MATH4321 — Game Theory

Topic One: Zero-sum games and saddle point equilibriums

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1.1 Definitions and examples

- *Players*: Individuals who make decisions among choices of actions. Each player's goal is to maximize his (expected) payoff by the choice of actions.
  - Nature is a pseudo-player who takes random actions at specified points in the game with specified probabilities or probability distribution. An example is the potential firing of the pistol with probability 1/6 in the Russian roulette.

- *Action* (move): Choice a_i made by player i.
  - Player i's action set A_i contains the entire set of actions available.
  - An action combination is an ordered set $\boldsymbol{a} = (a_1,\ a_2, ...,\ a_n)$ of one action chosen by each of the n players in the game.

- *Outcomes* are the possible consequences that can result from any combination of the players' actions.

Pure strategies and complete information

A pure strategy for player i is a deterministic plan of action. This is in contrast to choose randomly among plans of action, called mixed strategies, as one plays in the rock-paper-scissors game.

A game of complete information requires that the following components are common knowledge among all players of the game:

1. All possible actions of all the players,
2. All the possible outcomes.
3. How each combination of actions of all players affects which outcome that will materialize.

In a static game, each player simultaneously and independently chooses an action (once-and-for-all) without the knowledge of actions taken by other players. Conditional on the players' choices of actions, payoffs are distributed to the respective player.

Normal-form representation of a game

A normal-form representation of a game includes three components:

1. A finite set of players, $N = \{1, 2, \ldots, n\}$.
2. A collection of sets of pure strategies, $\{S_1, S_2, \ldots, S_n\}$.
3. A set of payoff functions, $\{v_1, v_2, \ldots, v_n\}$, each assigning a payoff value to each combination of chosen strategies, that is, a set of functions $v_i : S_1 \times S_2 \times \cdots \times S_n \to \mathbb{R}$ for each $i \in N$.

Example - Voting game

Players: $N = \{1, 2, 3\}$.
Strategy sets: $S_i = \{Y, N, A\}$ for $i \in \{1, 2, 3\}$.
Payoffs: Let P denote the set of strategy profiles for which the new agenda is chosen (at least two "yes" votes or one "yes" with two "obstain"), and let Q denote the set of strategy profiles for which the status quo remains (new proposal not passed). Voters 1 and 2 prefer passage while player 3 dislikes passage.

Each player has 3 choices of actions, so there are $3^3 = 27$ strategy profiles. The strategy profiles are categorized into 2 sets: P (passage) and Q (non-passage).

$$P = \begin{Bmatrix} (Y, Y, N), & (Y, N, Y), \\ (Y, Y, A), & (Y, A, Y), \\ (Y, A, A), & (A, Y, A), \\ (Y, Y, Y), & (N, Y, Y), \\ (A, Y, Y), & (A, A, Y) \end{Bmatrix} \quad \text{and}$$

$$Q = \begin{Bmatrix} (N, N, N), & (N, N, Y), & (N, Y, N), & (Y, N, N), \\ (A, A, A), & (A, A, N), & (A, N, A), & (N, A, A), \\ (A, Y, N), & (A, N, Y), & (N, A, Y), & (Y, A, N), \\ (N, Y, A), & (Y, N, A), & (N, N, A), & (N, A, N), \\ (A, N, N) & & & \end{Bmatrix}.$$

Then payoffs can be written as

$$v_i(s_1, s_2, s_3) = \begin{cases} 1 & \text{if } (s_1, s_2, s_3) \in P \\ 0 & \text{if } (s_1, s_2, s_3) \in Q \end{cases} \quad \text{for } i \in \{1, 2\},$$

$$v_3(s_1, s_2, s_3) = \begin{cases} -1 & \text{if } (s_1, s_2, s_3) \in P \\ 0 & \text{if } (s_1, s_2, s_3) \in Q. \end{cases}$$

This completes the normal-form representation of the voting game.

Two-person zero sum game

Payoff as represented by a game matrix

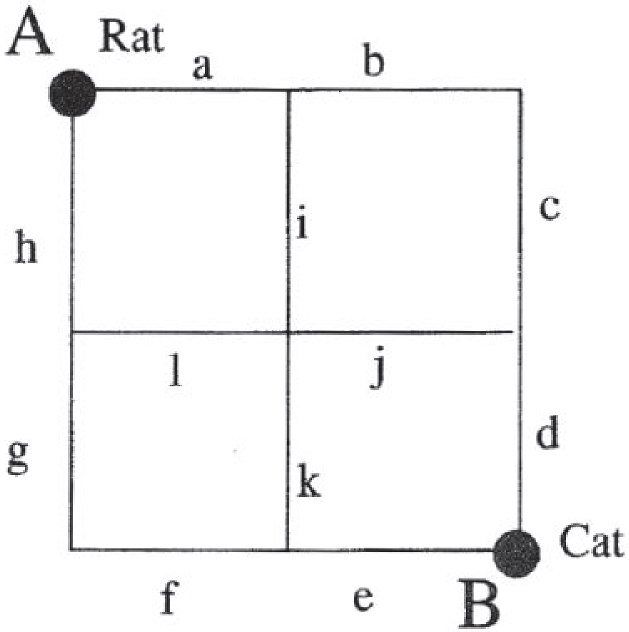
In a zero sum game, if a_{ij} is the payoff received by Player I, then Player II receives $-a_{ij}$.

	player II			
player I	Strategy 1	Strategy 2	$\cdots$	Strategy m
Strategy 1	a_{11}	a_{12}	$\cdots$	a_{1m}
Strategy 2	a_{21}	a_{22}		a_{2m}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
Strategy n	a_{n1}	a_{n2}	$\cdots$	a_{nm}

Player I (Row player) wants to choose a strategy to maximize the payoff in the game matrix, while Player II (Column player) wants to minimize the payoff.

Example Cat versus Rat in a maze

Each animal takes 4 steps concurrently



Cat/Rat	abcd	abcj	⋯	hlkf
dcba	1	1	⋯	0
dcbi	1	1	⋯	0
⋮	⋮	⋮	⋮	⋮
eklg	0	0	⋯	1

Game Matrix

If Cat finds Rat, Cat gets 1; and otherwise, Cat gets 0. For example, when Cat chooses $dcba$ and Rat $abcd$, they will meet at an intersection point.

Example - Nim with two piles of two coins

The 2×2 Nim game is represented in an *extensive form* - a game tree representing the successive moves of players.

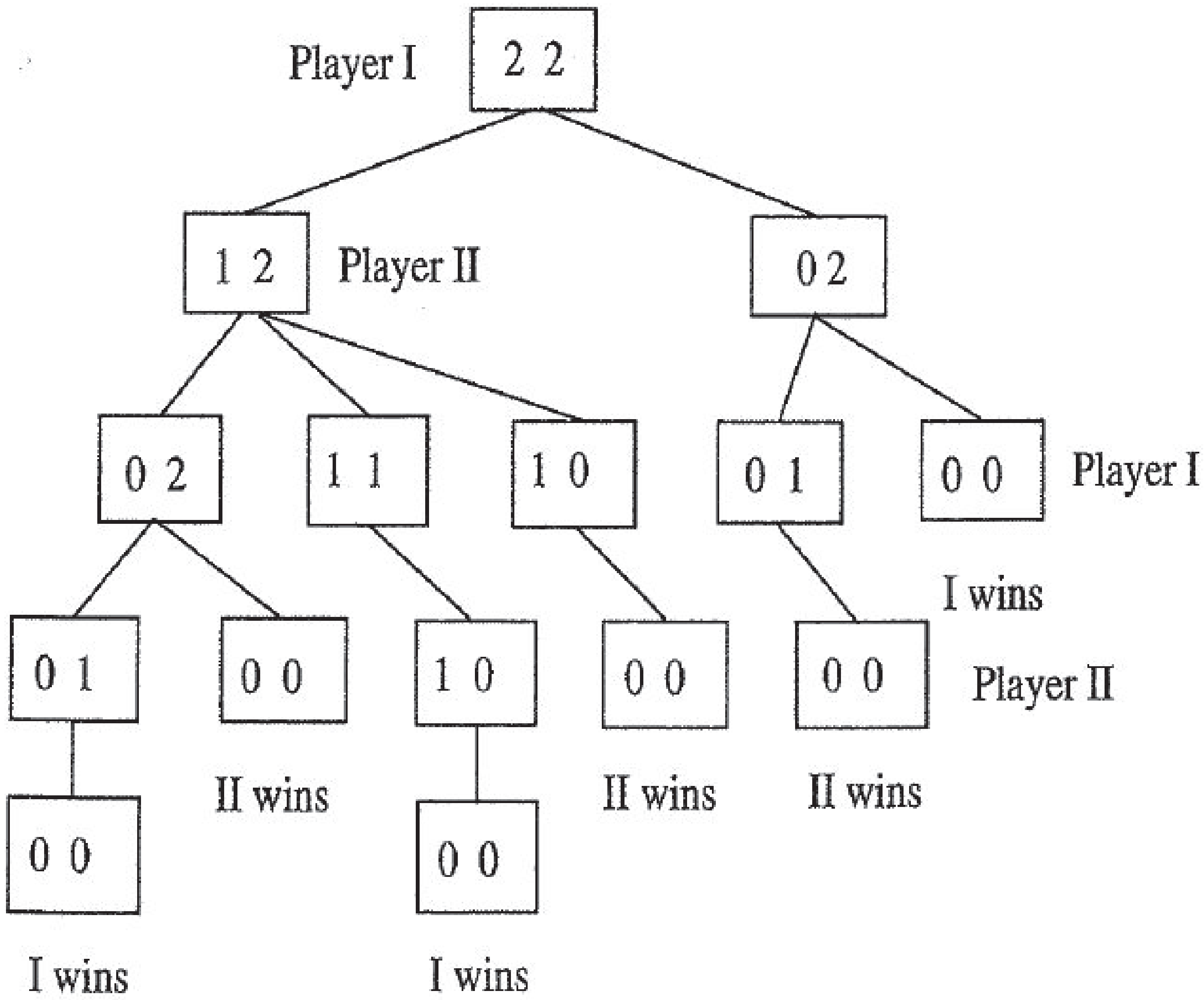
- Four pennies are placed in two piles of two pennies each.

- Each player chooses a pile and decides to remove one or two pennies.

- The loser is the one who removes the last penny (pennies).

The winning payoff is independent of the pile that the last penny (pennies) is taken.

Game tree representation



Strategies for player I
(1) Play (1,2) then, if at (0,2) $\longrightarrow$ (0,1).
(2) Play (1,2) then, if at (0,2) $\longrightarrow$ (0,0).
(3) Play (0,2).

If Player I plays $(1,2)$, then there are still 2 possible strategies in the later move. If $(0,2)$ is played by Player I, then there is no choice of strategy in the later move.

Strategies for player II
(1) If at (1,2) $\rightarrow$ (0,2); if at (0,2) $\rightarrow$ (0,1)
(2) If at (1,2) $\rightarrow$ (1,1); if at (0,2) $\rightarrow$ (0,1)
(3) If at (1,2) $\rightarrow$ (1,0); if at (0,2) $\rightarrow$ (0,1)
(4) If at (1,2) $\rightarrow$ (0,2); if at (0,2) $\rightarrow$ (0,0)
(5) If at (1,2) $\rightarrow$ (1,1); if at (0,2) $\rightarrow$ (0,0)
(6) If at (1,2) $\rightarrow$ (1,0); if at (0,2) $\rightarrow$ (0,0)

For Player II, there are $3 \times 2 = 6$ combinations of strategies, depending on whether $(1,2)$ or $(0,2)$ is played by Player I. There is no choice of strategy for Player II in the later move.

Game matrix representation

player I/player II	1	2	3	4	5	6
1	1	1	−1	1	1	−1
2	−1	1	−1	−1	1	−1
3	−1	−1	−1	1	1	1

Suppose Player I plays $(0,2)$ (strategy 3 and see the right half of the game tree), the first 3 strategies of Player II correspond to $(0,2) \to (0,1)$, so Player II wins. This gives the payoff of -1 in the first 3 entries in the 3rd row. Otherwise, if Player II plays the last 3 strategies, then Player I wins. This gives the payoff of 1 in the last 3 entries in the third row.

- No matter what Player I does, Player II always wins by playing Strategy 3. This is obvious since $(1,2) \to (1,0)$ and $(0,2) \to (0,1)$ both leave one penny behind for the opponent to remove the last penny.

  This strategy is said to be *weakly dominant*. It has the property that it is always at least as good and in some cases better in payoff when compared with any other strategies played by Player II. The game matrix representation helps in identifying the weakly dominant strategy at ease.

- Player II would never play Strategy 5 (*weakly dominated* strategy for Player II). Obviously, it always leads to loss of the game if $(1,2) \to (1,1)$ and $(0,2) \to (0,0)$.

Randomization of strategies in Evens or Odds game

In the Evens or Odds game, each player decides to show 1, 2 or 3 fingers. Player I wins \$1 if the sum of fingers is even; otherwise, Player II wins \$1. The game matrix is shown below. How should each player decide what number of fingers to show?

Evens	Odds		
I/II	1	2	3
1	1	−1	1
2	−1	1	−1
3	1	−1	1

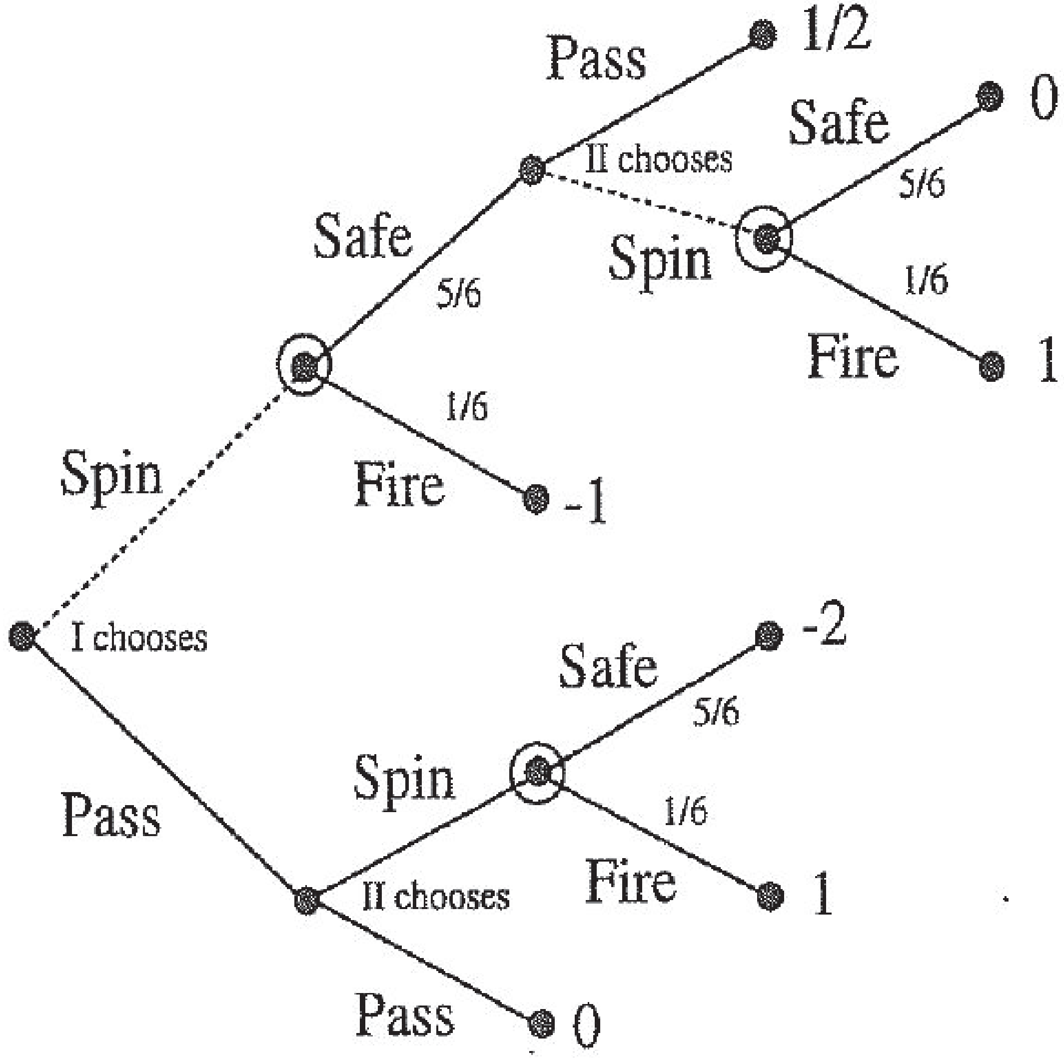
If a player always plays the same strategy, then the opponent can always win the game. She should mix or randomize the strategies. Later, we show that the saddle point mixed equilibrium strategies (an important concept in game theory to be discussed later) of player I are to play 50% chance of strategy 2 and combined 50% chance of strategy 1 and strategy 3 (note that strategy 1 and strategy 3 are identical in payoff). Due to symmetry, player 2 should also adopt the same mixed equilibrium strategies.

Example — Russian Roulette

The two players are faced with a 6-shot pistol loaded with one bullet. Both players put down $\$1$ and Player I goes first.

- At each play of the game, a player has the option of putting an additional $\$1$ into the pot and passing; or not adding to the pot, spinning the chamber and firing at his own head.
- If Player I chooses the option of spinning and survives, then he passes the gun to Player II, who has the same two options. Player II decides what to do, carries it out, and the game ends.
- If Player I fires and survives and then Player II passes, both will split the pot. In effect, Player II will pay Player I $\$0.5$ since they split the pot of $\$3$. The payoff to Player I is 0.5.
- If Player I chooses to pass and Player II chooses to fire, then if Player II survives, he takes the pot. Since Player I has put down \$2 and collects nothing at the end, the payoff to Player I is -2.

Game tree representation of the Russian Roulette



Nature comes in since the player survives with probability $\frac{5}{6}$ if he chooses to spin. One needs to consider the expected payoff based on the law of probabilities.

For Player I, he has two strategies: either spins (I1) or passes (I2).

For Player II, there are 4 strategies:

II1	If I1, then P; If I2, then S. (opposite play)
II2	If I1, then P; If I2, then P. (always pass)
II3	If I1, then S; If I2, then P. (same play)
II4	If I1, then S; If I2, then S. (always spin)

Even if the players always take the same actions, the random move by Nature means that the model would yield different realizations of a game.

The players devise *strategies* ($s_1,\ s_2, ...,\ s_n$) that pick actions depending on the information that has arrived at each moment so as to maximize their (expected) payoffs.

We assume that the bullet chamber of the pistol is reset when the pistol is passed to the second player. Therefore, the probability of survival of the second player remains to be 5/6. We compute the expected payoff to I under 8 cases:

I1 against II4 (both players spin): $\frac{5}{6}\left[\frac{5}{6}(0)+\frac{1}{6}(1)\right]+\frac{1}{6}(-1)=-\frac{1}{36}$.

I2 against II4 (Player I passes and Player II spins): $\frac{5}{6}(-2)+\frac{1}{6}(1)=$
$-\frac{3}{2}$;

I1 against II1 (Player I spins and Player II passes): $\frac{5}{6}\left(\frac{1}{2}\right)+\frac{1}{6}(-1)=$
$\frac{1}{4}$;

I2 against II1 (Player I passes and Player II spins): $\frac{5}{6}(-2)+\frac{1}{6}(1)=$
$-\frac{3}{2}$.

Game matrix of the Russian Roulette

I/II	II1	II2	II3	II4
I1	$\frac{1}{4}$	$\frac{1}{4}$	$-\frac{1}{36}$	$-\frac{1}{36}$
I2	$-\frac{3}{2}$	0	0	$-\frac{3}{2}$

Player II will only play II4 since this is a weakly dominant that strategy always yield positive payoff (negative value in the payoff matrix). Player I then plays I1 since Player II definitely plays II4. The expected payoff to Player I is $-\frac{1}{36}$. Otherwise, if player I plays I2, her expected payoff is $-\frac{3}{2}$ since player II will only play II4.

Later, we identify (I1, II4) as the saddle point strategies (observing the row-min and column-max property). Though under sequential moves of the players, their equilibrium strategies (with reference to some solution concept, later recognized as the Nash equilibrium in this case) can be identified at initiation of the game.

