MATH4321 — Game Theory

Topic Two: Nonzero sum games and Nash equilibrium

2.1 Nonzero sum games under pure strategies

- Dominant and dominated strategies
- Pure strategy Nash equilibrium
- Examples: Prisoner's dilemma; battle of sexes; coordination games; Cuban crisis; voters participation
- Iterated dominance: Battle of the Bismark Sea; beauty contest

2.2 Two-person nonzero sum games under mixed strategies

- Mixed strategy Nash equilibrium
- Best response functions
- Equality of payoff theorem (indifference principle)
- Examples: Welfare game, chicken game, civic duty game and expert diagnosis
- Safety values

2.1 Nonzero sum games under pure strategies

Dominated and dominant strategies

Let S_i denote the set of all pure strategies of player i. The strategy $s_i^d \in S_i$ is said to be strictly dominated if there exists a single $s_i^{'} \in S_i$ such that

$$\pi_i(s_i^d,\ s_{-i}) < \pi_i(s_i^{'},\ s_{-i}),\ \text{for all } s_{-i} \in S_{-i}.$$

Here, s_{-i} represents a profile of strategies for all players other than i and

$$S_{-i} = S_1 \times S_2 \times \cdots \times S_{i-1} \times S_{i+1} \times \cdots \times S_n,$$

where n is the total number of players. That is, s_i^d is strictly inferior to (dominated by) some other strategy $s_i^{'}$ for any combination of the other players' strategies s_{-i}.

There may exist some pure strategy that have better payoff to the player i compared to all his other pure strategies under any combination of strategies that can be played by all other opponents. The strategy s_i^* is a strictly *dominant strategy* if for any s_{-i}

$$\pi_i(s_i^*,\ s_{-i}) > \pi_i(s_i^{'},\ s_{-i}),\ \text{for all } s_i^{'} \neq s_i^* \ \text{ and } \ s_{-i} \in S_{-i}.$$

Let $\boldsymbol{s}^D \in S$ (set of all strategy profiles) and write $\boldsymbol{s}^D = (s_1^D, \ldots, s_n^D)$. We say $\boldsymbol{s}^D$ is a strict dominant strategy equilibrium if $s_i^D \in S_i$ is a strictly dominant strategy for all $i \in N$ (set of all players).

In other words, a strictly dominant strategy equilibrium is a strategy profile that consists of each player's strictly dominant strategy.

We may manage to find a plausible equilibrium using the concept of "weak dominance".

- Strategy $s_i^{'}$ is weakly dominated if there exists $s_i^{''}$ such that

  $$\pi_i(s_i^{''},\ s_{-i}) \geq \pi_i(s_i^{'},\ s_{-i}),\ \text{for all } s_{-i};$$

  with strict inequality for some s_{-i}.

- Similarly, a weakly dominant strategy is one that is always at least as good as every other strategy and better than some.

Best responses

The strategy $s_i \in S_i$ is player i's best response to his opponents' strategies $s_{-i} \in S_{-i}$ if

$$\pi_i(s_i, s_{-i}) \geq \pi_i(s_i', s_{-i}) \quad \forall s_i' \in S_i.$$

Claims

1. If s_i is a strictly dominated strategy for player i, then it cannot be a best response to any $s_{-i} \in S_{-i}$.

2. If s_i^* is a strictly dominant strategy, then s_i^* is a player's strictly best response to any strategies the other players might pick, even to wildly irrational actions of the other players.

3. The *best response correspondence* of player i selects for each $s_{-i} \in S_{-i}$ is denoted by $B_i(s_{-i})$, which is a subset of S_i where each strategy $s_i \in B_i(s_{-i})$ is a best response to s_{-i}.

Nash equilibrium under pure strategies

A solution concept is a method of analyzing games with the objectives of restricting the set of all possible outcomes to those that are more reasonable than others. An equilibrium refers to one of the strategy profiles that emerges as one of the solutions' predictions.

