MATH 4321 – Game Theory

Solution to Homework Four

Course instructor: Prof. Y.K. Kwok

---

1. To show equivalence of the game vectors, it suffices to consider the set of winning coalitions for each of the voting systems and check whether they are the same. It is seen that all of the 3-person games lead to the same set of winning coalitions, namely, $W = \{AB, AC, ABC\}$. Hence, all these 3-person games are equivalent.

2. Express the yes-no system as a weighted voting system.
   We let the voting weight of the j^{th} nonpermanent member N_j and the i^{th} permanent member P_i be 1 and w, respectively. Also, we let q denote the quota. According to the rule of passage of a bill, where each of P_i has veto power, we deduce the following pair of inequalities:
   $$3w + 4 \geq q \quad \text{and} \quad 2w + 8 < q.$$
   Combining the inequalities, we obtain
   $$3w + 4 > 2w + 8 \quad \text{giving} \quad w > 4.$$
   Suppose we set $w = 5$, then q satisfies
   $$19 \geq q > 18,$$
   giving $q = 19$. The corresponding weighted voting system is given by
   $$[19; 5, 5, 5, 1, 1, 1, 1, 1, 1, 1, 1].$$

3. (a) **Reasonable.** Adding more voters in a winning coalition X to form Y gives the enlarged coalition Y that remains winning.

   (b) **Non-reasonable.** Quote a counter-example. Consider the 3-person voting game in which approval is by majority vote. Take $X = \{A, B\}$ and $Y = \{B, C\}$, both are winning. However, $X \cap Y = \{B\}$ is losing.

   (c) **Reasonable.** If X and Y are disjoint, then Y is a subset of the complement of X. Suppose both X and Y are winning, the complement of X is also winning [by virtue of (a)]. However, it is not reasonable to have both X and its complement to be both winning.

   (d) **Reasonable.** Since $X \cup Y \supseteq X$ and X is winning, by virtue of (a), $X \cup Y$ is winning.

   (e) **Non-reasonable.** Quote a counter-example. Consider the 3-person voting game in which approval is by majority vote. Take $X = \{A, B\}$, $Y = \{A\}$ and $Z = \{B\}$. Obviously, both Y and Z are not winning coalitions.

4. (a) Consider the weighted voting system $[5; 4, 2, 1, 1, 1]$:

(i) Shapley-Shubik indexes
The 4-vote player is pivotal if there are one to three other players entering into a coalition before he enters. The number of such orderings is

$$O_4 = \sum_{n=1}^{3} c_n^4 n!(4-n)! = 72.$$

The 2-vote player is pivotal if the 4-vote player joins earlier or all three 1-vote player join earlier in a coalition. The number of such orderings is

$$O_2 = 1!3! + 3!1! = 12.$$

The 1-vote player is pivotal if either (1) the 4-vote player joins earlier, or (2) the 2-vote player and two 1-vote players join earlier in a coalition. The number of such orderings is

$$O_1 = 3(2!) + 3(2!) = 12.$$

The individual Shapley-Shubik indexes are

$$\Phi_4 = \frac{72}{5!} = \frac{3}{5}, \Phi_2 = \frac{12}{120} = \frac{1}{10} \text{ and } \Phi_1 = \frac{12}{120} = \frac{1}{10}.$$

Surprisingly, the 2-vote player and the three 1-vote players are equally powerful under the Shapley-Shubik power index.

(ii) Banzhaf indexes
The 4-vote player is marginal in the winning coalition if the winning coalition contains one to three other players. The number of such coalitions is

$$B_4 = \sum_{n=1}^{3} c_n^4 = 14.$$

The 2-vote player is marginal in the winning coalition if the winning coalition contains either (1) the 4-vote player only, or (2) all the three 1-vote players. The number of such coalitions is $B_2 = 2$. Lastly, any one of the 1-vote players is marginal if the winning coalition contains either (1) the 4-vote player or (2) the 2-vote player and both of the other two 1-vote players. The number of such coalitions is $B_1 = 1 + 1 = 2$. The individual Banzhaf indexes are given by

$$\beta_4 = \frac{14}{14 + 2 + 2 \times 3} = \frac{7}{11}, \beta_2 = \frac{1}{11} \text{ and } \beta_1 = \frac{1}{11}.$$

Surprisingly, the 2-vote player is equally powerful as any one of the three 1-vote players under the Banzhaf power index.

