2 "yes" votes from the 6 small states.

   (a) Find the weighted voting vector of the above game, specifying the quota and the number of votes held by each of the big states and small states.

   (b) Consider one of the big states, assuming the homogeneity assumption on voting probabilities among all 9 states, find the probability $\pi_b(p)$ that the vote of this particular big state makes a difference between approval or rejection of a bill. Here, p denotes the common homogeneous voting probability.

   (c) Using the above $\pi_b(p)$, or otherwise, compute the Shapley-Shubik index and Banzhaf index for any one of the big states. Then deduce the values for the above two power indexes for any one of the small states.

   (d) Suppose the 3 big states vote independently while the set of 6 smaller states vote as a homogeneous group. Show how to compute the Shapley-Shubik indexes.

8. Consider $\underset{}{[5; \underset{A}{3}, \underset{B}{2}, \underset{C}{1}, \underset{D}{1}]}$.

   Show that

$$
\begin{aligned}
\pi_A(p) &= p + (1-p)p^2 &&= p + p^2 - p^3 \\
\pi_B(p) &= p(1-p^2) &&= p - p^3 \\
\pi_C(p) &= p(1-p)p &&= p^2 - p^3.
\end{aligned}
$$

   Hence, check that the Banzhaf index and Shapley-Shubik index are given by

$$
\begin{aligned}
\beta &= \left(\frac{5}{8}, \frac{3}{8}, \frac{1}{8}, \frac{1}{8}\right), \\
\phi &= \left(\frac{7}{12}, \frac{3}{12}, \frac{1}{12}, \frac{1}{12}\right),
\end{aligned}
$$

   respectively.

9. A British game show involves a final prize. The two contestants may each choose either to Split the prize, or Claim the prize. If they Split the prize, they each get $\frac{1}{2}$; if they each Claim the prize, then each gets 0. If one player Splits, and the other player Claims, the player who Claims the prize gets 1 and the other player gets 0. The game matrix is

$$
\begin{bmatrix} (\frac{1}{2}, \frac{1}{2}) & (0, 1) \\ (1, 0) & (0, 0) \end{bmatrix}.
$$

   (a) Find the Nash bargaining solution.

   (b) If the game will only be played one time and one player announces that he will definitely Claim the prize and then split the winnings after the show is over, what must the other player do?

10. A classic bargaining problem involves a union and management in contract negotiations. If management hires $w \geq 0$ workers, the company produces $f(w)$ revenue units, where f is a continuous, increasing function. The maximum number of workers who are represented by the union is W. A person who is not employed by the company gets a payoff $p_0 \geq 0$, which is either unemployment benefits or the pay at another job. In negotiations with the union, the firm agrees to the pay level p and to employ $0 \leq w \leq W$ worker. We may consider the payoff function as

$$u(p, w) = f(w) - pw \text{ to the company}$$

and

$$v(p, w) = pw + (W - w)p_0 \text{ to the union.}$$

Assume the safety security point is $u^* = 0$ for the company and $v^* = Wp_0$ for the union.

(a) What is the nonlinear program to find the Nash bargaining solution?

(b) Assuming an interior solution (which means you can find the solution by taking derivatives), show that the solution (p^*, w^*) of the Nash bargaining solution satisfies

$$f'(w^*) = p_0 \textit{ and } p^* = \frac{w^* p_0 + f(w^*)}{2w^*}.$$

(c) Find the Nash bargaining solution for $f(w) = \ln(w + 1) + b$, where $a > 0$, $\frac{1}{a} > p_0$, $b > -\ln a$.