Modellers commonly use the solution concept of Nash equilibrium. The strategy profile $\boldsymbol{s}^*$ is a Nash equilibrium if no player has incentive to deviate from his part of the Nash strategy given that the other players do not deviate from their parts of Nash equilibrium strategies. That is

$$\pi_i({s_i}^*,\ s^*_{-i}) \geq \pi_i({s_i}^{'},\ s^*_{-i}),\ \text{for all } {s_i}^{'}.$$

- To define *strong* Nash equilibrium, we make the inequality strict:

$$\pi_i({s_i}^*,\ s^*_{-i}) > \pi_i({s_i}^{'},\ s^*_{-i}),\ \text{for all } {s_i}^{'} \neq s_i^*.$$

In other words, the strategy profile $\boldsymbol{s}^*$ is a Nash equilibrium of a game if and only if every player's action is a best response to the other players' strategy choices: s_i^* is in player i's best response correspondence to s^*_{-i}, denoted by $B_i(s^*_{-i})$, for every player i.

Nash equilibrium concept, being less restrictive, helps yield predictions in more games than the dominant strategy equilibrium concept.

Every (weakly) dominant strategy equilibrium is a Nash equilibrium, but not every Nash equilibrium is a dominant-strategy equilibrium. We may drop "weakly" for brevity in later discussion if there is no ambiguity.

- If a strategy is dominant, it is a best response to *any* strategies the other players pick, including their *Nash equilibrium* strategies.

- If a strategy is part of a Nash equilibrium, it needs only be a best response to the other players' Nash equilibrium strategies. In general, it fails to be a dominant strategy since it may not be the best response to all strategies of other players.

Prisoner's dilemma — Noncooperative game with conflict

- If each prisoner tries to provide convicting evidence of the crime committed by the other, each is sentenced to 8 years in prison.
- If both remain silent, each is sentenced to one year (say, charging on some other minor offenses).
- If just one provides convicting evidence, he is set free but the silent prisoner is sentenced to higher imprisonment of 10 years.

Each player has 2 possible actions: *Confess* (blaming the other) and *Deny* (silent).

		Column	
		Deny	*Confess*
Row	*Deny*	−1, −1	−10, 0
	Confess	0, −10	**−8, −8**

Payoffs to: (Row, Column).

Each player has a strictly dominant strategy of Confess since the payoff under Confess is better than Deny under all strategies (Confess and Deny) played by the other player. The strictly dominant strategy equilibrium is (Confess, Confess).

To show that the strategy profile (Confess, Confess) in the Prisoner's Dilemma is a strong Nash equilibrium, we test whether each player's strategy is a best response to the other's Nash equilibrium strategies.

- If Row chooses Confess, then Confess is the best response of Column.

- By symmetry, Confess is the best response of Row if Column chooses Confess.

Knowledge of the opponent's strategy

If Column is allowed to know Row's move before taking his own, the equilibrium is unchanged. Row still chooses Confess, knowing that Column will surely choose Confess afterwards.

Communication between players

What difference would it make if the two prisoners could talk to each other before making their decisions?

If promises are not biding, though the two prisoners might agree to *Deny*, they would Confess anyway when the time came to choose actions. Since (Deny, Deny) is not a Nash equilibrium, when one player chooses Deny, the other player is better off by deviating from playing Deny.

Mafia punishment

Suppose there is a mafia punishment, which subtracts z units of payoff for each player who confesses.

		Column	
		Deny	*Confess*
Row	*Deny*	$-1,\ -1$	$-10,\ -z$
	Confess	$-z,\ -10$	$-8-z,\ -8-z$

If z is larger than 2, then "deny" becomes the strictly dominant strategy for both players, so (deny, deny) is a strictly dominant strategy equilibrium and strong Nash equilibrium as well.

This example illustrates existence of an institutional design that results in both players becoming better off compared to those outcomes without the design.