(b) Consider the weighted voting system $[9; 5, 4, 3, 2, 1]$:

(i) Shapley-Shubik indexes
The 5-vote player is pivotal if players "4" or "4, 1" or "4, 2" or "4, 3" or "4, 1, 2" or "4, 1, 3" or "3, 1" or "3, 2" or "3, 1, 2" entering into a coalition before he enters. The number of such orderings is

$$O_5 = 1!3! + 5 \times 2!2! + 3 \times 3!1! = 44.$$

The 4-vote player is pivotal if players "5" or "5, 1" or "5, 2" or "5, 3" or "5, 1, 2" or "3, 2" or "3, 1, 2" entering into a coalition before he enters. The number of such orderings is

$$O_4 = 1!3! + 4 \times 2!2! + 2 \times 3!1! = 34.$$

The 3-vote player is pivotal if players "5, 1" or "5, 2" or "5, 1, 2" or "4, 2" or "4, 1, 2" entering into a coalition before he enters. The number of such orderings is

$$O_3 = 3 \times 2!2! + 2 \times 3!1! = 24.$$

(ii) Banzhaf indexes
The 5-vote player is marginal in a winning coalition if the winning coalition contains either "4" or "4, 1" or "4, 2" or "4, 3" or "4, 1, 2" or "4, 1, 3" or "3, 1" or "3, 2" or "3, 1, 2". The number of such coalitions is $B_5 = 9$.
The 4-vote player is marginal in a winning coalition if the winning coalition contains either "5" or "5, 1" or "5, 2" or "5, 3" or "5, 1, 2" or "3, 2" or "3, 1, 2". The number of such coalition is $B_4 = 7$.
The 3-vote player is marginal in a winning coalition if the winning coalition contains either "5, 1" or "5, 2" or "5, 1, 2" or "4, 2" or "4, 1, 2". The number of such coalitions is $B_3 = 5$.
The 2-vote player is marginal in a winning coalition if the winning coalition contains either "5, 3" or "4, 3" or "4, 3, 1". The number of such coalitions is $B_2 = 3$.
The 1-vote player is marginal in a winning coalition if the winning coalition contains either "5, 3" only. The number of such coalitions is $B_1 = 1$.
The individual Banzhaf indexes are given by

$$\beta_5 = \frac{9}{9+7+5+3+1} = \frac{9}{25}, \quad \beta_4 = \frac{7}{25}, \quad \beta_3 = \frac{5}{25} = \frac{1}{5},$$
$$\beta_2 = \frac{3}{25} \quad \text{and} \quad \beta_1 = \frac{1}{25}.$$

5. Let p be the voting probability of each of the four players, assuming homogeneity. We have

$$\begin{aligned}
\pi_A(p) &= P(B \text{ say "yes", zero, one or two of } C \text{ and } D \text{ say "yes"}) \\
&\quad + P(B \text{ say "no", both of } C \text{ and } D \text{ say "yes"}) \\
&= p[(1-p)^2 + 2p(1-p) + p^2] + (1-p)p^2 = p + p^2 - p^3; \\
\pi_B(p) &= P(A \text{ say "yes" and zero or one of } C \text{ and } D \text{ say "yes"}); \\
&= p[(1-p)^2 + 2p(1-p)] = p - p^3; \\
\pi_C(p) &= P(A \text{ say "yes", } B \text{ say "no", } D \text{ say "yes"}) = p^2(1-p) = p^2 - p^3; \\
\pi_D(p) &= \pi_C(p) = p^2 - p^3.
\end{aligned}$$

The Shapley-Shubik indexes and Banzhaf indexes of the players are found to be

$$\beta = \left(\pi_A\left(\frac{1}{2}\right), \pi_B\left(\frac{1}{2}\right), \pi_C\left(\frac{1}{2}\right), \pi_D\left(\frac{1}{2}\right)\right) = \left(\frac{5}{8}, \frac{3}{8}, \frac{1}{8}, \frac{1}{8}\right);$$

$$\Phi = \left(\int_0^1 \pi_A(p)\, dp, \int_0^1 \pi_B(p)\, dp, \int_0^1 \pi_C(p)\, dp, \int_0^1 \pi_D(p)\, dp\right) = \left(\frac{7}{12}, \frac{1}{4}, \frac{1}{12}, \frac{1}{12}\right).$$

6. According to the voting rule, the passage of a bill requires support from at least 4 members in the entire legislature and at least 2 votes from the 3-person committee.