1.2 Saddle points

Value of a zero sum game under pure strategies

Player 1's perspective

Given the game matrix $A = (a_{ij})$, for any given row i (i^{th} strategy of Player 1), Player 1 assumes that Player 2 chooses a column j so as to

$$\text{Minimize } a_{ij} \text{ over } j = 1,\ 2, ...,\ m.$$

Player 1 can choose the specific row i that will maximize among these minima along the rows. That is, Player 1 can *guarantee* that in the worst scenario he can receive at least

$$v^{-} = \max_{i=1,\ldots,n}\ \min_{j=1,\ldots,m}\ a_{ij}.$$

This is the *lower value* of the game (or Player 1's game floor).

Player 2's perspective

For any given column $j = 1,\ 2, ...,\ m$, Player 2 assumes that Player 1 chooses a row so as to

$$\text{Maximize } a_{ij} \text{ over } i = 1,\ 2, ...,\ n.$$

Player 2 can choose the column j so as to guarantee a loss of no more than

$$v^+ = \min_{j=1,...,m} \max_{i=1,...,n} a_{ij}.$$

This is the *upper value* of the game (or Player 2's loss ceiling).

Based on the *minimax criterion*, Player 2 chooses a strategy (a-mong all possible strategies) to minimize the maximum damage the opponent can cause.

$$
\begin{array}{cccc|l}
a_{11} & a_{12} & \cdots & a_{1m} & \longrightarrow \min_j a_{1j} \\
a_{21} & a_{22} & \cdots & a_{2m} & \longrightarrow \min_j a_{2j} \\
\vdots & \vdots & \cdots & \vdots & \\
a_{n1} & a_{n2} & \cdots & a_{nm} & \longrightarrow \min_j a_{nj} \\
\hline
\downarrow & \downarrow & \cdots & \downarrow & \\
\max_i a_{i1} & \max_i a_{i2} & \cdots & \max_i a_{im} & v^- = \text{largest min} \\
 & & & & v^+ = \text{smallest max}
\end{array}
$$

Proof of $\mathbf{v}^- \leq \mathbf{v}^+$

Observe that

$$v^- = \max_i \min_j a_{ij} \leq \max_i a_{ij}, \quad \text{for any } j.$$

The above inequality is independent of j, so it remains to be valid when we take $\min_j (\max_i a_{ij})$, which is precisely v^+. Hence, $v^- \leq v^+$.

Saddle point in pure strategies

We call a particular row i^* and column j^* a saddle point in pure strategies of the game

$$a_{ij^*} \leq a_{i^*j^*} \leq a_{i^*j},$$

for all rows $i = 1,\ 2, ...,\ n$ and columns $j = 1,\ 2, ...,\ m$.

We can spot a saddle point in a matrix (if there is one) as the entry that is simultaneously the smallest in a row and largest in a column.

In a later lemma, we show that $v^+ = v^-$ is equivalent to have the existence of a saddle point (may not be unique) under pure strategies (players do not randomize the choices of strategies).

The value of a zero sum game is the payoff at the saddle point. The game is said to have a value if $v^- = v^+$, and we write

$$v = v(A) = v^+ = v^-.$$

Nash equilibrium concept

Note that $(i^*,\ j^*)$ is a saddle point if when Player 1 deviates from row i^*, but Player 2 still plays j^*, then Player 1 will get less or at most the same (largest value in the strategy chosen by Column). This is because $a_{ij^*} \leq a_{i^*j^*}$.

Vice versa, if Player 2 deviates from column j^* but Player 1 sticks with i^*, then Player 2 will get less or at most the same (smallest value in the strategy chosen by Row). This is because $a_{i^*j} \geq a_{i^*j^*}$.

When a saddle point exists in pure strategies, if any player deviates from playing his part of the saddle, then the player would be worst off or at most the same. Later, we define such solution concept as the Nash equilibrium concept.

Row player would choose row i^* and column player would choose column j^* as their equilibrium strategies. The saddle point payoff would be the minimum assured payoff to both players.

Consider the following two-person zero sum game

		Colin			
		A	B	C	D
Rose	A	12	−1	1	0
	B	5	1	7	−20
	C	3	2	4	3
	D	−16	0	0	16

Note that Rose wants the payoff to be large (16 would be the best) while Colin wants the payoff to be small (−20 the smallest).

$v^+ = \min(12, 2, 7, 16) = 2$ and $v^- = \max(-1, -20, 2, -16) = 2$.

Experimental results on people's choices

Rose Strategy	Percent of time
A	31%
B	10%
C	49%
D	10%
	100%

Colin Strategy	Percent of time
A	20%
B	51%
C	2%
D	27%
	100%

It seems quite irrational for the Column player to choose C over B, where payoff to Colin is better or the same under all strategies of Rose if C is played by Colin. Strategy C is weakly dominated by Strategy B.

Apparently, in real experiments, participating players may not practise the minimax criterion. Interestingly, the average payoff is close to the game value of 2.0.

Best responses

Player i's *best response* to the strategies s_{-i} chosen by the other players is the strategy s_i^* that yields him the greatest payoff, where

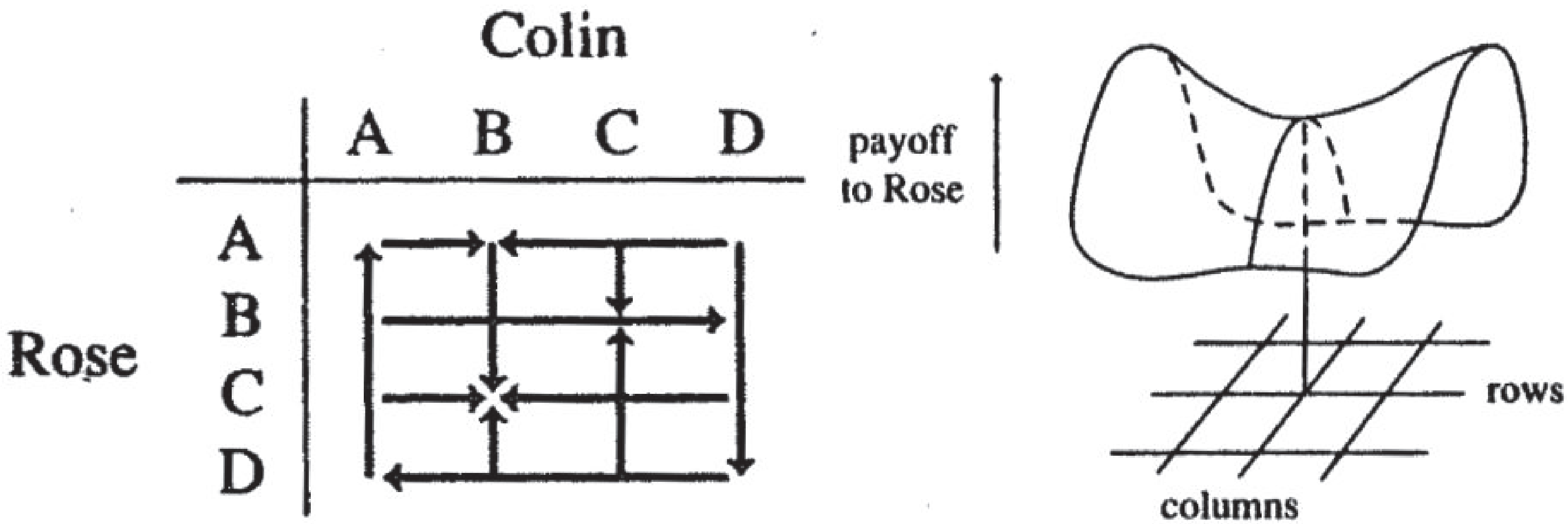
$$\pi_i(s_i^*,\ s_{-i}) \geq \pi_i(s_i^{'},\ s_{-i}),\ \forall\ s_i^{'} \neq s_i^*.$$

The best response is strongly best if no other strategies are equally good.

In a two-person zero-sum game, each part of a saddle point strategies is the best response to the other player.

Rose C - Colin B (saddle point) is an equilibrium outcome

If Colin knows or believes that Rose will play Rose C, then Colin would choose Colin B as the best response; similarly, Rose C is Rose's best response to Colin B. Once both players are playing these strategies, then neither player has any incentive to move to a different strategy.



Suppose Colin plays A, then Rose would choose to play A as the best response. In response to Rose playing A, the best response of Colin is to choose to play B. After then, the best response of Rose is choosing to play C.

Lemma – existence of saddle point

A game has a saddle point in pure strategies if and only if

$$v^- = \max_i \min_j a_{ij} = \min_j \max_i a_{ij} = v^+.$$

Proof

(i) existence of a saddle point $\Rightarrow$ $v^+ = v^-$

Suppose $(i^*,\ j^*)$ is a saddle point, we have

$$v^+ = \min_j \max_i a_{ij} \leq \max_i a_{ij^*} = a_{i^*j^*} = \min_j a_{i^*j} \leq \max_i \min_j a_{ij} = v^-.$$

The equality on the left (right) side arises from the column-max (row-min) property of the saddle point (i^*, j^*). When Column player plays j^*, the best response of the Row player is to play i^*.

However, $v^- \leq v^+$ always holds, so we have equality throughout and

$$v = v^+ = v^- = a_{i^*j^*}.$$

(ii) $v^+ = v^- \;\Rightarrow$ existence of a saddle point

Suppose $v^+ = v^-$, so

$$v^+ = \min_j \max_i a_{ij} = \max_i \min_j a_{ij} = v^-.$$

Let the specific column j^* be such that $v^+ = \max_i a_{ij^*}$ and the specific row i^* such that $v^- = \min_j a_{i^*j}$. We would like to establish that (i^*, j^*) is the saddlepoint.

Note that for any $i = 1,\ 2, ...,\ n$ and $j = 1,\ 2, ...,\ m$, we have

$$a_{i^*j} \geq \min_j a_{i^*j} = v^- = v^+ = \max_i a_{ij^*} \geq a_{ij^*}. \qquad \text{(i)}$$

Since the above inequality is valid for any i and j, by taking $j = j^*$ on the left inequality and $i = i^*$ on the right, we obtain

$$a_{i^*j^*} = v^+ = v^-.$$

Replacing v^- and v^+ by $a_{i^*j^*}$ in inequality (i), so $a_{i^*j} \geq a_{i^*j^*} \geq a_{ij^*}$. This satisfies the condition for $(i^*,\ j^*)$ to be a saddle point.

Multiple saddle points

A two-person zero sum game may have no saddle point or more than one saddle point.

Example 2×2 Nim

1	1	-1	1	1	-1	$\longrightarrow$	$\min = -1$
-1	1	-1	-1	1	-1	$\longrightarrow$	$\min = -1$
-1	-1	-1	1	1	1	$\longrightarrow$	$\min = -1$
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$		$\boxed{v^- = -1}$
$\max = 1$	$\max = 1$	$\max = -1$	$\max = 1$	$\max = 1$	$\max = 1$	$\boxed{v^+ = -1}$	

The third column is a weakly dominant strategy for the Column player since he receives payoff that is at least as good or better than his other strategies, irrespective to the strategies played by the opponent.

All entries in the third column are saddle points since all these entries observe the row-min and column-max property.

Example

		Colin			
		A	B	C	D
Rose	A	4	2	5	2
	B	2	1	−1	−20
	C	3	2	4	2
	D	−16	0	16	1

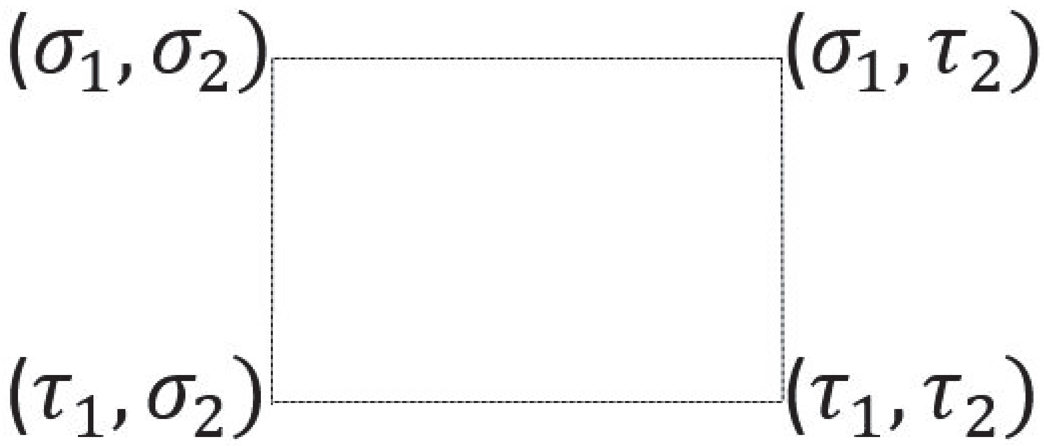
All four of the circled outcomes are saddle points. They are the corners of a rectangular block.

Note that "2" at Rose B - Colin A is not a saddle point. It just happens to have the same value as that of other saddle points but it does not possess the row-min and column-max property.

Lemma

In a two-person zero sum game, suppose $(\sigma_1,\ \sigma_2)$ and $(\tau_1,\ \tau_2)$ are two saddle strategies, then $(\sigma_1,\ \tau_2)$ and $(\tau_1,\ \sigma_2)$ are also saddle strategies. Also, their payoffs are the same (value lemma). That is,

$$a_{\sigma_1\sigma_2} = a_{\tau_1\tau_2} = a_{\sigma_1\tau_2} = a_{\tau_1\sigma_2}.$$



Form a rectangle having the two saddle points (σ_1, σ_2) and (τ_1, τ_2) as corners, then the other two corners in the same rectangle are saddle points as well. All these 4 saddle points share the same payoff.

Proof

Since $(\sigma_1,\ \sigma_2)$ is a saddle point, so $a_{\sigma_1\sigma_2} \geq a_{\tau_1\sigma_2}$ (largest value in a column). Similarly, we have $a_{\tau_1\sigma_2} \geq a_{\tau_1\tau_2}$ (smallest value in a row). Combining the results, we obtain

$$a_{\sigma_1\sigma_2} \geq a_{\tau_1\sigma_2} \geq a_{\tau_1\tau_2}.$$

In a similar manner, moving from $(\sigma_1,\ \sigma_2)$ to $(\sigma_1,\ \tau_2)$ along the row σ_1 and $(\tau_1,\ \sigma_2)$ to $(\tau_1,\ \tau_2)$ along the column τ_2, we can establish

$$a_{\sigma_1\sigma_2} \leq a_{\sigma_1\tau_2} \leq a_{\tau_1\tau_2}.$$

Combining the results, we obtain equality of the 4 payoffs:

$$a_{\sigma_1\sigma_2} = a_{\tau_1\tau_2} = a_{\sigma_1\tau_2} = a_{\tau_1\sigma_2}.$$

For any $\widehat{\sigma}_1$, we have $a_{\widehat{\sigma}_1\sigma_2} \leq a_{\sigma_1\sigma_2} = a_{\tau_1\sigma_2}$; and for any $\widehat{\sigma}_2$, we also have $a_{\tau_1\widehat{\sigma}_2} \geq a_{\tau_1\tau_2} = a_{\tau_1\sigma_2}$. Therefore, $a_{\tau_1\widehat{\sigma}_2} \geq a_{\tau_1\sigma_2} \geq a_{\widehat{\sigma}_1\sigma_2}$ and so $(\tau_1,\ \sigma_2)$ is a saddle strategy. Since the payoffs at these entries are the same, (τ_1,σ_2) inherits the row min property from (τ_1,τ_2) and column max property from (σ_1,σ_2). Similarly, we can also establish that $(\sigma_1,\ \tau_2)$ is a saddle strategy.

1.3 Mixed strategies for zero sum games

Define the set of n-component probability vectors

$$S_n = \left\{(z_1, z_2, ..., z_n) : \ 0 \le z_k \le 1, \ k = 1, 2, ..., n \text{ and } \sum_{k=1}^{n} z_k = 1\right\}.$$

A mixed strategy is characterized by a probability vector $X = (x_1, ..., x_n) \in S_n$ for Player I and $Y = (y_1, ..., y_m) \in S_m$ for Player II, where

$$x_i \ge 0, \ \sum_{i=1}^{n} x_i = 1 \text{ and } y_j \ge 0, \ \sum_{j=1}^{m} y_i = 1.$$

Here, $x_i = P[\text{I uses row } i]$ and $y_j = P[\text{II uses column } j]$. Let $A = (a_{ij})_{n\times m}$ be the game matrix. Each player's choices of strategies are dependent on a_{ij} but *no explicit dependence* on the opponent's strategies (the two players make their random choices of mixed strategies independently).

For example, in a rock-paper-scissors game, each player may use a fortune wheel (designed based on the probability distribution of the mixed strategies) to determine the show of rock, paper or scissors. The random experiments of spinning the fortune wheels by the two players are independent.

Matching pennies

This is a zero-sum game, reduces to the Evens and Odds game when the two players can show either one or two fingers. The first player wins if the two pennies match; otherwise he loses.

	Head	Tail
Head	1	-1
Tail	-1	1

The game can be used to model the choices of appearances for new products by an established producer and a new firm. The established producer (Player 2) prefers the newcomer's product to look different whereas the newcomer (Player 1) prefers that the products look alike.

The game has *no* pure strategy saddle point since none of the entries in the game matrix observes the row-min and column-max property. For the pairs of actions (Head, Head) and (Tail, Tail), Player 2 is better off by deviating unilaterally; for the pairs of actions (Head, Tail) and (Tail, Head), Player 1 is better off by deviating unilaterally.

Stochastic steady state action profile

The matching pennies game has a stochastic steady state action profile. Each player chooses his actions probabilistically according to the same unchanging distribution of the mixed strategies.

Let p be the probability that Player 1 chooses Head and q be the probability that Player 2 chooses Head. Assuming $q \neq \frac{1}{2}$, then

Player 1 gains $\$$ 1 with probability

$$p_{1,\text{gain}} = pq + (1-p)(1-q) = 1 - q + p(2q-1);$$

and loses $\$$ 1 with probability

$$p_{1,\text{lose}} = q + p(1-2q) = 1 - p_{1,\text{gain}}.$$

When $q < \frac{1}{2}$, $p_{1,\text{gain}}$ is decreasing in p and $p_{1,\text{lose}}$ is increasing in p. Therefore, the lower is p, the better is the outcome for Player 1. The best choice is $p = 0$. Player 1 chooses Tail with certainty when Player 2 plays Head less than 50% chance.

Similarly, if Player 2 chooses $q > \frac{1}{2}$, then the optimal choice for p is 1 (Player 2 chooses Head with certainty).

If Player 1 chooses an action with certainty, with $p = 0$ or $p = 1$, then the optimal policy of Player 2 is to choose an action with certainty. That is, Head if Player 1 chooses Tail and Tail if Player 1 chooses Head.

As a result, there is no steady state in which the probability that Player 2 chooses Head differs from $\frac{1}{2}$. The same conclusion is obtained when the probability that Player 1 chooses Head differs from $\frac{1}{2}$.

When $q = \frac{1}{2}$, then $p_{1,\text{gain}} = \frac{1}{2}p + \frac{1}{2}(1 - p) = \frac{1}{2}$.

Interestingly, the probability distribution over outcomes is independent of p. In that sense, every value of p is optimal. Player 1 can do no better than choosing Head with probability $\frac{1}{2}$ and Tail with probability $\frac{1}{2}$.

By symmetry, a similar analysis shows that the probability of winning of Player 2 is always $\frac{1}{2}$, independent of the choice of q, when Player 1 chooses $p = \frac{1}{2}$.

Question: How can the mixed strategies be sustained?

The pattern may cycle forever until Player 2 stumbles on trying $q = \frac{1}{2}$ and Player 1 taking $p = \frac{1}{2}$.

One player cannot be better off if the other player keeps playing his mixed strategy of equal probabilities and vice versa. This is related to the notion of a saddle point in mixed strategies defined later, where if one player plays her part of the saddle point strategy then the opponent cannot be better off if he deviates from his part of the saddle point strategy. We assume learning on both sides in the process of achieving the best response eventually as an interpretation of how a steady state might be reached.

The game then has a stochastic steady state in which each player chooses each outcome with probability $\frac{1}{2}$. The steady state pattern of behavior can be visualized as spinning a fixed wheel of fortune (unchanging probability distribution) of choosing Head or Tail in $50-50$ chance for both players.