Arm race as Prisoner's dilemma

Two players: US & USSR

Possible strategies: armed or disarmed

Rank the four possibilities (most preferred to least preferred)

1. Highly positive (payoff $= 3$)
   Self-armed and other's unilaterally disarmed (military superiority)

2. Moderately positive (payoff $= 1$)
   Mutual disarmament (parity without economic hardship)

3. Moderately negative (payoff $= -1$)
   An arm race (parity but with economic hardship)

4. Highly negative (payoff $= -3$)
   Self-disarmed and other's unilaterally armed (military inferiority)

		Soviet Union	
		disarm	arm
US	disarm	(1, 1)	(-3, 3)
	arm	(3, -3)	(-1, -1)*

- First entry for US's payoff
- Second entry for Soviet's payoff

Ordinal payoffs rather than cardinal payoffs: The payoff values reflect only the order of preference as opposed to the absolute magnitude of one's preference.

Both Soviet and US have dominant strategies leading to "arm-arm" outcome that is strictly worse (for both players) than the "disarm-disarm" outcome via mutual operation.

Remark

For analyzing pure strategies, ordinal (only order matters) payoffs representing the order of preference are sufficient. For mixed strategies, it is necessary to have cardinal (precise value) payoffs since the probabilities of playing various strategies depend on the actual values of outcomes.

Pareto dominance and Pareto efficiency

Outcome X is said to be (weakly) Pareto-dominating outcome Y if all players have higher (at least the same or higher) payoff under outcome X.

An outcome is Pareto efficient (optimal) if there is no other outcome where some players can increase their payoffs without decreasing the payoff of other players.

In the Prisoner's Dilemma, (confess, confess) $= (-8, -8)$ should be considered as a very likely outcome, assuming players are aware of the dominant strategy. However, it is Pareto dominated by (deny, deny) $= (-1, -1)$, so (confess, confess) is not Pareto-optimal. Note that (deny, deny), (deny, confess) and (confess, deny) are all Pareto efficient outcomes since no other profile dominates any of them. Though (confess, confess) is a dominant strategy equilibrium, it does not Pareto-dominates all other strategy profiles.

Modified Prisoner's Dilemma

Consider the modified Prisoner's Dilemma where the payoff to (Deny, Deny) is (0, 0) instead of (−1, − 1), assuming that the police does not have enough evidence to convict the prisoners of even a minor offense if the prisoner remains silent.

		Column		
		Deny		*Confess*
Row	*Deny*	**0**, **0**	↔	−10, 0
		↕		↓
	Confess	0, −10	→	$\boxed{\mathbf{-8}}$, $\boxed{\mathbf{-8}}$

Payoffs to: (Row, Column). Arrows show how a player can increase his payoff.

- It does not have a strictly dominant strategy equilibrium.

- It does have a weak dominant strategy equilibrium since Confess is still a weakly dominant strategy for each player.

- Note that Deny is a weakly dominated strategy of both players, and it is eliminated in the iterative process. The remaining single strategy is Confess for both players. (Confess, Confess) is said to be an iterated elimination equilibrium (formal definition and examples are given in more details later). Note that an iterated elimination equilibrium is not the same as a dominant strategy equilibrium (which requires the equilibrium strategies of all players are dominant strategies). Also, it is a strong Nash equilibrium.

- However, (Deny, Deny) is a Nash equilibrium. Its outcome is Pareto-superior since the payoffs to both players under (Deny, Deny) are at least as good or better than those in other outcomes.