   (i) Shapley-Shubik indexes
   A committee member A is pivotal if there are either (1) two other "A"s and exactly one "b", or (2) exactly one "A" and at least two "b"s entering into a coalition before A enters. The number of such orderings is

$$O_A = c_1^4 3!3! + \sum_{n=2}^{4} c_1^2 c_n^4 (n+1)!(4-n+1)! = 4! \times 50.$$

   The Shapley-Shubik indexes of A and b are given by

$$\Phi_A = \frac{4! \times 50}{7!} = \frac{5}{21} \quad \text{and} \quad \Phi_b = \frac{1}{4}(1 - 3\Phi_A) = \frac{1}{14}.$$

   Note that we have used the relation: sum of all Shapley-Shubik indexes equals one. The ratio of power (Shapley-Shubik) between a committee member and a non-committee member is $\frac{5/21}{1/14} = \frac{10}{3}$.

   (ii) Banzhaf indexes
   A committee member A is marginal in a winning coalition if the coalition contains either (1) other two "A"s and exactly one "b", or (2) exactly one "A" and at least two "b"s. The number of such coalitions is

$$B_A = c_1^4 + \sum_{n=2}^{4} c_1^2 c_n^4 = 26.$$

   A non-committee member is marginal in the winning coalition if the coalition also contains n "A"s and exactly $3-n$ other "b"s, where $n \geq 2$. The number of such coalitions is

$$B_b = \sum_{n=2}^{3} c_n^3 c_{3-n}^3 = 10.$$

   The individual Banzhaf indexes are given by

$$\beta_A = \frac{26}{26 \times 3 + 10 \times 4} = \frac{13}{59} \quad \text{and} \quad \beta_b = \frac{10}{26 \times 3 + 10 \times 4} = \frac{5}{59}.$$

   The ratio of power (Banzhaf) between a committee member and a non-committee member is $\frac{13}{5}$.

7. (a) With 5 other equally split stockholders, the proportion of shares held by each of these 5 stockholders is $\frac{100\% - 40\%}{5} = 12\%$. Together with the major stockholder, there are 6 players in the voting game.

   (i) Shapley-Shubik indexes
   L is pivotal if there are one, two, three or four other small stockholders (S) entering into the coalition before L enters. Out of 6 possible ordering of entry of

L, there are 4 orderings that L pivots, so the individual Shapley-Shubik indexes are given by

$$\Phi_L = \frac{4 \times 5!}{6!} = \frac{2}{3}, \quad \Phi_S = \frac{1}{5}(1 - \Phi_L) = \frac{1}{15}.$$

(ii) Banzhaf indexes

L is marginal in the winning coalition if the coalition also contains $1 \sim 4$ other stockholders. The number of such coalitions is $b_L = \sum_{n=1}^{4} c_n^5 = 30$.

S is marginal in the winning coalition if the coalition also contains either (1) L only or (2) exactly 4 S's. The number of such coalitions is $b_S = 1 + c_4^4 = 2$. The Banzhaf indexes are given by

$$\beta_L = \frac{30}{30 + 5 \times 2} = \frac{3}{4}, \; \beta_S = \frac{2}{30 + 5 \times 2} = \frac{1}{20}.$$

(b) With 7 other equally split stockholders, the proportion of shares held by each of these 7 stockholders is $\dfrac{100\% - 40\%}{7} = 8.57\%$.

(i) Shapley-Shubik indexes

L is pivotal if there are two, three, four or five other small stockholders (S) entering into the coalition before L enters. The number of such orderings is 4 out of 8 possible orderings, so the individual Shapley-Shubik indexes are given by

$$\Phi_L = \frac{4 \times 7!}{8!} = \frac{1}{2}, \quad \Phi_S = \frac{1}{7}(1 - \Phi_L) = \frac{1}{14}.$$

(ii) Banzhaf indexes

L is marginal in a winning coalition if the coalition also contains two, three, four or five small stockholders. The number of such coalitions is

$$B_L = \sum_{n=2}^{5} c_n^7 = 112.$$

S is marginal in a winning coalition if the coalition also contains either (1) L and exactly one other "S", or (2) exactly five other small stockholders. The number of such coalition is $B_5 = c_1^6 + c_5^6 = 12$.