Expected payoff under mixed strategies

The expected payoff to Player I of the matrix game under mixed strategies is given by

$$
\begin{aligned}
E(X,\ Y) &= \sum_{i=1}^{n}\sum_{j=1}^{m} a_{ij} P[\text{I uses } i \text{ and II uses } j] \\
&= \sum_{i=1}^{n}\sum_{j=1}^{m} a_{ij} P[\text{I uses } i] P[\text{II uses } j] \quad \text{(independence)} \\
&= \sum_{i=1}^{n}\sum_{j=1}^{m} x_i a_{ij} y_j = (x_1, ..., x_n) A \begin{pmatrix} y_1 \\ . \\ . \\ . \\ y_m \end{pmatrix} = XAY^T.
\end{aligned}
$$

Since it is a zero sum game, the expected payoff to Player II is simply $-E(X,\ Y)$.

For an $n \times m$ matrix $A = (a_{ij})$, we write the j^{th} column and i^{th} row as

$$A_j = \begin{pmatrix} a_{1j} \\ a_{2j} \\ \cdot \\ \cdot \\ \cdot \\ a_{nj} \end{pmatrix} \text{ and } {}_iA = (a_{i1},\ a_{i2}, ..., a_{im}).$$

If Player I decides to use the pure strategy with row i used 100%, and Player II uses the mixed strategy, then for a fixed i, we have

$$E(i,\ Y) = \ {}_iAY^T = \sum_{j=1}^{m} a_{ij} y_j.$$

Similarly, for a fixed j, we have

$$E(X,\ j) = XA_j = \sum_{i=1}^{n} x_i a_{ij}.$$

Also, note that

$$E(X,\ Y) = \sum_{i=1}^{n} x_i E(i,\ Y) = \sum_{j=1}^{m} y_j E(X,\ j).$$

Saddle point in mixed strategies for two-player zero-sum game

A *saddle point* in mixed strategies is a pair $(X^*,\ Y^*)$ of probability vectors $X^* \in S_n$, $Y^* \in S_m$, that satisfies

$$E(X,\ Y^*) \leq E(X^*,\ Y^*) \leq E(X^*,\ Y),\ \forall X \in S_n \text{ and } Y \in S_m.$$

If Player I uses a strategy other than X^* but Player II still uses Y^*, then Player I receives an expected payoff less than or equal to that obtainable by staying with X^*. A similar statement holds for Player II.

Given Y^*, Player I's expected payoff is maximized by using X^*. That is, $E(X^*, Y^*) = \max_{X \in S_n} E(X, Y^*)$. In other words, X^* is the Player I's best response if Player II plays Y^*.

On the other hand, given X^*, Player II's expected loss is minimized by using Y^*. That is, $E(X^*, Y^*) = \min_{Y \in S_m} E(X^*, Y)$. Again, Y^* is the Player II's best response if Player I plays X^*.

We define the upper value and lower value of the zero-sum game under mixed strategies by

$$v^+ = \min_{Y \in S_m} \max_{X \in S_n} XAY^T \quad \text{and} \quad v^- = \max_{X \in S_n} \min_{Y \in S_m} XAY^T.$$

Lemma Existence of $(X^*, Y^*) \Rightarrow v^+ = v^- = E(X^*, Y^*)$

Proof

In general, we observe $v^- \leq v^+$ since

$$v^- = \max_X \min_Y E(X, Y) \leq \max_X E(X, Y) \quad \text{for any } Y.$$

To explain more clearly, for a given X, we observe $\min_Y E(X, Y) \leq E(X, Y)$ for any Y. Therefore, the maximum among X for $E(X, Y)$ is always greater than or equal to the maximum among X for $\min_Y E(X, Y)$. Note that the choices of X that give the above two maxima may not be the same.

Therefore, it remains true even we take $\min_Y (\max_X E(X, Y))$, so

$$v^- \leq \min_Y \max_X E(X, Y) = v^+.$$

Given the existence of (X^*, Y^*), we deduce that

$$\begin{aligned} v^+ = \min_Y \max_X E(X,Y) &\leq \max_X E(X,Y^*) = E(X^*,Y^*) \\ &= \min_Y E(X^*,Y) \leq \max_X \min_Y E(X,Y) = v^-. \end{aligned}$$

Combining the results, we obtain

$$v^+ = v^- = E(X^*, Y^*).$$

Here, v^+ is the ceiling on the loss of Player 2 and v^- is the floor on the gain of Player 1. Existence of at least one saddle point (X^*, Y^*) in mixed strategies is guaranteed for zero-sum game. Under the saddle point equilibrium, $E(X^*, Y^*)$ equals $v^+ = v^-$. Each player is settled with the maximum among the worst payoffs that can be imposed by his opponent.

Theorem (von Neumann)

For any zero-sum game, there is at least one saddle point $X^* \in S_n$ and $Y^* \in S_m$ such that

$E(X,\ Y^*) \leq E(X^*,\ Y^*) = v(A) \leq E(X^*,\ Y)$ for all $X \in S_n$ and $Y \in S_m$,

where A is $n \times m$ game matrix.

The upper and lower values of the mixed game are equal, and it is called the value of the matrix game $v(A)$. That is,

$$v^+ = \min_{Y \in S_m} \max_{X \in S_n} XAY^T = v(A) = \max_{X \in S_n} \min_{Y \in S_m} XAY^T = v^-.$$

Note that mixed saddle point strategy (X^*, Y^*) may not be unique but value of the game $v(A)$ is always unique.

Proof of the von Neumann Theorem

The proof relies on the *Brouwer fixed point theorem*: Let $K \subset \mathbb{R}^p$ be a closed, bounded and convex set. If the mapping T: $K \to K$ is continuous, then there exists $\widehat{x} \in K$ such that $T(\widehat{x}) = \widehat{x}$.

The fixed point under some chosen mapping T is related to the saddle point equilibrium pair, which arises from the notion that X^* is the best response to Y^* and vice versa (maximin and minimax criteria).

We use the notation: $x^+ = \max(x, 0)$. For any given (X, Y), we define

$$_i\Delta(X,Y) = (E(i,Y) - E(X,Y))^+, \Delta_j(X,Y) = (E(X,Y) - E(X,j))^+.$$

Note that $_i\Delta(X,Y)$ measures the amount that strategy i of row player is better than X as a response to the mixed strategy Y played by the column player, if higher payoff is indeed achieved under strategy i. Similar interpretation can be used for $\Delta_j(X,Y)$.

Write $X = (x_1, x_2, \dots, x_n)$ and $Y = (y_1, y_2, \dots, y_m)$, and define

$$x_i' = \frac{x_i + {}_i\Delta(X,Y)}{1 + \sum_{k=1}^{n} {}_k\Delta(X,Y)} \quad \text{and} \quad y_j' = \frac{y_j + \Delta_j(X,Y)}{1 + \sum_{k=1}^{m} \Delta_k(X,Y)}.$$

Note that $\displaystyle\sum_{i=1}^{n} x_i' = 1$ and $\displaystyle\sum_{j=1}^{n} y_j' = 1$; $0 \le x_i' \le 1$ and $0 \le y_j' \le 1$.
Therefore, we observe

$$X' = (x_1', x_2', \dots, x_n') \in S_n \quad \text{and} \quad Y' = (y_1', y_2', \dots, y_m') \in S_m.$$

Define $T : S_n \times S_m \to S_n \times S_m$ be the mapping

$$T((X,Y)) = (X', Y'),$$

which is seen to be continuous. Under this clever construction, we would like to establish that if $(\widehat{X}, \widehat{Y})$ is a fixed point of T, then it must be a saddle point equilibrium pair.

By the Brouwer fixed point theorem, fixed point of T always exists. Therefore, saddle point equilibrium pair always exist.

Fixed point $\Rightarrow$ saddle point

Suppose $\left(\widehat{X},\widehat{Y}\right)$ is a fixed point of T. We recall

$$E\left(\widehat{X},\widehat{Y}\right)=\sum_{i=1}^{n}\widehat{x}_iE\left(i,\widehat{Y}\right),\qquad \widehat{X}=\left(\widehat{x}_1,\widehat{x}_2,\cdots,\widehat{x}_n\right).$$

We deduce that $E\left(\widehat{X},\widehat{Y}\right)<E\left(i,\widehat{Y}\right)$ cannot hold for all i such that $\widehat{x}_i>0$. If otherwise, suppose $E\left(\widehat{X},\widehat{Y}\right)<E\left(i,\widehat{Y}\right)$ holds for all i with $\widehat{x}_i>0$, then

$$E\left(\widehat{X},\widehat{Y}\right)=\sum_{i=1}^{n}\widehat{x}_iE(i,\widehat{Y})>E\left(\widehat{X},\widehat{Y}\right)\sum_{i=1}^{n}x_i=E\left(\widehat{X},\widehat{Y}\right),$$

a contradiction. Therefore, there exists at least one i_0 such that $\widehat{x}_{i_0}>0$ and $E\left(\widehat{X},\widehat{Y}\right)\geq E\left(i_0,\widehat{Y}\right)$, so that

$${}_{i_0}\Delta\left(\widehat{X},\widehat{Y}\right)=\left(E\left(i_0,\widehat{Y}\right)-E\left(\widehat{X},\widehat{Y}\right)\right)^{+}=0.$$

Since $\left(\hat{X}, \hat{Y}\right)$ is a fixed point of T, in particular for i_0, we have $x'_{i_0} = \widehat{x}_{i_0} > 0$. On the other hand, since ${}_{i_0}\Delta\left(\hat{X}, \hat{Y}\right) = 0$, this gives

$$x'_{i_0} = \frac{\widehat{x}_{i_0}}{1 + \sum_{i=1}^{n} {}_{i}\Delta\left(\hat{X}, \hat{Y}\right)}.$$

Given that $\widehat{x}_{i_0} > 0$, the denominator must be equal to one and ${}_{i}\Delta\left(\hat{X}, \hat{Y}\right) \geq 0$ for all i, we deduce that $\sum_{i=1}^{n} {}_{i}\Delta\left(\hat{X}, \hat{Y}\right) = 0$, for all i. More precisely, we obtain ${}_{i}\Delta\left(\hat{X}, \hat{Y}\right) = (E(i, \hat{Y}) - E(\hat{X}, \hat{Y}))^{+} = 0$ for all i. This gives $E(i, \hat{Y}) \leq E(\hat{X}, \hat{Y})$. That is, all pure strategies cannot achieve better payoff than $\hat{X}$, so does any mixed strategy.

Hence, $\hat{X}$ is a best response against $\hat{Y}$. Similarly, we can show that $\hat{Y}$ is at least as good a response against $\hat{X}$. Hence, $\left(\hat{X}, \hat{Y}\right)$ is a saddle point equilibrium pair.

Saddle point $\Rightarrow$ fixed point

We may also establish the converse statement that a saddle point equilibrium pair must be a fixed point of T. Suppose (X^*, Y^*) is a saddle point equilibrium pair, where X^* is the best response to Y^* and vice versa, then ${}_i\Delta(X^*, Y^*) = 0$ and $\Delta_j(X^*, Y^*) = 0$, for all i and j, so that

$$T((X^*, Y^*)) = (X^*, Y^*).$$

Hence, (X^*, Y^*) is a fixed point of T.

System of linear inequalities

For pure strategies, it is straightforward to find $\max\limits_i a_{ij}$ for each column j and compute $v^+ = \min\limits_j(\max\limits_i a_{ij})$ by finding the smallest value among all column maxima.

How to construct the algebraic and graphical procedures to compute $v(A)$ under mixed strategies (at least for $2 \times m$ and $n \times 2$ zero-sum games), where $v^+ = \min\limits_{Y \in S_m} \max\limits_{X \in S_n} E(X,\ Y)$ is to be computed? As part of the solution procedure, the saddle point in mixed strategies (X^*, Y^*) is found.

Based on the results of a later Theorem, the computational procedure can be established by finding X^* and Y^* such that the following set of inequalities hold with some value v:

$$E(i,\ Y^*) \leq v \leq E(X^*,\ j),\ \ i = 1, 2, ..., n,\ \ j = 1, 2, ..., m.$$

One can establish that (X^*, Y^*) is a saddle point in mixed strategies and $v = E(X^*, Y^*)$ is the value of the game. Existence of (X^*, Y^*) and v is guaranteed by virtue of the von Neumann Theorem.

Lemma

If $X \in S_n$ is any mixed strategy for Player I and a is any number such that $E(X,\ j) \geq a$, $j = 1, 2, ..., m$, then for any $Y \in S_m$, it is also true that $E(X,\ Y) \geq a$. That is, if X is a good strategy for Player I that achieves expected payoff greater than a under any pure strategy used by Player II, then it is still a good strategy for Player I even if Player II uses a mixed strategy. Similarly, suppose $E(i,\ Y) \leq b$, $\forall i = 1, 2, ..., n$, then $E(X,\ Y) \leq b$ for any $X \in S_n$.

The proof is straightforward. Note that for all j

$$E(X,\ j) \geq a \Leftrightarrow \sum_i x_i a_{ij} \geq a.$$

Multiplying both sides by $y_j \geq 0$ and summing on j, we obtain

$$E(X,\ Y) = \sum_j \sum_i x_i a_{ij} y_j \geq \sum_j a y_j = a \text{ since } \sum_j y_j = 1.$$

The proof for the other part of the lemma is similar.

Theorem

Let $A = (a_{ij})$ be an $n \times m$ game with value $v(A)$. Let w be a real number. If we can find X^* and Y^* such that $E(i,\ Y^*) = \ _iAY^{*T} \leq w \leq E(X^*,\ j) = X^*A_j,\ i = 1, 2, ..., n,\ j = 1, 2, ...m$, then $w = v(A)$ and $(X^*,\ Y^*)$ is a saddle point for the game.

Proof

Given that $E(i,\ Y^*) = \sum_j a_{ij} y_j^* \leq w \leq \sum_i a_{ij} x_i^* = E(X^*,\ j)$, for all i and j, then by the lemma on the last page, we have

$$E(X^*,\ Y^*) \leq w \quad \text{and} \quad E(X^*,\ Y^*) \geq w.$$

Therefore, $w = E(X^*,\ Y^*)$. We now have

$$E(i,\ Y^*) \leq E(X^*,\ Y^*) \leq E(X^*,\ j) \text{ for any } i \text{ and } j.$$

Taking any strategies $X \in S_n$ and $Y \in S_m$, and using the lemma again, we obtain

$$E(X,\ Y^*) \leq E(X^*,\ Y^*) \leq E(X^*,\ Y)$$

so that $(X^*,\ Y^*)$ is a saddle point and $v(A) = E(X^*,\ Y^*) = w$.

Example - Evens and Odds Revisited

Recall the following game matrix

	I/II	Odds 1	2	3
	1	1	−1	1
Evens	2	−1	1	−1
	3	1	−1	1.

We calculated: $v^- = \max_i \min_j a_{ij} = -1$ and $v^+ = \min_j \max_i a_{ij} = 1$, so this game does not have a saddle point in pure strategies. Let us find its value and saddle point $(X^*,\ Y^*) = \Big((x_1, x_2, x_3), (y_1, y_2, y_3)\Big)$ under mixed strategies.

Note that strategy 3 is identical to strategy 1. A smart game theorist would guess the solution to be

$$X^* = (\alpha, \frac{1}{2}, \frac{1}{2} - \alpha),\ 0 \le \alpha \le \frac{1}{2};\ Y^* = (\beta, \frac{1}{2}, \frac{1}{2} - \beta),\ 0 \le \beta \le \frac{1}{2}.$$

The system of inequalities derived from $E(i,\ Y) \leq v$ and $E(X,\ j) \geq v$, $\forall i, j$, are

$$y_1 - y_2 + y_3 \leq v, \quad -y_1 + y_2 - y_3 \leq v, \text{ and } y_1 - y_2 + y_3 \leq v,$$
$$x_1 - x_2 + x_3 \geq v, \quad -x_1 + x_2 - x_3 \geq v, \text{ and } x_1 - x_2 + x_3 \geq v.$$

First, we assume $v \geq 0$ and see whether we can obtain sensible solution. Substituting $x_1 = 1 - x_2 - x_3$ into the first two inequalities for $x_1,\ x_2$ and x_3, we obtain

$$1 - 2x_2 \geq v \text{ and } -1 + 2x_2 \geq v \Rightarrow -v \geq 1 - 2x_2 \geq v.$$

This gives $v = 0$ so $x_2 = \frac{1}{2}$. Given that $v = 0$ and $x_2 = \frac{1}{2}$, this would force $x_1 + x_3 = \frac{1}{2}$ as well. Alternatively, instead of substituting for x_1, we substitute $x_2 = 1 - x_1 - x_3$. Again, we obtain $x_1 + x_3 = \frac{1}{2}$.

If we assume $v \leq 0$, we consider the solution for $y_1,\ y_2$ and y_3. We obtain $-v \leq 1 - 2y_2 \leq v$. Given $v \leq 0$, we deduce that $v = 0$. The same set of solution for $y_1,\ y_2$ and y_3 can be obtained.

Apparently, the third row is redundant, so does the third column. If we drop row 3, we obtain $x_2 = \frac{1}{2} = x_1$ (like having $x_3 = 0$). The corresponding saddle point and value are $X^* = (\frac{1}{2}, \frac{1}{2}, 0)$ and $Y^* = (\frac{1}{2}, \frac{1}{2}, 0)$ and $v = 0$.

Infinite number of saddle points

Remember that there are 3 rows and 3 columns in the game matrix. This gives an infinite number of saddle points in mixed strategies: $X^* = (x_1, \frac{1}{2}, \frac{1}{2} - x_1)$, $0 \le x_1 \le \frac{1}{2}$ and $Y^* = (y_1, \frac{1}{2}, \frac{1}{2} - y_1)$, $0 \le y_1 \le \frac{1}{2}$. Nevertheless, there is always one value of the game, namely, $v = 0$.

It is not surprising to observe zero value of the game since the two players are indifferent to serve as the row player or column player. This is a symmetric game. Later, we show that the value of a symmetric game is always zero.

Algebraic method for finding saddle point in mixed strategies and value of the game

The above computational procedure involves solution of system of algebraic inequalities. It would be more tractable to solve a system of algebraic equations. This motivates an alternative algebraic method based on the renowned Equality of Payoff Theorem.

Equality of Payoff Theorem

Let (X^*, Y^*) be a saddle point in mixed strategies. Provided that $y_j^* > 0$ and $x_i^* > 0$, we have equality (instead of inequality) of $E(X^*,\ j)$ and $v(A)$, same for $E(i, Y^*)$ and $v(A)$. More precisely, we have (i) $y_j^* > 0 \Rightarrow E(X^*,\ j) = v(A)$; (ii) $x_i^* > 0 \Rightarrow E(i,\ Y^*) = v(A)$.

If a saddle point mixed strategy for a player has a strictly positive probability of using a row or a column, then that row or column played against the opponent's saddle point mixed strategy will yield the value of the game.

The contrapositive statement dictates that when $E(X^*, j) > v(A)$, y_j^* must be 0. That is, when the column player's set of best response to X^* does not include strategy j [since $E(X^*, j) > v(A)$], then strategy j should never be played by the column player in a saddle point equilibrium.

However, when $E(X^*, j) = v(A)$, it is still possible to have $y_j^* = 0$ in a saddle point in mixed strategies. An example is revealed in the above matching game, where $E(X^*, 3) = x_1^* - x_2^* + x_3^* = 0$ since $x_1^* + x_3^* = x_2^*$. However, we have seen that $Y^* = (\frac{1}{2}, \frac{1}{2}, 0)$, with $y_3^* = 0$, is a saddle point in mixed strategies as well.

Proof

We prove by contradiction. Let $(X^*,\ Y^*)$ be a saddle point in mixed strategies. Recall that for any i, we have $E(i, Y^*) \leq v(A)$. Suppose there is a component of $X^* = (x_1^*, ..., x_k^*, ..., x_n^*)$ where $x_k^* > 0$ but $E(k,\ Y^*) < v(A)$. Now, for any i other than k, we have

$$E(i,\ Y^*) \leq v(A) \Rightarrow x_i E(i,\ Y^*) \leq x_i v(A) \text{ for any } x_i,\ x_i \in [0, 1].$$

Since $E(k,\ Y^*) < v(A)$ and $x_k^* > 0$, we obtain the strict inequality

$$x_k^* E(k,\ Y^*) + \sum_{\substack{i=1 \\ i \neq k}}^{n} x_i E(i,\ Y^*) < v(A)$$

while $x_1 + \cdots + x_{k-1} + x_k^* + x_{k+1} + \cdots + x_n = 1$. This is valid for any probability vector $X = (x_1, \ldots, x_k^*, \ldots, x_n)$ with the inclusion of the strictly positive value x_k^* at the k^{th} component. When we choose the probability vector to be the saddle strategy X^*, we obtain a contradiction since

$$v(A) = E(X^*,\ Y^*) = \sum_{i=1}^{n} x_i^* E(i,\ Y^*) < v(A).$$

The left hand equality arises from the property of a saddle point in mixed strategies while the right hand strict inequality is deduced from above. Therefore, if $x_k^* > 0$, then $E(k,\ Y^*) = v(A)$ [since we always have $E(k, Y^*) \leq v(A)$ while $E(k, Y^*) < v(A)$ is ruled out].