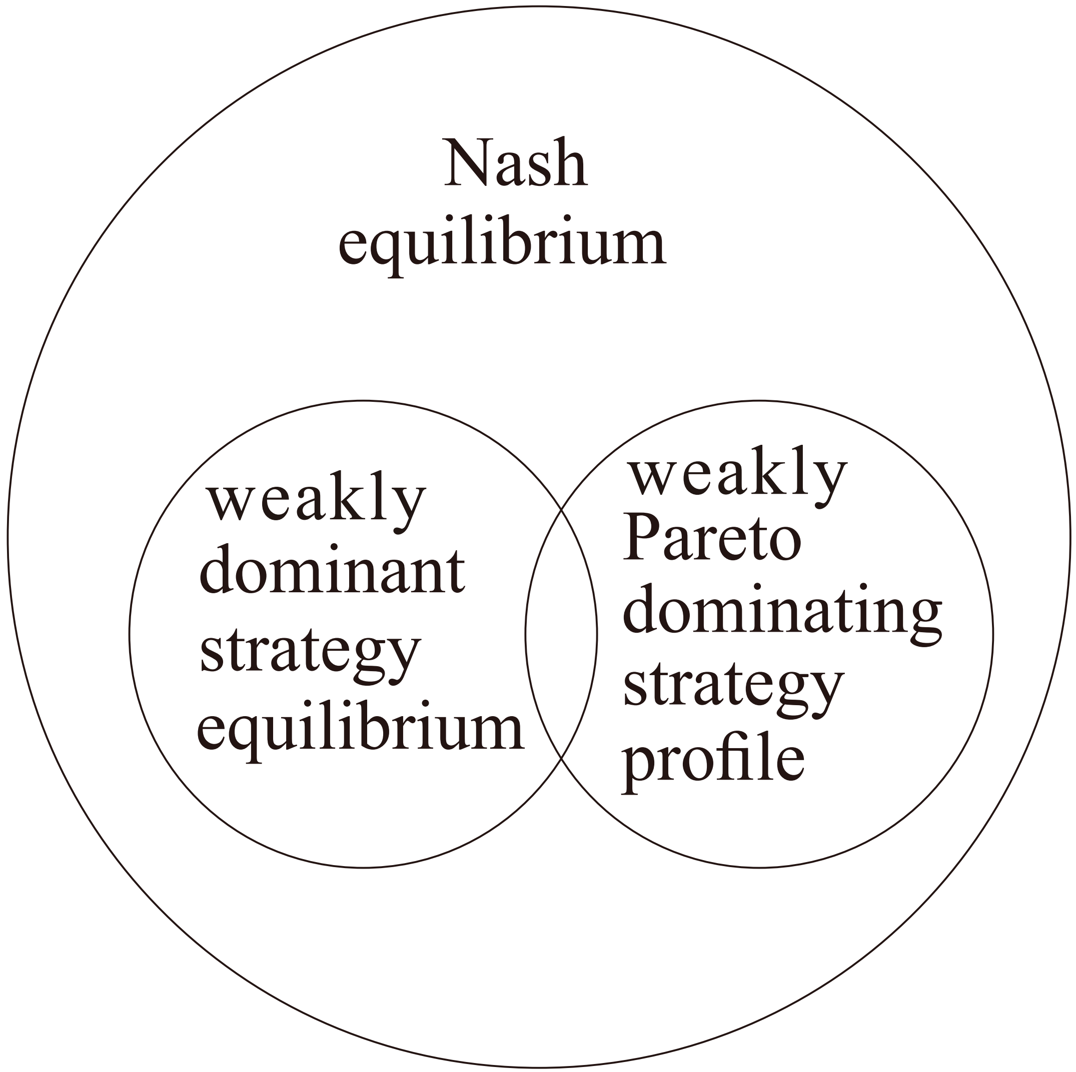
(Confess, Confess) is a strong Nash equilibrium but it is Pareto-dominated by another (not strong) Nash equilibrium (Deny, Deny).

1. Insist on a strong Nash equilibrium only, so rule out the other Nash equilibrium (Deny, Deny).
2. Rule out Nash equilibriums that are Pareto-dominated by other Nash equilibriums, and end up with (Deny, Deny).

Lemma
Suppose a strategy profile s weakly Pareto-dominating all other strategy profiles (payoffs in the outcome are higher or at least the same when compared with payoffs in all other outcomes), then it must be a Nash equilibrium.