The individual Banzhaf indexes are given by

$$\beta_L = \frac{112}{112 + 7 \times 12} = \frac{4}{7} \quad \text{and} \quad \beta_S = \frac{12}{112 + 7 \times 12} = \frac{3}{49}.$$

When the remaining proportion of shares are split among a large number of stockholders, the chance of forming a winning coalitions among the small stockholders against the major stockholder is less, so the major stockholder is less powerful when there are more equally split small stockholders.

8. (a) Let the voting weight of each small state be 1 and the voting weight of each big state be x. The quota q must satisfy

$$3x + 2 \geq q \quad \text{and} \quad q > 2x + 6.$$

Solving the inequalities yields $x > 4$. Suppose we take $x = 5$, then q satisfies $17 \geq q > 16$, so $q = 17$. The yes-no voting system can be written as the weighted voting system with voting vector specified as $[17; 5, 5, 5, 1, 1, 1, 1, 1, 1]$.

(b) $\pi_b(p) = P(\text{other 2 big states say "yes" and at least 2 small states say "yes"})$

$$= p^2 \left[\sum_{k=2}^{6} c_k^6 p^k (1-p)^{6-k} \right] = 15p^4 - 40p^5 + 45p^6 - 24p^7 + 5p^8;$$

$\pi_s(p) = P(\text{all 3 big states say "yes" and exactly one other small state says "yes"})$

$$= c_1^5 p^3 \left[p(1-p)^4 \right].$$

(c) The Shapley-Shubik index and Banzhaf index of any of the big states are given by

$$\begin{aligned}
\Phi_b &= \int_0^1 \pi_b(p)\ dp = \int_0^1 (15p^4 - 40p^5 + 45p^6 - 24p^7 + 5p^8)\ dp = \frac{20}{63}; \\
\Phi_s &= \int_0^1 \pi_s(p)\ dp \\
&= \int_0^1 c_1^5 p^4 (1-p)^4\ dp = c_1^5 \frac{4!4!}{9!} = \frac{1}{126}; \\
\beta_b' &= \pi_b\left(\frac{1}{2}\right) = \frac{57}{256}; \\
\beta_s' &= \pi_s\left(\frac{1}{2}\right) = \frac{5}{256}.
\end{aligned}$$

As a check, $3\Phi_b + 6\Phi_s = 3 \times \frac{20}{63} + 6 \times \frac{1}{126} = 1.$

Normalizing the Banzhaf indexes, we obtain $\beta_b = \dfrac{57}{201}$ and $\beta_s = \dfrac{5}{201}$.

(d) Assume that the 3 big states vote independently and the 6 smaller states vote as a homogeneous group. Let p_1, p_2 and p_3 be the voting probabilities of the 3 big states, respectively, and p be the common voting probability of the small states. We first compute $\pi_{b_k}(p, p_1, p_2, p_3)$, $k = 1, 2, 3$, and $\pi_s(p, p_1, p_2, p_3)$ in terms of p_1, p_2, p_3, p as follows:

(i) $\pi_{b_1}(p, p_2, p_3) = P(\text{other 2 big states say "yes" and at least 2 small states say "yes"})$

$$= p_2 p_3 \left[\sum_{k=2}^{6} c_k^6 p^k (1-p)^{6-k} \right];$$

(ii) $\pi_{b_2}(p, p_1, p_3) = p_1 p_3 \left[\sum_{k=2}^{6} c_k^6 p^k (1-p)^{6-k} \right];$

(iii) $\pi_{b_3}(p, p_1, p_2) = p_1 p_2 \left[\sum_{k=2}^{6} c_k^6 p^k (1-p)^{6-k} \right];$