Remark In the above Evens and Odds game, $Y^* = (\beta, \frac{1}{2}, \frac{1}{2} - \beta)$. We observe $E(2, Y^*) = (-1, 1, -1)Y^{*T} = -\beta + \frac{1}{2} - (\frac{1}{2} - \beta) = 0 = v$. Similarly, $E(1, Y^*) = E(3, Y^*) = 0$.

Example

Consider the game matrix

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{pmatrix}.$$

We first assume $x_i > 0$, $y_j > 0$, $i, j = 1, 2, 3$, and check whether we obtain feasible saddle point in mixed strategies (to be checked after solution to X and Y are obtained). If this assumption fails to give the saddle point in mixed strategies, then one has to resort to other approaches. We have the following system of equations for $Y = (y_1, y_2, y_3)$:

$$\begin{aligned} &E(1,\ Y) = y_1 + 2y_2 + 3y_3 = v \\ &E(2,\ Y) = 3y_1 + y_2 + 2y_3 = v \\ &E(3,\ Y) = 2y_1 + 3y_2 + y_3 = v \\ &y_1 + y_2 + y_3 = 1. \end{aligned}$$

This gives $y_1 = y_2 = y_3 = \frac{1}{3}$ and $v = 2$. A similar approach shows that $X = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ is the saddle point in mixed strategies for Player I.

Recall

$$v(A) = \min_{Y \in S_m} \max_{X \in S_n} E(X,Y) = \max_{X \in S_n} \min_{Y \in S_m} E(X,Y).$$

The following lemma shows that the computational procedure for calculating $v(A)$ can be simplified. For a given X, we only need to find the minimum among $E(X,\ j)$, $j = 1, 2, ..., m$, instead of finding $\min_{Y \in S_m} E(X,\ Y)$. Here, Player II plays a pure strategy while Player I plays mixed strategies.

Lemma

The value of a zero-sum matrix game is given by

$$v(A) = \min_{Y \in S_m} \max_{1 \le i \le n} E(i,\ Y) = \max_{X \in S_n} \min_{1 \le j \le m} E(X,\ j).$$

To prove the Lemma, it suffices to show that

$$\min_{Y \in S_m} E(X,Y) = \min_{1 \le j \le m} E(X,j).$$

Proof

Since every pure strategy is also a mixed strategy, we have

$$\min_{Y\in S_m} E(X,\ Y) \leq \min_{1\leq j\leq m} E(X,\ j).$$

We write $\min\limits_{1\leq j\leq m} E(X,j) = a$ so that $E(X,j) - a \geq 0$ for any j, we have

$$0 \leq \min_{Y\in S_m} \sum_{j=1}^{m} y_j \Big[E(X,\ j) - a\Big] = \min_{Y\in S_m} E(X,\ Y) - a.$$

Combining the two inequalities, we have

$$a \leq \min_{Y\in S_m} E(X,\ Y) \leq \min_{1\leq j\leq m} E(X,\ j) = a,$$

so we have the result.

Solving 2×2 games graphically

Consider the game matrix

$$A = \begin{pmatrix} 1 & 4 \\ 3 & 2 \end{pmatrix},$$

it is seen that $v^- = \max_i \min_j a_{ij} = \max(1, 2) = 2$ and $v^+ = \min_j \max_i a_{ij} = \min_j(4, 3) = 3 \neq v^-$, so the saddle point strategies must be mixed.

Playing $X = (x, 1 - x)$ against each column for Player II, we obtain

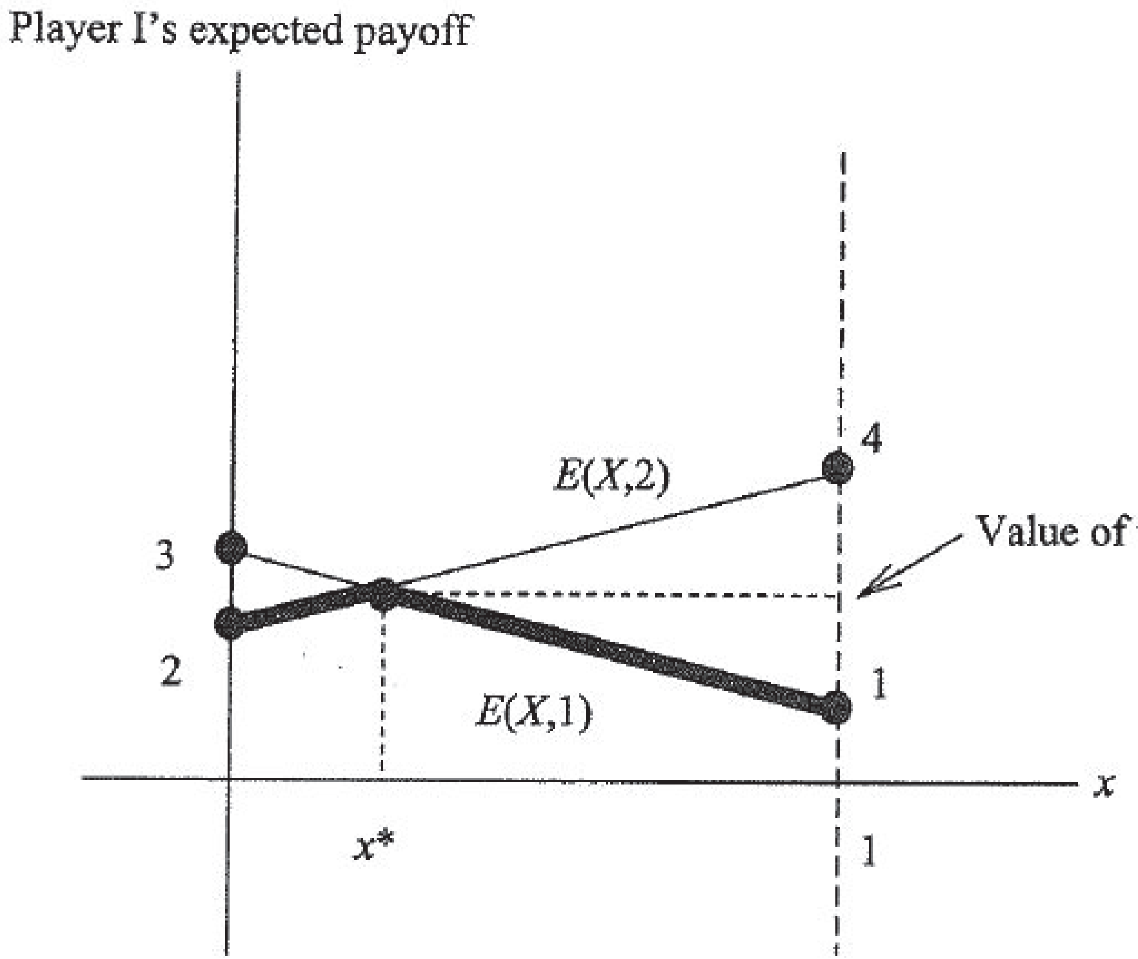
$E(X,\ 1) = XA_1 = x + 3(1 - x)$ and $E(X,\ 2) = XA_2 = 4x + 2(1 - x)$.

We plot $E(X,\ 1)$ and $E(X,\ 2)$ against x for $0 \leq x \leq 1$, and find the lower envelope $\min_j E(X, j)$. The two lines intersect at

$$x^* = \frac{1}{4} \text{ and } v = \frac{10}{4}.$$

Player I, assuming that Player II will be doing his best, will choose to play $X = (x^*,\ 1 - x^*) = (\frac{1}{4},\ \frac{3}{4})$.

Player I will choose x to achieve the maximum among minima of $E(X,1)$ and $E(X,2)$. For each $X=(x,1-x)$, $\min_{1\le j\le 2} E(X,j)$ lies on the lower envelope, represented by the bold lines. The maximum of minima is at the intersection, shown by the highest point of the lower envelope (shown by the bold line segments).



If Player I chooses an $x < x^*$, we deduce that Player II should definitely not to use column 1 but should use column 2 since $E(X,\ 2) < \frac{10}{4}$ for $x < x^*$. $E(x, 2)$ lies on the lower envelope when $x < x^*$.

Player I will choose a mixed strategy $X^* = \left(\frac{1}{4}, \frac{3}{4}\right)$ so that he will get $\frac{10}{4}$ no matter what Player II does.

Player I is indifferent to Y when $X^* = \left(\frac{1}{4}, \frac{3}{4}\right)$ since $E(X^*, 1) = E(X^*, 2) = \frac{10}{4}$ under this mixed strategy, so $E(X^*, Y) = y_1 E(X^*, 1) + y_2 E(X^*, 2) = \frac{10}{4}$, independent of y_1 and y_2.

To find $Y^* = (y_1^*, y_2^*)$, we use the Equality of Payoff Theorem to obtain

$$\begin{aligned} E(1, Y^*) &= y_1^* + 4y_2^* = v \\ E(2, Y^*) &= 3y_1^* + 2y_2^* = v \\ y_1^* + y_2^* &= 1 \end{aligned}$$

giving $v = \frac{10}{4}$ and $y_1^* = y_2^* = \frac{1}{2}$ or $Y^* = (\frac{1}{2}, \frac{1}{2})$.

Graphical solution of $2 \times \mathrm{m}$ games

Since there are only two rows, a mixed strategy of Player I is determined by the choice of $X = (x, 1-x)$, where $x \in [0, 1]$. Through minimizing $XA_j = E(X, j)$ over j for a given value of x, we define

$$f(x) = \min_{1 \leq j \leq m} XA_j = \min_{1 \leq j \leq m} \left\{ xa_{1j} + (1-x)a_{2j} \right\}.$$

This is called the lower envelope of all the straight lines $E(X, j)$ associated to each pure strategy j for Player II. Afterwards, we search for x^* that achieves maxima in the lower envelope such that

$$f(x^*) = \max_{0 \leq x \leq 1} f(x) = \max_x \min_j E(X,\ j).$$

Each line represents the payoff that Player I would receive by placing the mixed strategy $X = (x, 1-x)$ with Player II always playing a fixed column. This graphical method is consistent with the following result:

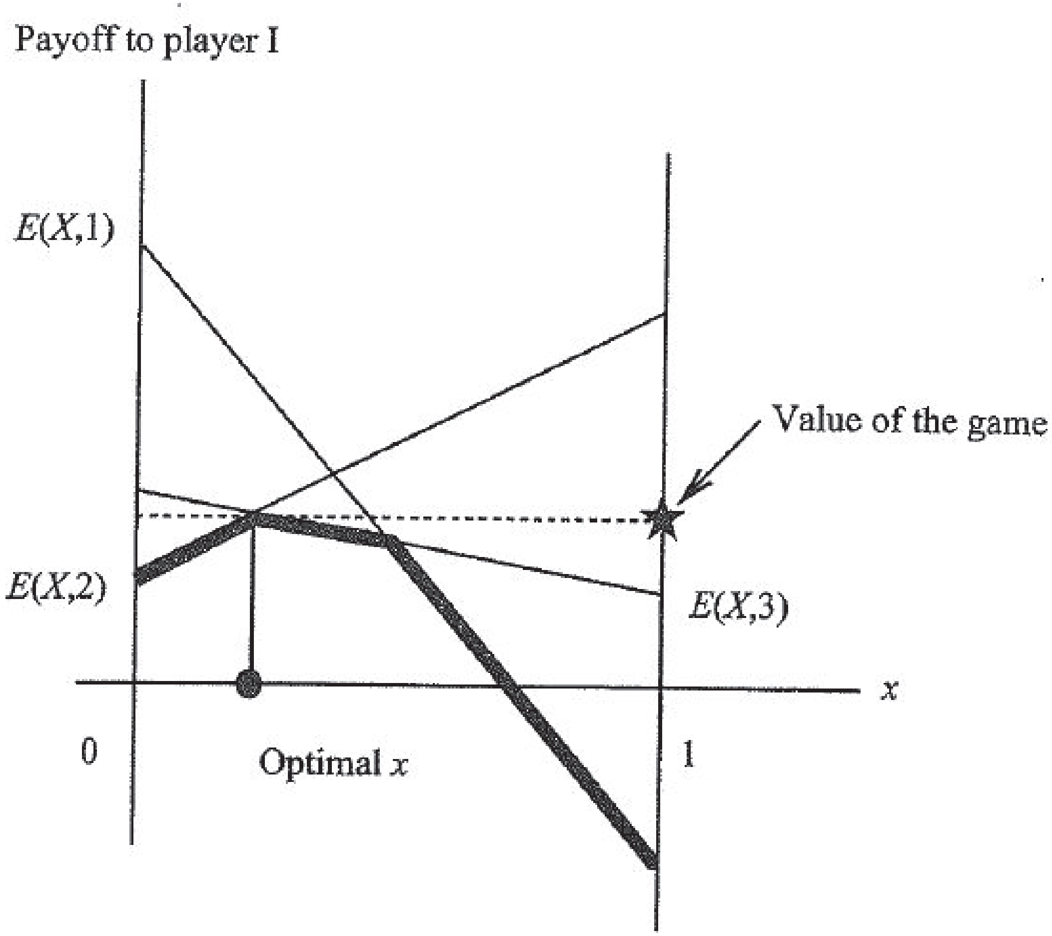
$$v(A) = \max_{X \in S_n} \min_{1 \leq j \leq m} E(X,\ j).$$

Graphical illustration when $m = 3$

Suppose there are 3 column strategies for Player II. We plot $E(X, 1)$, $E(X, 2)$ and $E(X, 3)$, and find the lower envelope of these 3 lines plotted with $0 \leq x \leq 1$.

For a given x, the best response by Player II is to play $\min_j E(X,\ j)$ (points along the bold line segments). Player I chooses x^* such that $\min_j E(X,\ j)$ [lower envelope of $E(X, 1)$, $E(X, 2)$ and $E(X, 3)$] is maximized among all choices of x.

The optimal x^* is NOT given by the maximum among all intersection points. To solve for $X^* = (x^*, 1 - x^*)$ for Player I, we only look at the lower envelope and find the highest point in the lower envelope.



Note that $E(X^*, 1) > v(A)$, $E(X^*, 2) = E(X^*, 3) = v(A)$, where $X^* = (x^*, 1 - x^*)$. Since $E(X^*, 1) > v(A)$, column strategy 1 is ruled out in mixed strategies played by Player II in any saddle point; that is, $y_1^* = 0$.

- If Player I decides to play the mixed strategy $X = (x, 1-x)$, where $x < x^*$, then Player II would choose to play column 2 since $E(X, 2)$ constitutes to part of the lower envelope when $x < x^*$.

- If Player I decides to play the mixed strategy $X = (x, 1-x)$, where $x > x^*$, then Player II would choose to play column 3, up to the intersection of $E(X,\ 1) = E(X,\ 3)$ and then switch to column 1.

- Once Player I chooses x^*, Player II would play some combination of columns 2 and 3. It would be a convex combination of these two columns (zero probability of playing column 1). Recall that column 1 is ruled out since $E(X^*, 1) > v(A)$.

- Suppose Player II chooses to play the pure strategy column 2, then Player I could do better by changing his mixed strategy from x^* to some $x > x^*$, taking $x = 1$. This explains why Player II should play mixed strategies in order that the saddle point equilibrium can substain.

Graphical solution of $\mathbf{n} \times \mathbf{2}$ game

For a $n \times 2$ game where Player II uses the mixed strategy $Y = (y, 1-y)$, $0 \le y \le 1$. Player II would choose y to minimize

$$\max_{1 \le i \le n} E(i,\ Y) = \max_{1 \le i \le n}\ {}_iAY^T = \max_{1 \le i \le n} y(a_{i1}) + (1-y)(a_{i2}).$$

As his best response to a given Player II's mixed strategy Y, Player I wants to achieve largest values for $E(i,\ Y)$ as much as possible. Player II responds optimally by playing Y that will give the lowest maximum.

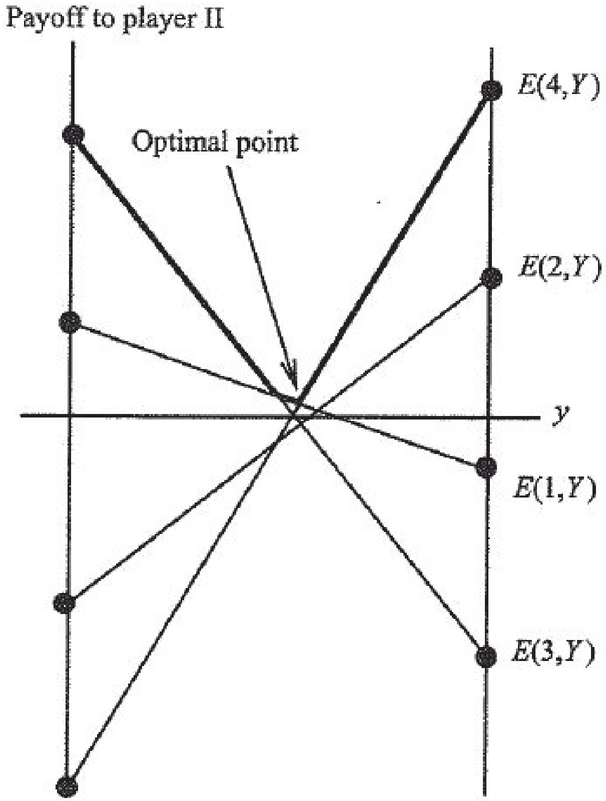
The optimality y^* will be the point giving the minimum of the upper envelope. Again, the graphical method is consistent with the following result:

$$v(A) = \min_{Y \in S_m} \max_{1 \le i \le n} E(i,\ Y).$$

That is, we find the lowest point in the upper envelope of the 4 lines: $E(1, Y)$, $E(2, Y)$, $E(3, Y)$ and $E(4, Y)$.

Example

$$A = \begin{pmatrix} -1 & 2 \\ 3 & -4 \\ -5 & 6 \\ 7 & -8 \end{pmatrix}$$



Mixed for player II versus 4 rows for player I.

The saddle point mixed strategy Y will be determined at the intersection point of

$$E(4,\ Y) = 7y - 8(1-y) \text{ and } E(1,\ Y) = -y + 2(1-y).$$

This occurs at the point $y^* = \frac{5}{9}$ and the corresponding $v(A) = \frac{1}{3}$.

Since the intersection point involves $E(4,\ Y)$ and $E(1,\ Y)$ only, so we may drop rows 2 and 3 in finding the saddle point mixed strategy for Player I. After dropping rows 2 and 3 from the game matrix, it reduces to a 2×2 game.

We calculate

$$E(X,\ 1) \;=\; \begin{pmatrix} x & 0 & 0 & 1-x \end{pmatrix} \begin{pmatrix} -1 \\ 3 \\ -5 \\ 7 \end{pmatrix} = -x + 7(1-x)$$

$$E(X,\ 2) \;=\; \begin{pmatrix} x & 0 & 0 & 1-x \end{pmatrix} \begin{pmatrix} 2 \\ -4 \\ 6 \\ -8 \end{pmatrix} = 2x - 8(1-x).$$

They intersect at $x = \frac{5}{6}$ and give $v(A) = \frac{1}{3}$ (same as before). Row 1 should be used with probability $\frac{5}{6}$ and row 4 with probability $\frac{1}{6}$, so $X^* = (\frac{5}{6}, 0, 0, \frac{1}{6})$. Note that $x_2^* = x_3^* = 0$.