Proof
Given that s weakly Pareto-dominates all other strategy profiles, any player i cannot benefit from deviating unilaterally. If otherwise, this violates the weakly Pareto-dominating property. Hence, s must be a Nash equilibrium.

In summary, the relations between Nash equilibrium, weakly dominant strategy equilibrium and weakly Pareto-dominating profile are summarized in the following diagram:



Examples

1. Unique Nash equilibrium

Colin

		A		B
Rose	A	(2, 3)	←	(3, 2)
		↑		↑
	B	(1, 0)	→	(0, 1)

(A, A) is a Nash equilibrium since neither player would gain by unilaterally changing his strategy of A while the other player remains playing A.

Note that A is a dominant strategy for Rose but not for Colin, so (A, A) is not a dominant strategy equilibrium. Also, (A, A) is strongly Pareto-dominating (B, A) and (B, B). Note that (A, A) is Pareto non-dominated by any other outcome since we cannot increase the payoff of one player without lowering the payoff of the other player. Therefore, (A, A) is Pareto efficient. Similarly, (A, B) is Pareto efficient.

2. Absence of Nash equilibrium

Colin

		A	B
Rose	A	(2, 4)	(1, 0)
	B	(3, 1)	(0, 4)

There is no Nash equilibrium. None of the outcomes is Pareto-dominating all other outcomes. For example, $(2, 4)$ Pareto-dominates $(1, 0)$, weakly Pareto-dominates $(0, 4)$ but does not Pareto-dominate $(3, 1)$. Also, none of the strategies is dominant or being dominated.

Suppose the outcome in (B, A) is changed to $(1, 1)$, then (A, A) becomes a weakly Pareto-dominating profile, so it is a Nash equilibrium.

3. Multiple Nash equilibriums

		Colin	
		A	B
Rose	A	(1, 1)	(2, 5)
	B	(5, 2)	(-1, -1)

Both (5, 2) and (2, 5) are Nash equilibriums. They are also Pareto efficient (optimal). Both (1, 1) and $(-1, -1)$ are Pareto dominated by (5, 2) and (2, 5), so they cannot be Pareto optimal. However, none of the strategies is dominant, so there is no dominant strategy equilibrium.

Coordination game

If the players can coordinate and hunt, then they can both do better. Gathering alone is preferred to gather together, but hunting along is much worse than gathering alone. If a player hunts alone, the payoff is zero due to loss of cooperation. The other player enjoys higher payoff on gathering due to avoidance of sharing of fruits gathered.

	hunt	gather
hunt	$(5,5)$	$(0,4)$
gather	$(4,0)$	$(2,2)$

Note that (hunt, hunt) strongly Pareto-dominates all other strategy profiles, but it is not a dominant strategy equilibrium.

Note that (hunt, hunt) is a Nash equilibrium in pure strategies since it is Pareto-dominating, while (gather, gather) is also a Nash equilibrium in pure strategies.

- The Nash equilibrium (hunt, hunt) is payoff dominant.

- The Nash equilibrium (gather, gather) *risk dominates* (hunt, hunt). If either player is not absolutely certain that the other player will join the hunt, then the player who was going to hunt sees that he can minimize the risk of getting zero by gathering.

Battle of the Sexes — Compromise and cooperation

		Woman		
		Prize Fight		*Ballet*
	Prize Fight	**2, 1**	←	0, 0
Man		↑		↓
	Ballet	0, 0	→	**1, 2**

Payoffs to: (Man, Woman). Arrows show how a player can increase his payoff.

Two Nash equilibriums in pure strategies

1. Strategy profile (Prize Fight, Prize Fight)
   Given that the man chooses Prize Fight, so does the woman.
   Given that the woman chooses Prize Fight, so does the man.

2. Strategy Profile (Ballet, Ballet)
   Both Nash equilibriums are Pareto-efficient, where no player gains without another player losing. Comparing $(2, 1)$ and $(1, 2)$, when man's payoff increases from 1 to 2, woman's payoff decreases from 2 to 1, and vice versa.