(iv) $\pi_s(p, p_1, p_2, p_3) = P(\text{3 big states say "yes" and exactly one small state say "yes"})$

$$= p_1 p_2 p_3 [c_1^5 p (1-p)^4].$$

The power index of the big state "1" and any small state are given by

$$
\begin{aligned}
\Phi_{b_1} &= E[\pi_{b_1}(p, p_1, p_2, p_3)] \\
&= \int_0^1 \int_0^1 \int_0^1 p_2 p_3 \left[\sum_{k=2}^{6} c_k^6 p^k (1-p)^{6-k} \right] dp_2 dp_3 dp \\
&= \int_0^1 p_2 \, dp_2 \int_0^1 p_3 \, dp_3 \left[\int_0^1 c_2^6 p^2 (1-p)^4 \, dp + \int_0^1 c_3^6 p^3 (1-p)^3 \, dp \right. \\
&\quad \left. + \int_0^1 c_4^6 p^4 (1-p)^2 \, dp + \int_0^1 c_5^6 p^5 (1-p) \, dp + \int_0^1 c_6^6 p^6 \, dp \right] \\
&= \left(\frac{1}{2}\right)^2 \left[c_2^6 \frac{2!4!}{7!} + c_3^6 \frac{3!3!}{7!} + c_4^6 \frac{4!2!}{7!} + c_5^6 \frac{5!}{7!} + c_6^6 \frac{6!}{7!} \right];
\end{aligned}
$$

$$
\begin{aligned}
\Phi_s = E[\pi_s(p, p_1, p_2, p_3)] &= \int_0^1 \int_0^1 \int_0^1 \int_0^1 p_1 p_2 p_3 \left[c_1^5 p (1-p)^4 \right] dp_1 dp_2 dp_3 dp \\
&= \left(\frac{1}{2}\right)^3 c_1^5 \int_0^1 p(1-p)^4 \, dp = \frac{c_1^5}{8} \frac{4!}{6!} = \frac{1}{40}.
\end{aligned}
$$

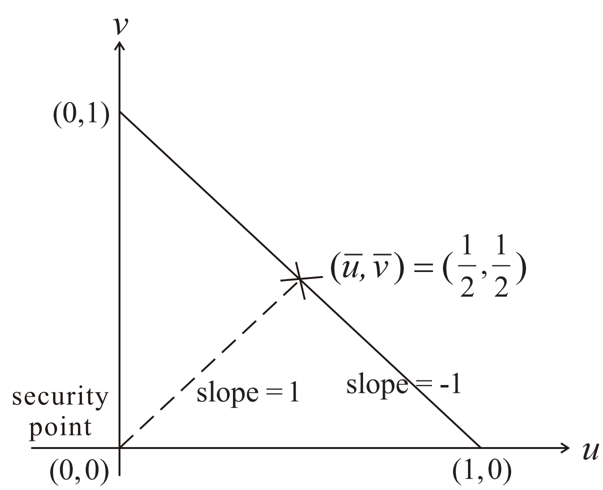
9. (a) The payoff matrices of the corresponding zero-sum games are $A = \begin{pmatrix} \frac{1}{2} & 0 \\ 1 & 0 \end{pmatrix}$, $B^T = A$. Performing the standard calculations gives

$$\text{value}(A) = \text{value}(B^T) = 0.$$

Hence, the security point is $(0, 0)$. The Nash bargaining problem is then

$$
\begin{aligned}
&\text{Maximize } uv \\
&\text{subject to } u + v \le 1, \ 0 \le u \le 1, \ 0 \le v \le 1.
\end{aligned}
$$

By calculus, the solution is $\bar{u} = \frac{1}{2}$, $\bar{v} = \frac{1}{2}$. The bargaining solution is that each contestant should split. That seems fair and natural. Note that the slope of the line joining the security point and bargaining solution is negative to that of the Pareto-optimal boundary line.



How to achieve the negotiated outcome $(\frac{1}{2}, \frac{1}{2})$?

(i) Both *cooperate* to play *(split, split)* to get $(\frac{1}{2}, \frac{1}{2})$.

(ii) They agree to play (claim, split) and (split, claim) **50% of the time** for each pair of pure strategies. The expected payoff is

$$\frac{1}{2}(0,1) + \frac{1}{2}(1,0) = \left(\frac{1}{2}, \frac{1}{2}\right).$$

In a cooperative game, we implicitly assume that the game can be *repeated many times.*

(b) With the change of the game matrix, the new zero-sum game matrices are

$$A = \begin{pmatrix} \alpha & 0 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad B^T = \begin{pmatrix} 1-\alpha & 0 \\ 1 & 0 \end{pmatrix}.$$

The values of the games remain the same, where value$(A) = 0$ and value$(B^T) = 0$. Therefore, the formulation of the bargaining game is identical to that of part (a). The bargaining solution remains to be $(\frac{1}{2}, \frac{1}{2})$; and for $\alpha \neq \frac{1}{2}$, it does not correspond to (split, split) as before.