It is most likely that the intersection point involves only two of $E(i,\ Y)$, $i = 1, 2, ..., n$. In this example, the optimal point is given by the intersection of $E(1, Y)$ and $E(4, Y)$. As a result, only x_1^* and x_4^* are strictly positive. Recall the result: $x_i^* > 0 \Rightarrow E(i, Y^*) = v(A)$, or equivalently, $E(i, Y^*) < v(A) \Rightarrow x_i^* = 0$. Graphically, we observe $E(2, Y^*) = -1/9 < v$ and $E(3, Y^*) = -1/9 < v$.

We would like to check whether

$$E(i,\ Y^*) \leq v(A) \leq E(X^*,\ j)$$

for all rows and columns. Note that the components of X^*A give $E(X^*,\ j)$, $j = 1, 2$, where

$$X^*A = (\frac{5}{6}\ 0\ 0\ \frac{1}{6}) \begin{pmatrix} -1 & 2 \\ 3 & -4 \\ -5 & 6 \\ 7 & -8 \end{pmatrix} = (\frac{1}{3}\ \frac{1}{3}) = (E(X^*, 1), E(X^*, 2)).$$

Similarly, the components of AY^{*T} give $E(i,\ Y^*)$, $i = 1, 2, 3, 4$, where

$$AY^{*T} = \begin{pmatrix} -1 & 2 \\ 3 & -4 \\ -5 & 6 \\ 7 & -8 \end{pmatrix} \begin{pmatrix} \frac{5}{9} \\ \frac{4}{9} \end{pmatrix} = \begin{pmatrix} \frac{1}{3} \\ -\frac{1}{9} \\ -\frac{1}{9} \\ \frac{1}{3} \end{pmatrix} = \begin{pmatrix} E(1, y^*) \\ E(2, y^*) \\ E(3, y^*) \\ E(4, y^*) \end{pmatrix}.$$

Here, $E(1,\ Y^*) = E(4,\ Y^*) = \frac{1}{3} = v(A)$ and $E(2,\ Y^*) = E(3,\ Y^*) = -\frac{1}{9} < v(A)$, confirming that $x_2^* = x_3^* = 0$. This is because when $E(i,\ Y^*) < v(A)$, then $x_i^* = 0$.

Note that even $E(i, Y^*) = v(A)$, it may still be possible to choose $x_i^* = 0$ in the saddle point mixed strategy for Player I (see the Evens and Odds game).

Invertible matrix games (provided that A^{-1} exists)

Let the game matrix A be a square matrix with equal number of rows and columns. Let (X^*, Y^*) be a saddle point mixed strategies. Suppose that Player I has a saddle point strategy that is completely mixed, $X^* = (x_1^*, ..., x_n^*)$, and $x_i^* > 0$, $i = 1, 2, ..., n$. We then have

$$E(i,\ Y^*) = \ _iAY^{*T} = v(A),\ \forall\ i = 1, 2, ..., n.$$

That is, the saddle point mixed strategy Y^* of Player II played against any row will give the value of the game. We write $J_n = (1\ 1\ ...\ 1)$ and

$$AY^{*T} = \begin{pmatrix} v(A) \\ . \\ . \\ . \\ v(A) \end{pmatrix} = v(A) \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = v(A) J_n^T.$$

Based on the assumptions that (i) $x_i^* > 0$, $\forall i$, and (ii) the game matrix is invertible, we obtain

$$Y^{*T} = v(A) A^{-1} J_n^T.$$

To determine $v(A)$, we apply the condition that sum of probabilities equals 1: $J_n Y^{*T} = 1$. This gives

$$J_n Y^{*T} = 1 = v(A) J_n A^{-1} J_n^T$$

so that

$$v(A) = \frac{1}{J_n A^{-1} J_n^T} \text{ and } Y^{*T} = \frac{A^{-1} J_n^T}{J_n A^{-1} J_n^T}.$$

In a similar manner, by assuming that Player II has a saddle point strategy that is completely mixed, $Y^* = (y_1, ..., y_n)$, $y_i^* > 0$, $i = 1, 2, ..., n$, we have

$$X^* A = v(A) J_n \quad \text{so} \quad X^* = \frac{J_n A^{-1}}{J_n A^{-1} J_n^T}.$$

If $v(A) = 0$, then $AY^{*T} = 0$. Suppose A^{-1} exists, then $Y^{*T} = \mathbf{0}$. This is impossible if Y^* is a probability vector. Therefore, the value of the game cannot be zero under (i) existence of A^{-1} and (ii) completely mixed strategy. As an example, in the Evens and Odds game, the value is zero and A^{-1} does not exist.

For $A = \begin{pmatrix} 1 & 4 \\ 3 & 2 \end{pmatrix}$, we obtain $A^{-1} = \dfrac{1}{\begin{vmatrix} 1 & 4 \\ 3 & 2 \end{vmatrix}} \begin{pmatrix} 2 & -4 \\ -3 & 1 \end{pmatrix} = -\dfrac{1}{10} \begin{pmatrix} 2 & -4 \\ -3 & 1 \end{pmatrix}$.

Note that $1/v(A) = J_n A^{-1} J_n^T = (1\ 1) \left(-\dfrac{1}{10}\right) \begin{pmatrix} 2 & -4 \\ -3 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \dfrac{4}{10}$, so $v(A) = \dfrac{10}{4}$ and

$$
\begin{aligned}
X^* &= \frac{(1\ 1)\left(-\frac{1}{10}\right)\begin{pmatrix} 2 & -4 \\ -3 & 1 \end{pmatrix}}{4/10} = \begin{pmatrix} \frac{1}{4} & \frac{3}{4} \end{pmatrix} \\
Y^* &= \frac{\left(-\frac{1}{10}\right)\begin{pmatrix} 2 & -4 \\ -3 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}}{4/10} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \end{pmatrix}.
\end{aligned}
$$

We observe that $y_1^* > 0$ and $y_2^* > 0$, implying $E(X^*, 1) = E(X^*, 2) = v(A)$. These results are verified as follows:

$$
\begin{aligned}
E(X^*, 1) &= \begin{pmatrix} \frac{1}{4} & \frac{3}{4} \end{pmatrix} \begin{pmatrix} 1 & 4 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{10}{4}, \\
E(X^*, 2) &= \begin{pmatrix} \frac{1}{4} & \frac{3}{4} \end{pmatrix} \begin{pmatrix} 1 & 4 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{10}{4}.
\end{aligned}
$$

Remark

Qn 7 in HW1 shows an example where the computational formulas remain applicable even when the saddle point strategies may not be completely mixed. That is, even though $x^*_{i_0} = 0$ or $y^*_{j_0} = 0$ for some i_0 or j_0, it may still be possible to have

$$E(i_0, Y^*) = v \quad \text{and} \quad E(X^*, j_0) = v.$$

Note that $x^*_i > 0$ and $y^*_j > 0$, for any i and j, are sufficient conditions to ensure

$$E(i, Y^*) = v \quad \text{and} \quad E(X^*, j) = v.$$

In other word, positivity of x^*_i or y^*_j, for any i and j, may not be the required condition for $E(i, Y^*) = v$ or $E(X^*, j) = v$.

Calculus solution for 2×2 games

Consider the following 2×2 game matrix:

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

and mixed strategies: $X^* = (x, 1-x)$ for Player I and $Y^* = (y, 1-y)$ for Player II. Recall

$$A^{-1} = \frac{1}{\det A} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}, \text{ where } \det A = a_{11}a_{22} - a_{12}a_{21}.$$

For any mixed strategies (X^*, Y^*) in the 2×2 game, the expected payoff is a quadratic function in x and y, where

$$\begin{aligned} E(X^*,\ Y^*) &= X^* A Y^{*T} \\ &= \begin{pmatrix} x & 1-x \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} y \\ 1-y \end{pmatrix} \\ &= xy(a_{11} - a_{12} - a_{21} + a_{22}) + x(a_{12} - a_{22}) \\ &\quad + y(a_{21} - a_{22}) + a_{22} \\ &= f(x, y). \end{aligned}$$

Recall that pure strategy means that one of the components in the probability vector is zero. We assume that there are no saddle point in pure strategies. First, we seek for the interior critical points of f that are found inside the unit square region: $0 < x, y < 1$. Consider

$$\frac{\partial f}{\partial x} = y\alpha + \beta = 0 \text{ and } \frac{\partial f}{\partial y} = x\alpha + \gamma = 0$$

where $\alpha = a_{11} - a_{12} - a_{21} + a_{22}$, $\beta = a_{12} - a_{22}$ and $\gamma = a_{21} - a_{22}$.

There exists one critical point that satisfies

$$\frac{\partial f}{\partial x} = 0 \quad \text{and} \quad \frac{\partial f}{\partial y} = 0.$$

Later, we show that the critical point is a saddle point.

If $\alpha = 0$, the partial derivatives are never zero (assuming $\beta \neq 0$ and $\gamma \neq 0$). Note that saddle point in pure strategies always exists when $\alpha = 0$ (see the later proof). This is ruled out in our assumption.

Assuming $\alpha \neq 0$, then

$$x^* = -\frac{\gamma}{\alpha} = \frac{a_{22} - a_{21}}{a_{11} - a_{12} - a_{21} + a_{22}},$$
$$y^* = -\frac{\beta}{\alpha} = \frac{a_{22} - a_{12}}{a_{11} - a_{12} - a_{21} + a_{22}}.$$

In order to ensure that x^* and y^* is a saddle point in completely mixed strategies, it is necessary to check: $0 < x^* < 1$ and $0 < y^* < 1$. If otherwise, there will be no completely mixed saddle point.

The expected payoff to Player II is indifferent to x when $y = y^*$ since $\frac{\partial f}{\partial x} = 0$ when $y = -\beta/\alpha$. Similarly, the expected payoff to Player I is indifferent to y when $x = x^*$ since $\frac{\partial f}{\partial y} = 0$ when $x = -\gamma/\alpha$. See the example on P.61-63 as an illustration of such properties.

When one player plays his part of the saddle point strategy, his expected payoff is indifferent to the strategy played by his opponent.

Provided that A^{-1} exist (det $A \neq 0$) and $\alpha \neq 0$, $X^* = (x, 1-x)$ and $Y^* = (y^*, 1-y^*)$ admit the following representation:

$$X^* = \frac{(1\ 1)A^{-1}}{(1\ 1)A^{-1}\begin{pmatrix}1\\1\end{pmatrix}} = \begin{pmatrix} -\frac{\gamma}{\alpha} \\ 1+\frac{\gamma}{\alpha}\end{pmatrix} \text{ and } Y^* = \frac{A^{-1}\begin{pmatrix}1\\1\end{pmatrix}}{(1\ 1)A^{-1}\begin{pmatrix}1\\1\end{pmatrix}} = \begin{pmatrix} -\frac{\beta}{\alpha} \\ 1+\frac{\beta}{\alpha}\end{pmatrix}.$$

The corresponding value of the game is given by

$$v(A) = \frac{1}{(1\ 1)A^{-1}\begin{pmatrix}1\\1\end{pmatrix}} = \frac{\det\ A}{(1\ 1)\begin{pmatrix}a_{22} & -a_{12}\\ -a_{21} & a_{11}\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix}} = \frac{\det\ A}{\alpha}.$$

Two mathematical queries:

1. What happens when $\alpha = 0$?

2. Why an interior critical point inside $0 < x, y < 1$ must be a saddle point but not a local min or local max of f?

1. We check that a saddle point in pure strategies exists when $\alpha = 0$. As an illustration, we rewrite the game matrix as

$$\begin{pmatrix} a_{11} & a_{11} - u \\ a_{11} - v & a_{11} - u - v \end{pmatrix},$$

observing $\alpha = 0$, where $u > 0$ and $v > 0$ are assumed. We have $v^+ = \min_j \max_i a_{ij} = \min(a_{11}, a_{11} - u) = a_{11} - u$ and $v^- = \max_i \min_j a_{ij} = \max(a_{11} - u, a_{11} - u - v) = a_{11} - u = v^+$, so a saddle point in pure strategies exists that observes the row-min column-max property. Similarly, suppose $u \leq 0$ and $v > 0$, then a_{11} is a saddle point in pure strategies. The remaining two other cases (i) $u > 0$ and $v \leq 0$, (ii) $u \leq 0$ and $v \leq 0$, also lead to pure strategy saddle point.

2. Recall a theorem in multivariate calculus which states that an interior critical point with negative determinant of Hessian is a saddle point. To verify the result, we observe that

$$\det\, H = \det \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} = \det \begin{pmatrix} 0 & \alpha \\ \alpha & 0 \end{pmatrix} = -\alpha^2 < 0.$$

Elimination by dominance

We may reduce the size of the game matrix A by eliminating rows or columns (that is, strategies) that will never be used because there is always a better row or column to use. This procedure is called the *elimination by dominance*.

Suppose in row i and row k, we observe $a_{ij} \geq a_{kj}$, $j = 1, \ldots, m$, and strict inequality for at least one j, then the row player I would never play row k. In this case, row k is a dominated strategy, so we can drop it from the game matrix.

Similarly, suppose in column j and column k, we observe $a_{ij} \leq a_{ik}$, $i = 1, 2, \ldots, n$, and strict inequality for at least one i, then the column player II would never play column k and so we can drop it from the matrix.

Redundancy associated with a duplicate row

A *dominated* row is dropped and it is played in a mixed strategy with zero probability. But a row that is dropped because it is equal to another row may not have zero probability of being played (for example, see the Evens and Odds game).

Suppose that we have a matrix with three rows and row 1 is the same as row 3. If we drop row 3, we now have two rows and the resulting saddle point mixed strategy will look like $X^* = (x_1, x_2)$ for the reduced game. For the original game, the set of all saddle point mixed strategies for player I would consist $X^* = (\lambda x_1, x_2, (1-\lambda)x_1)$ for any $0 \le \lambda \le 1$, and this is the most general solution. While the probability of playing row 2 remains the same, the probability x_1 of playing row 1 in the reduced matrix is split into the probabilities of playing row 1 and row 3 in the original matrix.

A duplicate row is a redundant row and may be dropped to reduce the size of the matrix. However, one may need to account for the redundant strategies in the most general solution.

Dominance through convex combination of strategies

Another solution method used to reduce the size of a matrix is to drop rows or columns by dominance through a *convex combination* of other rows or columns. If a row (or column) is (strictly) dominated by a convex combination of other rows (or columns), then this row (column) can be dropped from the matrix.

For example, row k is dominated by a convex combination of two other rows, say, p and q, then we can drop row k. This means that if there is constant $\lambda \in [0, 1]$ so that

$$a_{kj} \leq \lambda a_{pj} + (1 - \lambda) a_{qj}, \quad j = 1, \ldots, m,$$

then row k is dominated and can be dropped.

If the constant $\lambda = 1$, then row p dominates row k. If $\lambda = 0$ then row q dominates row k. In both cases, we can drop row k. More than two rows can be involved in the convex combination.

Example

Consider the 3×4 game

$$A = \begin{pmatrix} 10 & 0 & 7 & 4 \\ 2 & 6 & 4 & 7 \\ 5 & 2 & 3 & 8 \end{pmatrix}.$$

In column 4, every number in that column is larger than each corresponding number in column 2. Player II should drop column 4 and the game matrix is reduced to

$$\begin{pmatrix} 10 & 0 & 7 \\ 2 & 6 & 4 \\ 5 & 2 & 3 \end{pmatrix}.$$

There is no obvious dominance of one row by another or one column by another. However, we suspect that row 3 is dominated by a convex combination of rows 1 and 2. If that is true we must have, for some $0 \leq \lambda \leq 1$, the inequalities

$$5 \leq 10(\lambda) + 2(1-\lambda),\ 2 \leq 0(\lambda) + 6(1-\lambda),\ 3 \leq 7(\lambda) + 4(1-\lambda).$$

Simplifying, $5 \le 8\lambda + 2$, $2 \le 6 - 6\lambda$, $3 \le 3\lambda + 4$. We see that any λ that satisfies $\frac{3}{8} \le \lambda \le \frac{2}{3}$ will work. So, there is a λ that works to cause row 3 to be dominated by a convex combination of rows 1 and 2, and row 3 may be dropped from the matrix (i.e., an optimal mixed strategy will play row 3 with probability 0).

To ensure dominance by a convex combination, all we have to show is that there are λ's lying in $[0, 1]$ that satisfy all the inequalities.

The new reduced matrix is

$$\begin{pmatrix} 10 & 0 & 7 \\ 2 & 6 & 4 \end{pmatrix}.$$

Again there is no obvious dominance, but it is a reasonable guess that column 3 is a bad column for player II and that it might be dominated by a combination of columns 1 and 2. To check, we need to have

$$7 \ge 10\lambda + 0(1 - \lambda) = 10\lambda \quad \text{and} \quad 4 \ge 2\lambda + 6(1 - \lambda) = -4\lambda + 6.$$

These inequalities require that $\frac{1}{2} \le \lambda \le \frac{7}{10}$, which is fine. So there are λ's that work, and column 3 may be dropped. Finally, we are down to a 2×2 matrix

$$\begin{pmatrix} 10 & 0 \\ 2 & 6 \end{pmatrix}.$$

This reduced game can be solved graphically or via the computational formulas. The value of the game is found to be $v(A) = \frac{30}{7}$ and the saddle point mixed strategies for the original game are $X^* = \left(\frac{2}{7}, \frac{5}{7}, 0\right)$ and $Y^* = \left(\frac{3}{7}, \frac{4}{7}, 0, 0\right)$. As a check, we compute

$$X^*A = \left(\frac{2}{7}, \frac{5}{7}, 0\right) \begin{pmatrix} 10 & 0 & 7 & 4 \\ 2 & 6 & 4 & 7 \\ 5 & 2 & 3 & 8 \end{pmatrix} = \left(\frac{30}{7}, \frac{30}{7}, \frac{34}{7}, \frac{43}{7}\right).$$

Since $E(X^*, 3) = \frac{34}{7} > v(A)$ and $E(X^*, 4) = \frac{43}{7} > v(A)$, thus $y_3^* = y_4^* = 0$. We can perform a similar procedure to compute AY^{*T} to verify that $E(3, Y^*) < v$, so $x_3^* = 0$.

Example – non-strict dominance

We may lose solutions when we reduce by nonstrict dominance. Consider the game with matrix

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix}.$$

Again there is no obvious dominance, but it is easy to see that row 1 is dominated (nonstrictly) by a convex combination of rows 2 and 3. In fact $a_{1j} = \frac{1}{2}a_{2j} + \frac{1}{2}a_{3j}$, $j = 1, 2, 3$.

If we drop row 1, then column 1 is dominated (nonstrictly) by a convex combination of columns 2 and 3 and may be dropped. This leaves us with the reduced matrix $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$.

The solution for this reduced game is $v = 1$, $X^* = \left(\frac{1}{2}, \frac{1}{2}\right) = Y^*$. Consequently, a solution of the original game is $v(A) = 1$, $X^* = \left(0, \frac{1}{2}, \frac{1}{2}\right) = Y^*$.

There is a saddle point in pure strategies for this game given by $X^* = (1, 0, 0)$, $Y^* = (1, 0, 0)$ and this is missed by using non-strict dominance. Playing row 1 gives the same expected payoff as playing row 2 and row 3 in equal probability.