Nature of the game: Failure to cooperate creates lower payoff to both players. However, any cooperative agreement results in unequal distribution of payoff (lower payoff for the player who compromises).

Who moves first is important — first mover advantage. The woman receives a higher payoff if both attend the same event (best response to man's action), then the man would choose to buy the fight ticket if he is allowed to be the first mover.

If they do not communicate beforehand, and they choose to compromise where the man might go to the ballet and the woman to the fight, each mistaken about the other's beliefs. Repeating the game night after night, eventually they settle on one of the Nash equilibriums with the given distribution (see the discussion of mixed strategy Nash equilibrium later).

Economic analogy: Choice of an industry-wide standard when both firms have different preferences but both want a common standard to encourage consumers to buy the product.

Iterated elimination equilibrium: Battle of the Bismarck Sea

General Imamura has been ordered to transport Japanese troops across the Bismarck Sea to New Guinea in the South Pacific in 1943.

General Kenney wants to bomb the troop transports.

Imamura must choose between a *shorter* northern route or a *longer* southern route to New Guinea. Since the southern route is longer, the potential number of days of bombing is larger than that of the northern route counterpart.

Kenney must decide where to send his planes to look for the Japanese. If Kenney sends his plane to the wrong route, he can recall them but the number of days of bombing is reduced by one.

Players: Kenney and Imamura; action set = {North, South}.

Payoffs to Kenney in the two-person zero sum game

Battle of the Bismarck Sea

		Imamura	
		North	*South*
Kenney	*North*	2	2
	South	1	3

Neither player has a dominant strategy.

- Kenney would choose *North* if he thought Imamura would choose *North*, but *South* if he thought Imamura would choose *South*.
- Imamura would choose North if he thought Kenney would choose South and he would be *indifferent* between actions if he thought Kenney would choose North.

An *iterated elimination equilibrium* is a strategy profile found by deleting a weakly dominated strategy from the strategy set of one of the players, recalculating to find which remaining strategies are weakly dominated, deleting one of them, and continuing the process until one strategy remains for each player. Note that iterated elimination equilibrium may not exist. Most likely such equilibrium is unique, except in the unlikely scenario where two or more strategy profiles left after all iterated elimination steps happen to be the same payoff. They are all iterated elimination equilibriums.

- Using the iterated elimination equilibrium concept, Kenney eliminates "Imamura chooses South" from consideration since South is a weakly dominated strategy for Imamura.

- Having deleted one column of the payoff table, Kenney has a strongly dominant strategy: he chooses *North*, which achieves payoff strictly greater than South.

- The strategy profile (North, North) is an iterated elimination equilibrium, and indeed (North, North) was the outcome in 1943.

Sequential moves – order of play

Once Kenney has chosen North, Imamura is indifferent between North and South. The outcomes are dependent on which player moves first. This is unlike a dominant strategy equilibrium, where the equilibrium outcome is irrelevant to the sequence of moves.

1. If *Kenney moved first*, (North, North) would remain an iterated dominance equilibrium, but (North, South) would also become one.
   Though the payoffs would be the same for both equilibriums, but the strategy profiles are different.

2. If Imamura moved first, (North, North) would be the only iterated dominance equilibrium.

Example

Consider the following two-player nonzero sum game:

		Player 2		
		L	C	R
	U	4, 3	5, 1	6, 2
Player 1	M	2, 1	8, 4	3, 6
	D	3, 0	9, 6	2, 8

A quick observation reveals that there is no strictly dominant strategy, neither for player 1 nor for player 2. Also note that there is no strictly dominated strategy for player 1. There is, however, a strictly dominated strategy for player 2: the strategy C is strictly dominated by R because $2 > 1$ (row U), $6 > 4$ (row M), and $8 > 6$ (row D).

Both players know that the strategy C is eliminated from player 2's strategy set, which results in the following reduced game:

		Player 2	
		L	R
	U	4, 3	6, 2
Player 1	M	2, 1	3, 6
	D	3, 0	2, 8

In this reduced game, both M and D are strictly dominated by U for player 1, allowing us to perform a second round of eliminating strategies, this time for player 1.