Aa a summary, the optimal cooperative outcome cannot be achieved by playing (split, split). It can be achieved by playing 50% of the time for (claim, split) and (split, claim).

To find the threat strategies, we note that $m_p = -1$ (since the Pareto-optimal boundary is $u + v = 1$). We consider

$$-m_p A - B = A - B = \begin{pmatrix} \alpha & 0 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 1-\alpha & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 2\alpha - 1 & -1 \\ 1 & 0 \end{pmatrix}, \quad 0 < \alpha < 1.$$

Note that (claim, claim) is the saddlepoint of $A - B$ (observing row min and column max). It is quite natural for both players to choose this optimal threat strategy with outcome $(0, 0)$ since playing "claim" is the weakly dominant strategy. We obtain the same bargaining solution $(\frac{1}{2}, \frac{1}{2})$ as part (a). The Nash bargaining solution improves the outcomes of both players.

(c) (i) If the second player trusts the first player, the outcome is $(\frac{1}{2}, \frac{1}{2})$, same as that of the optimal cooperative outcome.

(ii) If otherwise, the second player should play "claim" as the threat strategy. Without trust, the first player responds by playing "claim" as well, which is a weakly dominant strategy. Since the game is played only once and there is no cooperation, the worse outcome (0,0) is resulted.

10. (a) The Nash bargaining problem is formulated as

$$\begin{aligned} \text{Maximize } (u - u^*)(v - v^*) &= [f(w) - pw][pw + (W - w)p_0 - Wp_0] \\ &= [f(w) - pw](p - p_0)w \end{aligned}$$

with $(p, w) \in S$, where

$$\begin{aligned} S &= \{(u,v) | u \geq u^*, \ v \geq v^*\} \\ &= \{(u,v) | u \geq 0, \ v \geq Wp_0\} \\ &= \{(p,w) | f(w) - pw \geq 0, \ p \geq p_0, \ 0 \leq w \leq W\}. \end{aligned}$$

(b) Set $h(p,w) = [f(w) - pw](p - p_0)w$. The first order conditions are found to be

$$\begin{aligned}\frac{\partial h}{\partial p} &= w[(p_0 - 2p)w + f(w)] = 0\\ \frac{\partial h}{\partial w} &= (p - p_0)\{f(w) + w[f'(w) - 2p]\} = 0.\end{aligned}$$

Since we seek solution for $p > p_0$, so we can cancel the factor $p - p_0$ in $\frac{\partial h}{\partial w}$. Solving the first equation for p gives the relation between the pay level p and size of the work force w:

$$p = \frac{wp_0 + f(w)}{2w}.$$

Substitute this p into the second equation to obtain

$$\begin{aligned}f(w) + w\left[f'(w) - \frac{wp_0 + f(w)}{w}\right] &= f(w) + wf'(w) - p_0 w - f(w)\\ &= w[f'(w) - p_0] = 0.\end{aligned}$$

This gives $f'(w) = p_0$. The critical point occurs at which the marginal revenue $f'(w)$ is equal to the minimum pay level. The increase of revenue for the last worker balances the minimum pay to this last worker. Recall that the marginal revenue $f'(w)$ is decreasing in w. The management stops hiring worker when the gain in hiring one more worker becomes lower than the minimum wage paid to the worker. This gives

$$p^* = \frac{w^* p_0 + f(w^*)}{2w^*},$$

where $w^* = (f')^{-1}(p_0)$.

(c) Applying the formula: $p_0 = f'(w^*)$ to $f(w) = \ln(w + a) + b$, we obtain

$$p_0 = f'(w^*) = \frac{1}{w^* + a}.$$

This gives the optimal number of workers hired

$$w^* = \frac{1}{p_0} - a > 0$$

and the optimal salary level

$$p^* = \frac{w^* p_0 + f(w^*)}{2w^*} = \frac{p_0[ap_0 - \ln(\frac{1}{p_0} - a) - b - 1]}{2ap_0 - 2}.$$