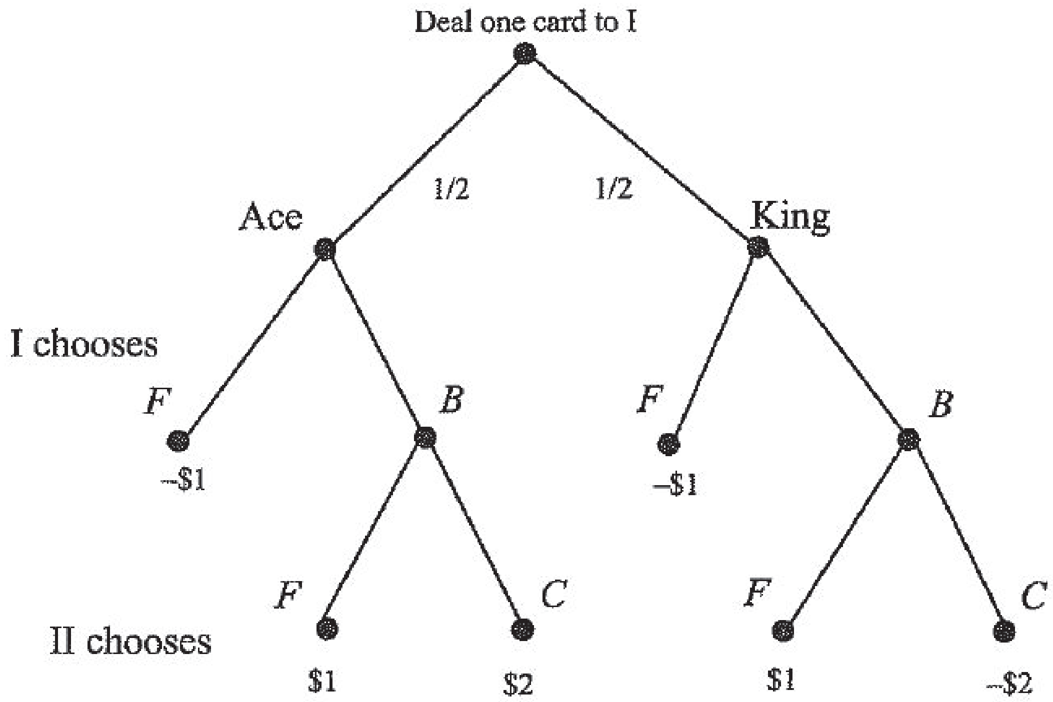
The probabilities of playing row 2 and row 3 remain the same, so for column 2 and column3. The most general solution is

$$X^* = \left(1 - \alpha,\ \frac{\alpha}{2},\ \frac{\alpha}{2}\right),\ 0 \leq \alpha \leq 1 \text{ and } Y^* = \left(1 - \beta,\ \frac{\beta}{2},\ \frac{\beta}{2}\right),\ 0 \leq \beta \leq 1.$$

The above two solutions correspond to $\alpha = 1$ and $\alpha = 0$, respectively.

Dropping rows or columns that are strictly dominated means that the dropped row or column is never played. However, dropping rows or column under non-strict dominance may miss some saddle point strategies.

Endgame in a poker



- If I folds, he has to pay II $\$1$.
- If I bets, Player II may choose to fold or call.
- If II folds, he pays I $\$1$. If Player II calls and the card is a king, then I pays II $\$2$; if the card is an ace, then Player II pays Player I $\$2$.

Player I has 4 strategies (fold or bet upon receipt of ace or king):

FF = fold on ace and fold on king (always fold, why enter the game)
FB = fold on ace and bet on king (insensible)
BF = bet on ace and fold on king
BB = bet on ace and bet on king

Note that FF and FB are eliminated by dominance argument.

Player II has 2 strategies:

F = fold or C = call

Player II knows that I has bet, but he does not know which branch the "bet" came from.

$E(FF,\ C) = E(FF, F) = -1$ since Player I always lose.

$E(FB,\ F) = \frac{1}{2}(-1) + \frac{1}{2} \cdot 1 = 0$; $E(FB,\ C) = \frac{1}{2}(-1) + \frac{1}{2}(-2) = -\frac{3}{2}$;

$E(BB,\ C) = 0$ since 50% chance of winning;

$E(BB,\ F) = 1$ since winning \$1 for sure;

$E(BF,\ C) = \frac{1}{2} \cdot 2 + \frac{1}{2}(-1) = \frac{1}{2}$; $E(BF,\ F) = \frac{1}{2} \cdot 1 + \frac{1}{2}(-1) = 0$.

I/II	C	F
FF	-1	-1
FB	$-\frac{3}{2}$	0
BF	$\frac{1}{2}$	0
BB	0	1

- The lower and upper values are $v^- = \max\left(-1, -\frac{3}{2}, 0, 0\right) = 0$ and $v^+ = \min\left(\frac{1}{2}, 1\right) = \frac{1}{2}$, so there is no saddle point in pure strategies.
- Row 2 is strictly dominated by Row 4, and it can be dropped. After dropping Row 2, Row 1 becomes a strictly dominated strategy, so we may drop it.

We are left behind with the reduced game matrix

$$A = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix}.$$

Suppose Player II plays $Y = (y, 1-y)$, then

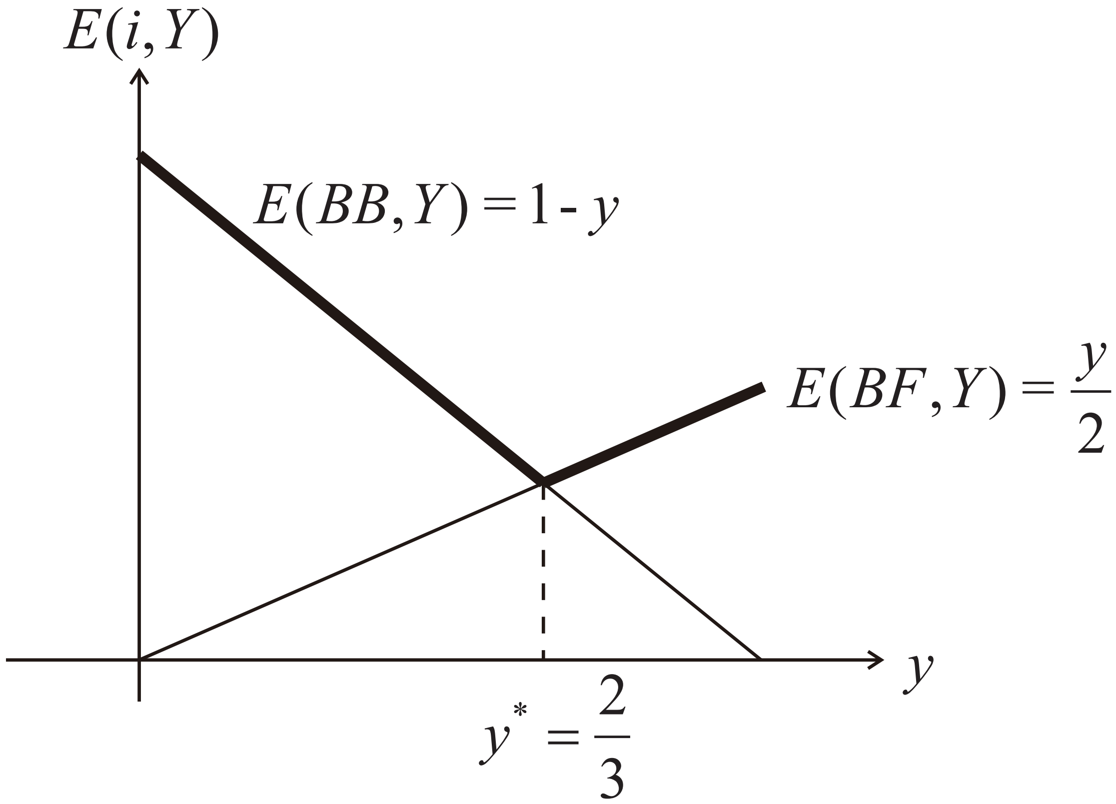
$$E(BF,\ Y) = \frac{y}{2} \text{ and } E(BB,\ Y) = 1 - y.$$

These two lines intersect at $y^* = \frac{2}{3}$. The saddle point mixed strategy for Player II is $Y^* = (\frac{2}{3}, \frac{1}{3})$. As a result, II should call $\frac{2}{3}$ of the time and fold $\frac{1}{3}$ of the time. The value of the game is $v = \frac{1}{3}$.

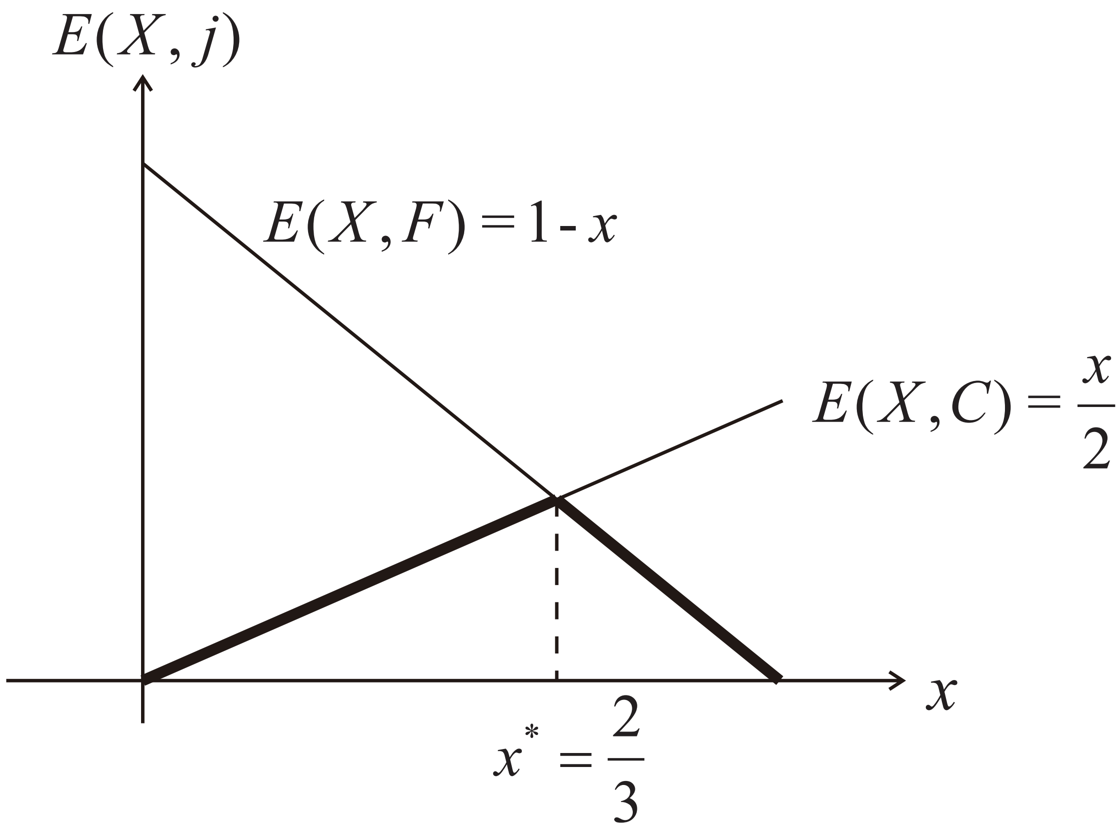
For Player I, suppose he plays $X = (x, 1-x)$, then

$$E(X,\ C) = \frac{x}{2} \text{ and } E(X,\ F) = 1 - x.$$

We find the intersection point and obtain $x^* = \frac{2}{3}$ so that $X^* = (0, 0, \frac{2}{3}, \frac{1}{3})$.



The bold line segments show the upper envelope of $E(BB, Y)$ and $E(BF, Y)$. The lowest point of the upper envelope is attained at $y^* = \frac{2}{3}$.



The bold line segments show the lower envelope of $E(X, C)$ and $E(X, F)$. The highest point of the lower envelope is attained at $x^* = \frac{2}{3}$. When the second player plays the pure strategy "call", it is optimal for the first player to play the pure strategy "bet on ace and fold on king" as the best response since $E(X, C)$ is maximized at $x = 1$.

Intuition behind the solution

Given 50% chance of getting either card for Player I, it is intuitive to deduce that Player II should choose to call with higher probability since “fold” always leads to loss of \$1. Indeed, we obtain Prob(call) $= 2/3$ and Prob(fold) $= 1/3$.

However, Player II should not choose “call” for sure since the best response of Player I becomes BF for sure. This is because $E(X, C) = \frac{x}{2}$ is maximized by choosing $x = 1$ by Player I (playing BF: “bet on ace and fold on king” 100% for sure). Accordingly, Player I gains \$2 with probability 50% and loses \$1 with probability 50%. The expected value is $1/2$, which is greater than $v = 1/3$.

All these results can be deduced directly from the game matrix by finding the best response of the opponent player when one player uses a pure strategy.

Since the value of the game is $v = \frac{1}{3}$, Player II is at a distinct disadvantage. Player II would never be induced to play the game unless I pays II exactly $\frac{1}{3}$ before the game begins. It is not surprising that this game is advantageous to Player I since he is informationally advantageous. Player I always bets if he receives ace, and chooses to bet with probability 1/3 or fold with probability 2/3 if he receives king.

Interestingly, the saddle point mixed strategy for I has him betting $\frac{1}{3}$ of the time when he has a losing card (king). *Bluffing with an appropriate probability* is a part of an optimal strategy.

To verify that $X^* = (0, 0, \frac{2}{3}, \frac{1}{3})$ and $Y^* = (\frac{2}{3}, \frac{1}{3})$ is a legitimate saddle point equilibrium, it suffices to check validity of

$$E(i, Y^*) \leq v \leq E(X^*, j), \text{ for all } i \text{ and } j.$$

Consider

$$
\begin{aligned}
(E(X^*,1),E(X^*,2)) &= X^*A \\
&= \begin{pmatrix} 0,0,\frac{2}{3},\frac{1}{3} \end{pmatrix} \begin{pmatrix} -1 & -1 \\ -\frac{2}{3} & 0 \\ \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{3},\frac{1}{3} \end{pmatrix};
\end{aligned}
$$

thus $E(X^*,j) \geq v = \dfrac{1}{3}$, $j = 1,2$, is verified.

$$
\begin{pmatrix} E(1,Y^*) \\ E(2,Y^*) \\ E(3,Y^*) \\ E(4,Y^*) \end{pmatrix} = AY^{*T} = \begin{pmatrix} -1 & -1 \\ -\frac{2}{3} & 0 \\ \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{2}{3} \\ \frac{1}{3} \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ \frac{1}{3} \\ \frac{1}{3} \end{pmatrix};
$$

thus $E(i,Y^*) \leq v = \dfrac{1}{3}$, $i = 1,\ldots,4$, is verified. Furthermore, since $E(1,Y^*) < \dfrac{1}{3}$ and $E(2,Y^*) < \dfrac{1}{3}$, we must have $x_1^* = x_2^* = 0$.

Submarine versus Bomber game

Consider the Submarine versus Bomber game. The board is a 3×3 grid.

1	2	3
4	5	6
7	8	9

Submarine-Bomber game in 3×3 grid

A submarine (which occupies two squares) is trying to hide from a bomber plane that can deliver torpedoes. The bomber can fire one torpedo at a square in the grid. If it is occupied by a part of the submarine, the submarine is destroyed (score 1 for the bomber). If the bomber fires at an unoccupied square, the submarine escapes (score 0 for the bomber). We take the bomber to be the row player.

The submarine has the size of two squares and can hide in a pair of adjacent squares out of the nine squares. There are 2 strategies for the submarine for each row, so total of 6 strategies if the submarine lies in a row. Similarly, there are 6 strategies if the submarine lies in a column. Altogether, there are $6+6=12$ strategies for the submarine. The bomber can bomb any one of the nine squares. The payoff matrix can be represented as follows:

Bomber \ Submarine	12	23	36	69	98	87	74	41	25	65	85	45
1	1	0	0	0	0	0	0	1	0	0	0	0
2	1	1	0	0	0	0	0	0	1	0	0	0
3	0	1	1	0	0	0	0	0	0	0	0	0
4	0	0	0	0	0	0	1	1	0	0	0	1
5	0	0	0	0	0	0	0	0	1	1	1	1
6	0	0	1	1	0	0	0	0	0	1	0	0
7	0	0	0	0	0	1	1	0	0	0	0	0
8	0	0	0	0	1	1	0	0	0	0	1	0
9	0	0	0	1	1	0	0	0	0	0	0	0

Using MATLAB to solve the corresponding linear programming problem, we can obtain the following solution (among many other solutions):

$$X^* = (0,\ \frac{1}{4},\ 0,\ \frac{1}{4},\ 0,\ \frac{1}{4},\ 0,\ \frac{1}{4},\ 0)$$
$$Y^* = (\frac{1}{8},\ \frac{1}{8},\ \frac{1}{8},\ \frac{1}{8},\ \frac{1}{8},\ \frac{1}{8},\ \frac{1}{8},\ \frac{1}{8},\ 0,\ 0,\ 0,\ 0).$$

Using symmetry, the submarine's pure strategies can be reduced to

S1: hide in a pair of squares that include the center square
S2: hide in a pair of squares that does not include the center square

The bomber's pure strategies can be reduced to

B1: fire at a corner
B2: fire at a square in the middle of each side
B3: fire at the center square

The payoff matrix is then reduced to

Bomber \ Submarine	S1	S2
B1	0	$\frac{1}{4}$
B2	$\frac{1}{4}$	$\frac{1}{4}$
B3	1	0

Why the payoff equals $\dfrac{1}{4}$ under the profile (B1, S2)? Suppose one plays B1 against S2, the bomber will fire at one of the four corners while the submarine will hide along an edge. This results in 25% chance of hitting the submarine.

When the torpedo fires at the center square, it always hits the submarine when it hides in a pair of squares that include the center square. Therefore, the payoff is unity.

(B2, S2) is a saddle point in pure strategies (row-min and column-max). Also, B1 is dominated by B2.

(i) Saddle point in pure strategies
The reduced game has a saddle point in pure strategies at (B2, S2) and value of the game $v(A) = \frac{1}{4}$. The bomber fires at one of the four middle side squares with equal probability. The submarine hides with equal probability in one of the eight locations that does not include the center square. This is the solution given by MATLAB.

(ii) Saddle point in mixed strategies
Let $X^* = (x_1, x_2, x_3)$ and $Y^* = (y_1, y_2)$ be the probability vectors. Knowing $v(A) = \dfrac{1}{4}$, the set of algebraic inequalities and equation for finding X^* are given by

$$\begin{aligned}
&E(X^*, 1) = \frac{1}{4}x_2 + x_3 \geq \frac{1}{4} \\
&E(X^*, 2) = \frac{1}{4}x_1 + \frac{1}{4}x_2 \geq \frac{1}{4} \\
&x_1 + x_2 + x_3 = 1.
\end{aligned}$$

The inequalities can be simplified to become

$$4x_1 + 3x_2 \leq 3 \quad \text{and} \quad x_1 + x_2 \geq 1.$$

We deduce that

$$3x_1 + 3x_2 \geq 3 \geq x_1 + 3(x_1 + x_2)$$

so that $x_1 \leq 0$. Since x_1 is non-negative, so $x_1 = 0$. Subsequently, we then obtain $x_2 \geq 1$ so $x_2 = 1$.

The solution is seen to be $X^* = (0, 1, 0)$. This is a bit surprising that the bomber chooses strategy 2 of firing at a square in the middle of each side with 100% certainty, though strategy B3 may deliver payoff of unity when S1 is played by the submarine. Unlike B1, B3 is not dominated by B2. However, the bomber should never fire at the center square.

The set of algebraic inequalities and equations for finding Y^* are given by

$$
\begin{aligned}
\left(0,\ \frac{1}{4}\right)\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} &= E(1, Y^*) = \frac{1}{4}y_2 \leq \frac{1}{4} \\
\left(\frac{1}{4},\ \frac{1}{4}\right)\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} &= E(2, Y^*) = \frac{1}{4}y_1 + \frac{1}{4}y_2 \leq \frac{1}{4} \\
(1,\ 0)\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} &= E(3, Y^*) = y_1 \leq \frac{1}{4} \\
& \quad\ y_1 + y_2 = 1.
\end{aligned}
$$

In this calculation, it is necessary to find $v(A)$ first using either pure strategy saddle point or computational formula. Note that $v(A)$ is unique while mixed strategy saddle points may not be unique.

The inequalities can be simplified to become

$$y_1 + y_2 \leq 1, \quad y_1 \leq \frac{1}{4} \quad \text{and} \quad y_2 \leq 1.$$

The solution is $Y^* = (\alpha, 1-\alpha)$, $0 \leq \alpha \leq \frac{1}{4}$.

To summarize the properties of the saddle point in mixed strategies, the bomber chooses to deliver the torpedo in pure strategy at a square that is in the middle of a side (position 2, 4, 6 or 8). The submarine chooses mixed strategy. It hides in a pair of squares that include the center square with probability α and in a pair of squares that do not include the center square with probability $1-\alpha$, where $0 \leq \alpha \leq \frac{1}{4}$. Since $0 \leq \alpha \leq \frac{1}{4}$, there is always a higher chance for the submarine to choose S2. As a check, we observe

$$E(X^*, Y^*) = E(2, Y^*) = \left(\frac{1}{4}, \quad \frac{1}{4}\right)\begin{pmatrix} \alpha \\ 1-\alpha \end{pmatrix} = \frac{1}{4}.$$

When $\alpha = 0$, it reduces to the pure strategies: $(X^*, Y^*) = (B2, S2)$.