Eliminating these two strategies yields the following trivial game, where player 2 has a strictly dominated strategy of playing R.

		Player 2	
		L	R
Player 1	U	**4, 3**	6, 2

Finally, the iterated elimination of strictly dominated strategies lead to the strategy profile (U, L). This is called the *iterated elimination equilibrium*. As a check, (U, L) is a Nash equilibrium.

Suppose the outcome in (U, R) is changed to $(4, 3)$, all iterated elimination steps remain the same. Both (U, L) and (U, R) are iterated elimination equilibriums. This represents the unlikely scenarios where iterated elimination equilibrium is not unique.

Relations between Nash equilibrium and iterated elimination equilibrium

1. Every iterated elimination equilibrium is a Nash equilibrium.

   We prove by contradiction. Suppose that an iterated elimination equilibrium $\boldsymbol{s}^*$ is not a Nash equilibrium, due to violation of the best response property in a Nash equilibrium, then there exists s_i' of some player i such that

   $$\pi_i(s_i', s_{-i}^*) > \pi_i(s_i^*, s_{-i}^*),$$

   where s_{-i}^* represents the exclusion of player i's equilibrium strategy s_i^* from the iterated elimination equilibrium $\boldsymbol{s}^*$. In this case, s_i^* is seen to be dominated by s_i', so it should have been eliminated in the iterated elimination procedure before arriving at (s_i^*, s_{-i}^*). Hence, $\boldsymbol{s}^*$ cannot be an iterated elimination equilibrium. This leads to a contradiction.

   Indeed, an iterated elimination of dominated strategy equilibrium are made up of strategies that are not dominated. If otherwise, such profile cannot survive under all iterated elimination steps.

2. Not every Nash equilibrium can be generated by iterated elimination.

   Consider the following two-person nonzero-sum game

I \ II	II1	II2
I1	(3,3)	(-1,-1)
I2	(-1,-1)	(1,1)

   Note that $(1,1)$ is a Nash equilibrium. Note that I1 and I2 do not weakly dominate each other, so do II1 and II2. Therefore, $(1,1)$ would NOT be generated by iterated dominance.

Coordination games — coordinate on one of the multiple Nash equilibriums

Unlike the Battle of the Sexes where the compromiser always has a lower payoff than that of his opponent, the payoffs to the two players are the same when they coordinate, one of the Nash equilibriums is Pareto-superior. One can use the size of the payoffs to choose between Nash equilibriums, where the Pareto-superior equilibrium is more preferred?

Players Smith and Jones decide whether to design the computers they sell to use large or small floppy disks.

		Jones		
		Large		*Small*
	Large	**2, 2**	$\leftarrow$	$-1, -1$
Smith		$\uparrow$		$\downarrow$
	Small	$-1, -1$	$\rightarrow$	**1, 1**

Payoffs to: (Smith, Jones). Arrows show how a player can increase his payoff.

Assuming no pre-game communication, how to make the Pareto-efficient equilibrium (Large, Large) be more plausible?

Though $(2, 2)$ Pareto-dominates all other strategy profiles. However, "Large" is not a dominant strategy for both players, so it is not a dominant strategy equilibrium.

Suppose we change the outcome of (Large, Small) from $(-1, -1)$ to $(1.1, -1)$. Now, Small is a dominated strategy of Row, so it is eliminated. Since Row always plays Large, then Column chooses to play Large. The strategy profile (Large, Large) becomes an iterated elimination equilibrium.

Though (Large, Large) is an iterated elimination equilibrium, it is not a dominant strategy equilibrium. This is because Large is not a dominant strategy for Column, though Large is a dominant strategy for Row. A dominant strategy equilibrium requires that all equilibrium strategies of all players are dominant strategies.

Dangerous coordination — extreme out-of-equilibrium payoff

The out-of-equilibrium payoff may affect which Nash equilibrium would be played out.