We have shown that $x_3 = 0$ (B3 is never used). As expected, we observe

$$E(3, Y^*) = (1,\ 0)\begin{pmatrix} \alpha \\ 1-\alpha \end{pmatrix} = \alpha \leq \frac{1}{4} = v.$$

Recall that we have

$$x_3 > 0 \Rightarrow E(3, Y) = v \Leftrightarrow E(3, Y) < v \Rightarrow x_3 = 0.$$

However, it is still allow to have $x_3 = 0$ while $E(3, Y^*) = v$ (which occurs at $\alpha = \frac{1}{4}$).

Use of computational formula [*work only under the assumption that* $E(2, Y^*) = E(3, Y^*) = v$]

1. We delete B1 since it is dominated by B2. The game matrix is then reduced to 2×2 matrix, where

	S1	S2
B2	$\frac{1}{4}$	$\frac{1}{4}$
B3	1	0

Note that $\gamma = a_{11} + a_{22} - a_{21} - a_{12} = \frac{1}{4} - 1 - \frac{1}{4} = -1$. By the direct computational formula, we obtain

$$x_2 = \frac{a_{22} - a_{21}}{\gamma} = 1, \; x_3 = 0 \text{ so that } X^* = (0, 1, 0);$$
$$y_1 = \frac{a_{22} - a_{12}}{\gamma} = \frac{1}{4}, \; y_2 = \frac{3}{4} \text{ so that } Y^* = \left(\frac{1}{4}, \; \frac{3}{4}\right).$$

2. To solve for the saddle point mixed strategy for the bomber: $X^* = (0, x_2, x_3)$, we consider

$$E(X^*, 1) = E(X^*, 2) \text{ giving } \frac{1}{4}x_2 + x_3 = \frac{1}{4}x_2.$$

This gives $x_3 = 0$ and $x_2 = 1$. This agrees with the earlier result.

Next, we solve for the saddle point mixed strategy for the submarine: $Y^* = (y_1,\ y_2)$ by considering

$$E(2, Y^*) = E(3, Y^*) \text{ so that } y_1 + y_2 = 4y_1 \text{ or } y_2 = 3y_1.$$

Together with $y_1 + y_2 = 1$, we obtain $y_1 = \frac{1}{4}$ and $y_2 = \frac{3}{4}$.

Recall that the most general solution is $Y^* = (\alpha, 1 - \alpha)$. $0 \leq \alpha \leq \frac{1}{4}$. This solution method gives only single solution that corresponds to $\alpha = \frac{1}{4}$. Recall that $E(3, Y^*) = \alpha \leq \frac{1}{4}$ and $E(3, Y^*)$ equals $E(2, Y^*)$ when $\alpha = \frac{1}{4}$.

As a remark, MATLAB gives single solution: $Y^* = (0, 1)$, which corresponds to $\alpha = 0$.

Summary

The most general solution for the saddle point in mixed strategies is

$$X^* = (0, 1, 0) \text{ and } Y^* = (\alpha, 1 - \alpha),\ 0 \le \alpha \le \frac{1}{4}.$$

1. The MATLAB solution and saddle point in pure strategies give the same single solution

$$X^* = (0, 1, 0) \text{ and } Y^* = (0, 1),$$

which corresponds to $\alpha = 0$.

2. The direct computational formula and graphical method assume $E(2, Y^*) = E(3, Y^*) = v$. Indeed, $E(3, Y^*)$ equals the value of the game when $\alpha = \frac{1}{4}$. As a result, both give the single solution

$$X^* = (0, 1, 0) \text{ and } Y^* = \left(\frac{1}{4},\ \frac{3}{4}\right),$$

which corresponds to $\alpha = \frac{1}{4}$.

Optimal target takeover and defense

- Suppose Player I is seeking to takeover one of n companies that have values to Player I: $a_1 > a_2 > ... > a_n > 0$. These companies are managed by a private equity firm that can defend exactly one of the companies from takeover.

- Suppose that an attack made on any company has probability $1 - p$ of being taken over if it is defended. Player I can attack exactly one of the n companies and Player II can choose to defend exactly one of the n companies.

- If an attack is made on an undefended company, the payoff to Player I is a_i. If an attack is made on a defended company, Player I's expected payoff is $(1 - p)a_i$.

The payoff matrix is given by

I/II	1	2	3	$\cdots$	n
1	$(1-p)a_1$	a_1	a_1	$\cdots$	a_1
2	a_2	$(1-p)a_2$	a_2	$\cdots$	$\vdots$
3	a_3	a_3	$(1-p)a_3$	$\cdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
n	a_n	a_n	a_n	$\cdots$	$(1-p)a_n$

It seems sensible that the saddle point mixed strategy for Player II takes the form $Y = (y_1, y_2, ...y_k, 0, ...0)$. That is, Player II may choose not to provide defense for companies with too low values.

We verify by the following numerical calculations with $n = 3$ that the assumption of completely mixed strategy for Player II leads to negative probability values in the solution probability vector Y^*.

Consider $n = 3$, we obtain

$$\det(A) = a_1 a_2 a_3 p^2 (3 - p) > 0,$$
$$v(A) = \frac{1}{J_3 A^{-1} J_3^T} = \frac{(3-p) a_1 a_2 a_3}{a_1 a_2 + a_1 a_3 + a_2 a_3}.$$

The formula for $Y^{*T} = v(A) A^{-1} J_3^T$ gives

$$Y^* = \left(\frac{\frac{1}{a_3} + \frac{1}{a_2} + (p-2)\frac{1}{a_1}}{p(\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3})}, \frac{\frac{1}{a_3} + \frac{1}{a_1} + (p-2)\frac{1}{a_2}}{p(\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3})}, \frac{\frac{1}{a_1} + \frac{1}{a_2} + (p-2)\frac{1}{a_3}}{p(\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3})} \right).$$

Since $p - 2 < 0$, Y^* may not be a legitimate strategy for Player II.

For example, if we take $p = 0.1$, $a_i = 4 - i$, $i = 1, 2, 3$, we obtain

$$v(A) = 1.58, \ X^* = (0.18, 0.27, 0.54), \ Y^* = (4.72, 2.09, -5.81),$$

which is obviously that it cannot be a solution of the strategy for Player II. The mistake arises since we assume apriori that both players would play completely mixed strategies.

It seems sensible to assume that the attacker attacks the first k companies that have higher values with positive probability. Suppose $x_1 > 0, x_2 > 0, \ldots, x_k > 0$, and $x_i = 0$ for $i > k$, the best response of the defender would lead him not to defend companies that correspond to $j > k$ so that $y_j = 0$ for $j > k$.

We have $Y^* = (y_1, \ldots, y_k, 0, \ldots, 0)$ and the Equality of Payoff Theorem gives

$$E(1,\ Y^*) = v, E(2,\ Y^*) = v, \ldots, E(k,\ Y^*) = v.$$

Both k and v are to be determined subsequently. For $i = 1, 2, \ldots, k$, and observing $\sum_{i=1}^{k} y_i = 1$, we have

$$E(i,\ Y^*) \ = \ a_i y_1 + \cdots + (1-p) a_i y_i + \cdots + a_i y_k = a_i(1 - p y_i) = v,$$

so that $a_i(1 - p y_i) = v$, or

$$y_i = \frac{1}{p}\left(1 - \frac{v}{a_i}\right),\ \ i = 1, 2, \ldots, k.$$

To find v, we use $\sum_{i=1}^{k} y_i = 1$ to obtain

$$1 = \frac{1}{p}\left(k - v\sum_{i=1}^{k}\frac{1}{a_i}\right).$$

This gives $v = \dfrac{k-p}{G_k}$, where $G_k = \displaystyle\sum_{i=1}^{k}\frac{1}{a_i}$. We obtain

$$y_i = \begin{cases} \frac{1}{p}\left(1 - \frac{k-p}{a_i G_k}\right), & i = 1, 2, ..., k; \\ 0, & \text{otherwise.} \end{cases}$$

Note that $a_1\left(\dfrac{1}{a_1} + \cdots + \dfrac{1}{a_k}\right) > k$ so $0 < \dfrac{k-p}{a_1\left(\frac{1}{a_1} + \cdots + \frac{1}{a_k}\right)} < 1$, giving $y_1 = \dfrac{1}{p}\left[1 - \dfrac{k-p}{a_1\left(\frac{1}{a_1} + \cdots + \frac{1}{a_k}\right)}\right] > 0$. Since $\dfrac{k-p}{a_i G_k}$ is increasing in i, so y_i is decreasing in i. The defender chooses to defend the smaller company (up to k) with smaller probability. Once k has been determined, it is necessary to check that the choice of k guarantees positivity of y_k.

To determine X^*, we use the Equality of Payoff Theorem again to obtain

$$E(X^*, j) = \frac{k-p}{G_k} = a_1 x_1 + \cdots + (1-p) a_j x_j + \cdots + a_k x_k,\ j = 1, 2, \ldots, k.$$

One can check that $x_j = \frac{1}{a_j}\frac{1}{G_k}$, $j = 1, 2, \ldots, k$, satisfies these equation, and together $\sum_{j=1}^k x_j = 1$ is observed. We obtain

$$X^* = \left(\frac{\frac{1}{a_1}}{G_k}, \frac{\frac{1}{a_2}}{G_k}, ..., \frac{\frac{1}{a_k}}{G_k}, 0, 0, ..., 0 \right).$$

Note that x_i is increasing in i, so the attacker attacks company k^* with the higher probability.

How to determine k? We use the criterion that the attacker chooses to takeover up to the k^{th} firm such that $v(A) = \frac{k-p}{G_k}$ is maximized. Next, we establish the procedure to find k, where k lies between $1, 2, \ldots, n$, that maximizes $v(A)$.

Technical results on $v(k) = \dfrac{k-p}{G_k}$ *and* $k^* = \underset{k}{\text{argmax}}\ \dfrac{k-p}{G_k}$

Define the auxiliary function $f(k) = k - a_k G_k$, and observe $a_k > a_{k+1}$, so

$f(k+1) - f(k) = 1 - a_{k+1}G_{k+1} + a_k G_k > 1 - a_{k+1}(G_{k+1} - G_k) = 0.$

Therefore, $f(k)$ is a strictly increasing function of k. As a check of the increasing property of $f(k)$, we observe that

$$
\begin{aligned}
f(1) &= 1 - a_1 G_1 = 0, \\
f(2) &= 2 - a_2\left(\frac{1}{a_1} + \frac{1}{a_2}\right) = 1 - \frac{a_2}{a_1} > 0, \\
f(3) &= \left(1 - \frac{a_3}{a_1}\right) + \left(1 - \frac{a_3}{a_2}\right).
\end{aligned}
$$

The first term of $f(3)$ is greater than $f(2)$ while the second term of $f(3)$ is positive.

Lemma

When $f(k^*) \leq p < f(k^*+1)$, we observe

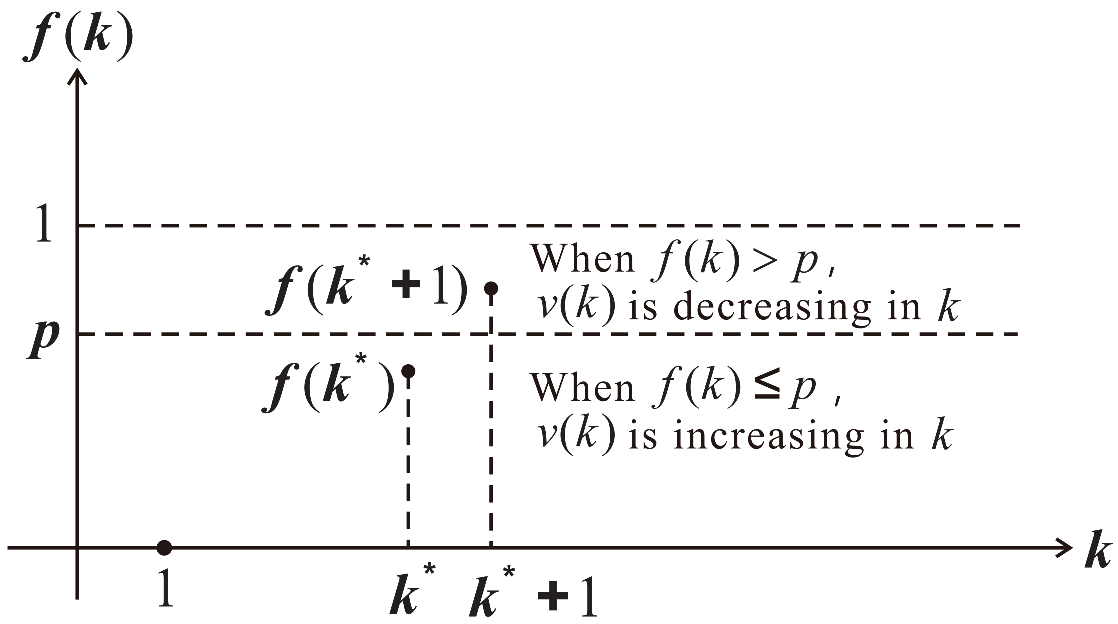
$$v(k^*-1) \leq v(k^*) \quad \text{and} \quad v(k^*+1) < v(k^*).$$

Here, k^* is the value of i achieving $\max_{1 \leq i \leq n} \frac{i-p}{G_i} = \frac{k^*-p}{G_{k^*}}$.

We establish that when $f(k) \leq p$, we observe $v(k) \geq v(k-1)$, so $v(k)$ is increasing in k; when $f(k) > p$, we observe $v(k) < v(k+1)$, so $v(k)$ is decreasing in k. In conclusion, when k^* satisfies $f(k^*) \leq p < f(k^*+1)$, then $v(k)$ is maximized at k^*; that is,

$$k^* = \operatorname*{argmax}_k \frac{k-p}{G_k}.$$

As a remark, when $f(n) \leq p$, we should take $k^* = n$. This is because $v(k)$ remains to be increasing for all k under such scenario.



1. For $f(n) > p$, $f(k)$ is strictly increasing in k and we locate k^* without ambiguity such that $f(k^*) \leq p < f(k^* + 1)$.

2. For $f(n) \leq p$, we take $k^* = n$.

In both cases, $v(k)$ achieves its maximum value at $k = k^*$.

Proof

1. $v(k^* + 1) < v(k^*) \Leftrightarrow p < f(k^* + 1)$

$$v(k^* + 1) = \frac{k^* + 1 - p}{G_{k^*+1}} < \frac{k^* - p}{G_{k^*}} = v(k^*)$$

$$\Leftrightarrow \quad p(G_{k^*+1} - G_{k^*}) < k^* G_{k^*+1} - (k^* + 1) G_{k^*} = \frac{k^*}{a_{k^*+1}} - G_{k^*}$$

$$\Leftrightarrow \quad p < k^* - a_{k^*+1} G_{k^*} = k^* + 1 - a_{k^*+1} G_{k^*+1} = f(k^* + 1).$$

2. $v(k^* - 1) \le v(k^*) \Leftrightarrow p \ge f(k^*)$

$$v(k^* - 1) = \frac{k^* - 1 - p}{G_{k^*-1}} \le \frac{k^* - p}{G_{k^*}} = v(k^*)$$

$$\Leftrightarrow \quad (k^* - 1) G_{k^*} - k^* G_{k^*-1} \le p(G_{k^*} - G_{k^*-1}) = \frac{p}{a_{k^*}}$$

$$\Leftrightarrow \quad \frac{p}{a_{k^*}} \ge (k^* - 1) G_{k^*} - k^* \left(G_{k^*} - \frac{1}{a_{k^*}} \right) = \frac{k^*}{a_{k^*}} - G_{k^*}$$

$$\Leftrightarrow \quad p \ge k^* - a_{k^*} G_{k^*} = f(k^*).$$

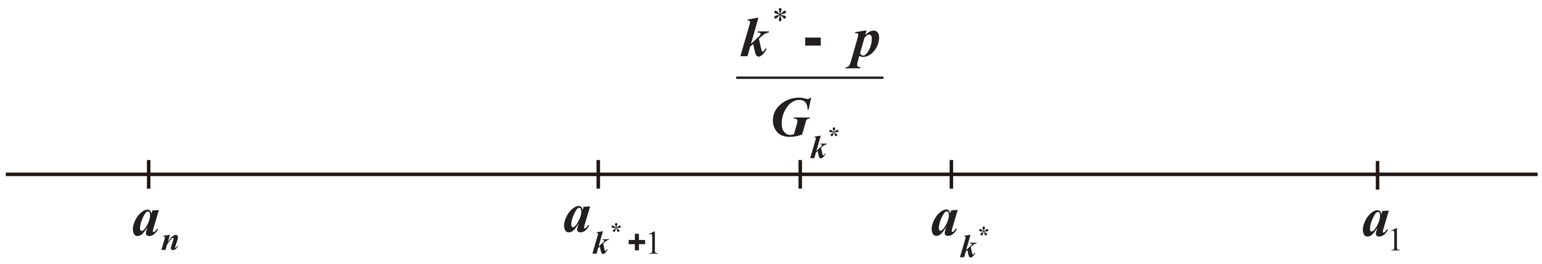
From $f(k^*) = k^* - a_{k^*}G_{k^*} \leq p < f(k^*+1)$

$$= k^* + 1 - a_{k^*+1}G_{k^*+1} = k^* - a_{k^*+1}G_{k^*},$$

The above pair of inequalities give

$$a_{k^*+1} < \frac{k^* - p}{G_{k^*}} \leq a_{k^*}.$$

Note that $E(i, Y^*) = a_i$, $i = k^*+1, \ldots, n$, since the attacker receives the full a_i since company with $i > k^*$ is not defended. We observe $v(A) = \dfrac{k^* - p}{G_{k^*}} > a_i = E(i, Y^*)$ for $i = k^* + 1, k^* + 2, \ldots, n$. Recall that $E(i, Y^*) < v(A)$ implies $x_i = 0$, $i = k^* + 1, k^* + 2, \ldots, n$.



Recall that

$$y_j = \frac{1}{p}\left[1 - \frac{v(A)}{a_j}\right], \quad j = 1, 2, \ldots, k^*.$$

Since $\frac{v(A)}{a_j} < 1$, $j = 1, 2, \ldots, k^* - 1$; in particular, $\frac{v(A)}{a_{k^*}} \leq 1$, so

$$y_j > 0 \text{ for } j = 1, 2, \ldots, k^* - 1 \text{ and } y_{k^*} \geq 0.$$

Remark Recall that we set $E(X^*, j) = v$, $j = 1, 2, \ldots, k^*$. We allow the possibility that y_{k^*} may assume zero value since it is still admissible to have $y_{k^*} = 0$ while $E(X^*, k^*) = v(A)$.

Summary of the saddle point in mixed strategies

Let k^* be the value of i giving $\max_{1\le i\le n} \frac{i-p}{G_i}$. We argue that

$$X^* = (x_1, ..., x_n), \text{ where } x_i = \begin{cases} \frac{1}{a_i G_{k^*}}, & \text{if } i = 1, 2, ..., k^* \\ 0, & \text{otherwise} \end{cases}$$

is the saddle point mixed strategy for Player I, and $Y^* = (y_1, y_2, ..., y_n)$,

$$y_j = \begin{cases} \frac{1}{p}\left(1 - \frac{k^*-p}{a_j G_{k^*}}\right), & \text{if } j = 1, 2, ..., k^*; \\ 0, & \text{otherwise.} \end{cases}$$

is the saddle point mixed strategy for Player II. In addition, $v(A) = \frac{k^*-p}{G_{k^*}}$ is the value of the game, with $a_{k^*+1} \le \dfrac{k^* - p}{G_{k^*}} < a_{k^*}$.

Note that y_j is decreasing in j while x_i is increasing in i.

To prove that X^* and Y^* are legitimate saddle point mixed strategies, it suffices to show that the following inequalities are satisfied:

$$E(i, Y^*) \le v(A) = E(X^*, Y^*) \le E(X^*, j), \text{ for all } i \text{ and } j.$$

We consider

$$
\begin{aligned}
E(X^*,\ j) &= \frac{1}{G_{k^*}}\left(\frac{1}{a_1}, \frac{1}{a_2}, ..., \frac{1}{a_{k^*}}, 0, 0, ..., 0\right)\begin{pmatrix} a_1 \\ a_2 \\ . \\ . \\ (1-p)a_j \\ . \\ . \\ a_n \end{pmatrix} \\
&= \begin{cases} \frac{k^*-p}{G_{k^*}} = v(A) \text{ if } j = 1, 2, ..., k^* \\ \frac{k^*}{G_{k^*}} > v(A) \quad \text{if } j = k^*+1, ..., n, \end{cases}
\end{aligned}
$$

Note that when $j > k^*$, the term $(1-p)a_j$ in the column vector A_j does not contribute to $E(X^*, j)$.

For $j > k^*$, we have $E(X^*, j) > v(A)$, so $y_j = 0$. For $j \leq k^*$, we have $E(X^*, j) = v(A)$. This is consistent with $y_j > 0$ for $j < k^*$ and $y_{k^*} \geq 0$. Recall that it is still admissable to have $y_{k^*} = 0$ even when $E(X^*, k^*) = v$.

We have derived y_j, $j = 1, 2, ..., k^*$, based on setting $v = E(i,\ Y^*)$, $i = 1, 2, ..., k$.

Since Player II will not defend company i, $i > k^*$, Player I is sure to capture a_i fully when he attacks the i^{th} company that has size less than that of the $k^{*\text{th}}$ company, so $E(i,\ Y^*) = a_i$, $i > k^*$. Recall that we use $E(i, Y^*) = v$, $i = 1, 2, \ldots, k^*$, to solve for Y^* in the first step of our procedure. Hence, we obtain

$$E(i,\ Y^*) = \begin{cases} \frac{k^*-p}{G_{k^*}} = v(A) \text{ if } i = 1, 2, ..., k^* \\ a_i < v(A) \qquad \text{if } i = k^* + 1, ..., n \end{cases},$$

consistent with $E(i, Y^*) < v(A)$ for $x_i = 0$, $i = k^* + 1, ..., n$; and $E(i, Y^*) = v(A)$ for $x_i > 0$, $i = 1, 2, ..., k^*$.

Symmetric games

	rock	paper	scissors
rock	0	-1	1
paper	1	0	-1
scissors	-1	1	0

Note that the payoff in row 1 in the rock-paper-scissors game is negative to that of column 1. The two players can switch roles in a symmetric game and $A = -A^T$ (matrix A is said to be skew symmetric).

Theorem

For any symmetric game, $v(A) = 0$. Also, if X^* is a saddle point mixed strategy for Player I, then it is also a saddle point mixed strategy for Player II.

Proof

For any mixed strategies $(X,\ Y)$ for a symmetric game, we have

$$\begin{aligned} E(X,\ Y) &= XAY^T = -XA^TY^T = -(XA^TY^T)^T \\ &= -YAX^T = -E(Y,\ X). \end{aligned}$$

This is obvious since the original expected payoff $E(X,Y)$ to Player I becomes $E(Y,X)$ when expressed as expected value to Player II after swapping the roles of X and Y. These two expected values must be the same due to symmetry. In terms of expected payoff to Player I, we obtain $E(X,Y) = -E(Y,X)$ (note the swap of sign).

By setting $X = Y$, we deduce that

$$E(X,\ X) = -E(X,\ X),$$

so $E(X,\ X) = 0$ for any mixed strategy. In a symmetric game, it is obvious that the expected payoff is zero when both players play the same strategy.

(i) $v(A) = 0$

Let $(X^*,\ Y^*)$ be a saddle point in mixed strategies for the zero-sum game so that

$$E(X,\ Y^*) \le E(X^*,\ Y^*) \le E(X^*,\ Y) \text{ for any } X \text{ and } Y.$$

Note that $E(X,Y^*) = -E(Y^*,X)$ and $E(X^*,Y) = -E(Y,X^*)$, we obtain

$$\begin{aligned} -E(Y^*,\ X) &= E(X,\ Y^*) \le E(X^*,\ Y^*) \\ &\le E(X^*,\ Y) = -E(Y,\ X^*). \end{aligned}$$

Observing $E(X^*,\ Y^*) = -E(Y^*,\ X^*)$, we finally obtain

$$E(Y,\ X^*) \le E(Y^*,\ X^*) \le E(Y^*,\ X).$$

This shows that (Y^*,X^*) is also a saddle point in mixed strategies and both $E(X^*,\ Y^*)$ and $E(Y^*,\ X^*)$ are equal to $v(A)$ since the value of the game is the same under any saddle point equilibrium. On the other hand, we always have $E(X^*,\ Y^*) = -E(Y^*,\ X^*)$, so $v(A) = 0$.

(ii) Suppose (X^*, Y^*) is a saddle point in mixed strategies, then $(X^*,\ X^*)$ is also a saddle point in mixed strategy.

Recall that $E(X, X) = 0$ for any X, so $E(X^*,\ X^*) = 0$. Also, $v(A) = 0$ so that $E(X^*, X^*) = E(X^*, Y^*) = v(A) = 0$. It then suffices to show that

$$E(X,\ X^*) \leq E(X^*,\ X^*) \leq E(X^*,\ Y) \text{ for all } X,\ Y.$$

Since (X^*, Y^*) is a saddle point in mixed strategies, we observe $E(X^*,\ X^*) = E(X^*,\ Y^*) = 0 \leq E(X^*,\ Y)$ for all Y. Therefore the right inequality is established.

Similarly, since (Y^*, X^*) is also a saddle point in mixed strategies, we observe $E(X^*,\ X^*) = E(Y^*,\ X^*) \geq E(X,\ X^*)$ for all X, so the left inequality is also established.

Remark

If (X^*, Y^*) is a saddle point for a symmetric game, then (X^*, X^*), (Y^*, Y^*) and (Y^*, X^*) are all saddle points of the game.

Rock-paper-scissors game

I/II	Rock	Paper	Scissors
Rock	0	$-a$	b
Paper	a	0	$-c$
Scissors	$-b$	c	0

Assuming $a, b, c > 0$, then $v^- = \max(-a, -c, -b) < 0$ and $v^+ = \min(a, c, b) > 0$. Therefore, this game does not have a pure strategy saddle point. We seek a saddle point in mixed strategy (X^*, X^*), where both players share the same saddle point mixed strategy. Recall that the value of a symmetric game is zero. To find $X^* = (x_1, x_2, x_3)$, we set $E(X^*,\ j) \geq v(A) = 0$, $j = 1, 2, 3$.

$$\begin{aligned} E(X^*,\ 1) &= (x_1, x_2, x_3)A_1 = ax_2 - bx_3 \geq 0, \text{ so } x_2\frac{a}{b} \geq x_3; \\ E(X^*,\ 2) &= (x_1, x_2, x_3)A_2 = -ax_1 + cx_3 \geq 0, \text{ so } x_3 \geq \frac{a}{c}x_1; \\ E(X^*,\ 3) &= (x_1, x_2, x_3)A_3 = bx_1 - cx_2 \geq 0, \text{ so } x_1 \geq \frac{c}{b}x_2. \end{aligned}$$

Combining the 3 inequalities, we obtain

$$x_2 \frac{a}{b} \geq x_3 \geq \frac{a}{c}\frac{c}{b} x_2.$$

Hence, we have equality throughout, where $\frac{x_2}{b} = \frac{x_3}{a}$. Similarly, we obtain $\frac{x_1}{c} = \frac{x_2}{b}$. Combining the results, we obtain

$$\frac{x_1}{c} = \frac{x_2}{b} = \frac{x_3}{a} = \frac{x_1 + x_2 + x_3}{a+b+c} = \frac{1}{a+b+c}$$

so that

$$x_3 = \frac{a}{a+b+c}, \; x_1 = \frac{c}{a+b+c} \text{ and } x_2 = \frac{b}{a+b+c}.$$

In the standard game, we take $a = b = c = 1$, so the saddle point mixed strategy for Player I is

$$X^* = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}).$$

and same for Player 2 by virtue of the above Theorem for symmetric game.

	Rock	Paper	Scissors	
Rock	0	-1	2	gain to loss is 2 : 1
Paper	1	0	-3	gain to loss is 1 : 3
Scissors	-2	3	0	gain to loss is 3 : 2

When the payoffs are not the same, say $a = 1$, $b = 2$, $c = 3$, then

$$X^* = (\frac{3}{6}, \frac{2}{6}, \frac{1}{6}).$$

- Though Scissors wins Paper with the highest winning payoff of 3, but the ratio of gain to loss is only 3 : 2. This strategy is played with the lowest probability 1/6.
- Rock is played with the highest probability 3/6, where the ratio of winning payoff to losing payoff is 2 : 1, the best among the three strategies.
- Paper strategy has the worst ratio of gain to loss of 1 : 3, however it is played with the medium probability of 2/6. This may be due to the observation paper beats rock while rock is played with a high probability of 3/6.

Best response to an opponent's strategy

If your opponent is assumed to use a particular strategy, then what should you do?

To be specific, suppose Player I knows or assumes that Player II is using the mixed strategy Y^0, which is not a saddle point mixed strategy for Player II. In this case, Player I should play the mixed strategy X that maximizes $E(X, Y^0)$ for the given Y^0. This strategy would be a *best response to* the use of Y^0 by Player II.

Since Player II is not playing a saddle point mixed strategy, such best response would not be a part of a saddle point.

Best response mixed strategy

A mixed strategy $\widehat{X} = (\widehat{x}_1, \ldots, \widehat{x}_n)$ for Player I is a *best response strategy* to the strategy Y^0 for Player II if it satisfies

$$\max_{X \in S_n} E(X, Y^0) = \max_{X \in S_n} \sum_{i=1}^{n} \sum_{j=1}^{m} \widehat{x}_i a_{ij} y_j^0 = E(\widehat{X}, Y^0).$$

A mixed strategy $\widehat{Y} = (\widehat{y}_1, \ldots, \widehat{y}_n)$ for Player II is a best response strategy to the strategy X^0 for Player I if it satisfies

$$\min_{Y \in S_m} E(X^0, Y) = \min_{Y \in S_m} \sum_{i=1}^{n} \sum_{j=1}^{m} x_i^0 a_{ij} \widehat{y}_j = E(X^0, \widehat{Y}).$$

Recall that if (X^*, Y^*) is a saddle point in mixed strategies of the game, then X^* is the best response to Y^*, and vice versa.

Example

Consider the 3×3 game

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix}.$$

Row 1 is replicated by $\frac{1}{2}$ of Row 2 and $\frac{1}{2}$ of Row 3. Similarly, Column 1 can be replicated by $\frac{1}{2}$ of Column 2 and $\frac{1}{2}$ of Column 3. Therefore, the general solution for the saddle point in mixed strategies is given by

$$X^* = \left(1 - \alpha,\ \frac{\alpha}{2},\ \frac{\alpha}{2}\right),\ 0 \le \alpha \le 1;\ Y^* = \left(1 - \beta,\ \frac{\beta}{2},\ \frac{\beta}{2}\right),\ 0 \le \beta \le 1.$$

As a check, we observe

$$E(i, Y^*) = 1,\ i = 1, 2, 3;\ E(X^*, j) = 1,\ j = 1, 2, 3.$$

For example, $E(2, Y^*) = (1, 2, 0)\left(1 - \beta, \frac{\beta}{2}, \frac{\beta}{2}\right)^T = 1.$

How to deduce $X^* = \left(1-\alpha, \frac{\alpha}{2}, \frac{\alpha}{2}\right)$, $0 \le \alpha \le 1$, using intuitive argument? Since row 1 is replicated by $\frac{1}{2}$ of row 2 and $\frac{1}{2}$ of row 3, Player I is indifferent to playing row 1 or mixed strategy of 50% chance of row 2 and 50% chance of row 3.

Now suppose that Player II, for some reason, thinks she can do better by playing $Y^0 = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{2}\right)$. Let $X = (x_1, x_2, 1 - x_1 - x_2)$ denote the best response of Row player. We calculate

$$E(X, Y^0) = XAY^{0T} = -\frac{x_1}{4} - \frac{x_2}{2} + \frac{5}{4}.$$

We want to maximize $E(X, Y^0)$ as a function of x_1 and x_2 with the constraints $0 \le x_1$, $x_2 \le 1$. We see that $E(X, Y^0)$ is maximized by taking $x_1 = x_2 = 0$ since x_1 and x_2 have both negative coefficients, then necessarily $x_3 = 1$. Hence, the best response strategy for Player I if Player II uses $Y^0 = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{2}\right)$ is $\hat{X} = (0, 0, 1)$.

Using this strategy, the expected payoff to Player I is

$$E(\widehat{X}, Y^0) = (1, 0, 2) \begin{pmatrix} \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{2} \end{pmatrix} = \frac{5}{4},$$

which is larger than the value of the game $v(A) = 1$. This shows that any deviation from a saddle point strategy could result in a better payoff for the opponent player.

If one player knows that the other player will not use her part of the saddle point, then the best response may not be the strategy used in the saddle point. In other words, if (X^*, Y^*) is a saddle point, the best response to $Y \neq Y^*$ would not be X^* in general, but some other X.

Usual scenario: best response strategy is a pure strategy

Since $E(X, Y^0)$ is linear in each strategy when the other mixed s-trategy is fixed, the best response strategy for player I will usually be a pure strategy.

For instance, if Y^0 is given, then $E(X, Y^0) = ax_1 + bx_2 + cx_3$, where $a = E(1, Y^0)$, $b = E(2, Y^0)$ and $c = E(3, Y^0)$.

The maximum payoff is then achieved by looking at the largest of a, b, c, and taking $x_i = 1$ for the x multiplying the largest of a, b, c, and the remaining values of $x_j = 0$.

It can be seen easily that

$$\begin{aligned} &\max\{ax_1 + bx_2 + cx_3 \mid x_1 + x_2 + x_3 = 1,\ x_1, x_2, x_3 \geq 0\} \\ =\ &\max\{a, b, c\}. \end{aligned} \tag{i}$$

What happens when some or all of the coefficients a, b and c are equal?

It is possible to get a mixed strategy best response but only if some or all of the coefficients a, b, c are equal. For instance, if $b = c$, then

$$\max\{ax_1 + bx_2 + cx_3 \mid x_1 + x_2 + x_3 = 1,\ x_1, x_2, x_3 \geq 0\} = \max\{a, c\}.$$

To see this, suppose that $a < c = b$. We compute

$$\begin{aligned}
& \max\{ax_1 + bx_2 + cx_3 \mid x_1 + x_2 + x_3 = 1,\ x_1, x_2, x_3 \geq 0\} \\
= \ & \max\{ax_1 + c(x_2 + x_3) \mid x_1 + x_2 + x_3 = 1\} \\
= \ & \max\{ax_1 + c(1 - x_1) \mid 0 \leq x_1 \leq 1\} \\
= \ & \max\{x_1(a - c) + c \mid 0 \leq x_1 \leq 1\} \\
= \ & c \text{ by choosing } x_1 = 0.
\end{aligned}$$

This maximum is achieved at $\widehat{X} = (0, 1 - \alpha, \alpha)$, $0 \leq \alpha \leq 1$. For $\alpha \in [0, 1]$, the expected payoff for the player is the same. Under this case, we can get a mixed strategy as a best response.

In general, *if the mixed strategy of one player, say,* Y^0 *is given and known*, then

$$\max_{X \in S_n} E(X, Y^0) = \max_{1 \le i \le n} E(i, Y^0).$$

In other words, it suffices to find maxima among $E(i, Y^0)$, $i = 1, 2, \ldots, n$. This result has been established in an earlier Lemma. Suppose the maxima among $E(i, Y^0)$ occurs at i^*, then the best response strategy is $x_{i^*} = 1$ and $x_i = 0$, for all $i \neq i^*$.

Recall that $E(X, Y^0) = \sum_{i=1}^n x_i E(i, Y^0)$, so $E(X, Y^0)$ is maximized according to the above choice of best response strategy.

When $E(i_1^*, Y^0) = E(i_2^*, Y^0) > E(i, Y^0)$, $i \neq i_1^*, i_2^*$, then $x_{i_1^*} + x_{i_2^*} = 1$, $0 \le x_{i_1^*} \le 1$ and $0 \le x_{i_2^*} \le 1$.

Note that $v(A) = \min_{Y \in S_m} \max_{1 \le i \le n} E(i, Y) \le \max_{1 \le i \le n} E(i, Y^0)$, so Player I can achieve value higher than $v(A)$ when Player II chooses his mixed strategy as Y^0 rather than choosing among all possible mixed strategies $Y \in S_m$ to minimize $\max_{1 \le i \le n} E(i, Y)$.

Investment strategies

Suppose that Player I has three investment options: stock (S), bonds (B), or CDs (certificates of deposit). The rate of return depends on the state of the market for each of these investments. Stock is considered risky, bonds have less risk than stock, and CDs are riskless.

The market can be in one of the three states: good (G), neutral (N), or bad (B), depending on factors such as the direction of interest rates, the state of the economy, prospects for future growth. Here is a possible game matrix in which the numbers represent the annual rate of return (in percentage) to the investor who is the row player:

I/II	G	N	B
S	12	8	-5
B	4	4	6
CD	5	5	5

The column player is the market. This game does not have a saddle point in pure strategies. We should not assume the market to be the opponent with the goal of minimizing the investor's rate of return, since the market cannot be an active player.

On the other hand, if the investor thinks that the market may be in any one of the three states with equal likelihood, then the market can be interpreted as playing the fixed mixed strategy $Y^0 = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$. The investor must choose how to respond to that; that is, the investor seeks an X^* for which $E(X^*, Y^0) = \max_{X \in S_3} E(X, Y^0)$, where $Y^0 = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$.

Suppose $Y^0 = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ is the probability distribution of the states of the market, then we have $E(1, Y^0) = \dfrac{12 + 8 - 5}{3} = 5$, $E(2, Y^0) = \dfrac{4 + 4 + 6}{3} = \dfrac{14}{3}$ and $E(3, Y^0) = 5$. Since $E(1, Y^0) = E(3, Y^0)$, the best response for Player I is $X = (\beta, 0, 1 - \beta)$, $0 \leq \beta \leq 1$, with payoff to Player I being equal to 5.

If $Y^0 = \left(\frac{2}{3}, 0, \frac{1}{3}\right)$, then $E(1, Y^0) = 6\frac{1}{3}$, $E(2, Y^0) = 4\frac{2}{3}$, $E(3, Y^0) = 5$. Since $E(1, Y^0)$ has the largest value, the best response is $X = (1, 0, 0)$, that is, invest in the stock if there is $\frac{2}{3}$ chance of good market and $\frac{1}{3}$ chance of bad market. The payoff then is $\frac{2}{3}(12) + \frac{1}{3}(-5) = \frac{19}{3} > 5$.

Belief in God

Player I has two possible strategies: believe or not believe in God. The opponent is God, who either plays "God exists" or "God does not exist". God does not play game, so this is not quite a zero-sum game. God plays the strategy $Y^0 = (\frac{1}{2}, \frac{1}{2})$ is interpreted as Player I's view: God's existence is 50-50 chance.

Player/God	God exists	God does not exist
believe	α	$-\beta$
not believe	$-\gamma$	0

Here, we first assume $\gamma \geq 0$, $\beta \geq 0$ and $\alpha \geq 0$. One may argue that γ should be much larger than α and β in spiritual currency (down to hell since God exists but you choose not to believe).

We observe that $v^+ = \min(\alpha, 0) = 0$ and $v^- = \max(-\beta, -\gamma) \leq 0$, so this game has no saddle point in pure strategies unless $\gamma = 0$ or $\beta = 0$.

Taking $Y^0 = (\frac{1}{2}, \frac{1}{2})$, what should be the player's best response strategy?

We calculate $f(x) = E(X,\ Y^0) = (x \quad 1-x)\begin{pmatrix} \alpha & -\beta \\ -\gamma & 0 \end{pmatrix}\begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} = x\dfrac{\alpha+\gamma-\beta}{2} - \dfrac{\gamma}{2}$. The maximum of $f(x)$ over $x \in [0,1]$ is

$$f(x^*) = \begin{cases} \frac{\alpha-\beta}{2} \text{ at } x^* = 1, \text{ when } \alpha+\gamma > \beta, \\ -\frac{\gamma}{2} \text{ at } x^* = 0, \text{ when } \alpha+\gamma < \beta, \\ -\frac{\gamma}{2} \text{ at any } 0 \le x \le 1, \text{ when } \alpha+\gamma = \beta. \end{cases}$$

Since $\gamma \gg \beta$, so the best response strategy to Y^0 would be $X^* = (1,0)$. Any rational person who thinks that God's existence is as likely as not (50-50 chance) would choose to play "believe".

As the last remark, since God should be more satisfied with a higher value of α, it is doubtful to set the payoff to God under (believe, exist) to be $-\alpha$. Actually, this is not quite a zero-sum game.