			Jones	
		Large		*Small*
Smith	*Large*	**2, 2**	←	−1000, −1
		↑		↓
	Small	−1, −1	→	**1, 1**

Payoffs to: (Smith, Jones). Arrows show how a player can increase his payoff.

Under this case, the Pareto-dominated Nash equilibrium (Small, Small) would be much more likely to occur than the Pareto-dominating Nash equilibrium (Large, Large).

If Smith cannot trust Jones to be rational to pick the Pareto-dominating (payoff dominant) Nash equilibrium, then Smith will be reluctant to pick "Large" since his payoff if Jones pick "Small" is then $-1,000$. Smith suffers greatly if the agreement of coordination is not honored by Jones.

Guessing two-thirds of the average – Beauty Context

Three persons are going to choose an integer from 1 to N. The person who chooses closest to two-thirds of the average of all the numbers chosen wins $\$1$.

If two or more people choose the same number closest to two-thirds of the average, they split $\$1$. The payoff function is given by

$$\pi_i(x_1, x_2, x_3) = \begin{cases} 1 \text{ if } x_1 \neq x_2 \neq x_3, \text{ and } x_i \text{ is closest to } \bar{x}; \\ \frac{1}{2} \text{ if } x_i = x_j \text{ for some } j \neq i, \text{ and } x_i \text{ is closest to } \bar{x}; \\ \frac{1}{3} \text{ if } x_1 = x_2 = x_3; \\ 0 \text{ otherwise.} \end{cases}$$

Here, $\bar{x} = \frac{2}{3}\frac{x_1+x_2+x_3}{3}$. The possible choices for x_i are $\{1, 2, ..., N\}$.

Claim: $(x_1^*, x_2^*, x_3^*) = (1, 1, 1)$ is a Nash equilibrium, independent of N.

It suffices to show that

$$\pi_1(1, 1, 1) \geq \pi_1(x_1, 1, 1), \; x_1 \neq 1, \; x_1 \in \{2, 3, ..., N\}.$$

Suppose Player 1 chooses $x_1 = k \geq 2$, then

$$\frac{2}{3}\left(\frac{1+1+k}{3}\right) = \frac{4}{9} + \frac{2}{9}k$$

and the distance between two-thirds of the average and k is

$$|k - \frac{4}{9} - \frac{2}{9}k| = \frac{7}{9}k - \frac{4}{9},$$

which is increasing with positive integer k.

If Player 1 had chosen $x_1 = 1$, then $|1 - \frac{2}{3}| = \frac{1}{3}$. We cannot find $k > 1$ such that

$$|\frac{7}{9}k - \frac{4}{9}| < \frac{1}{3}.$$

We can also show that none of the players can do better by switching to another integer. Therefore, $(1, 1, 1)$ is a Nash equilibrium.

Keynes' beauty contest

This game model is related to the famous Keynes' beauty contest where each participant is given 100 pictures of humans and chooses 6 pictures considered to be most attractive. The winner is the one whose 6 chosen pictures match closest to the top 6 choices of all participants. Here, the intelligences of the players are devoted to anticipating what average opinion expects the average opinion to be.

As an application in predicting stock price, Keynes postulated that people pricing stock prices not based on what they think their fundamental value is, but rather on what they think everyone else thinks their value is, or what everybody else would predict the average assessment of value to be.

Rational thinking process

Suppose a player believes the average play will be X (including his own integer). That player's optimal strategy is to play the closest integer to $\frac{2}{3}X$. Take $N = 100$, then the optimal strategy of any player has to be no more than 67. If X is no more than 67, then the optimal strategy of any player has to be no more than $\frac{2}{3} \times 67$. Going further, it should be no more than $(\frac{2}{3})^2 \times 67$. Iteratively, the unique Nash equilibrium is for everyone to announce 1.

This is an example of iterated elimination. Players eliminate all the strategies that are dominated, resulting in a "smaller" reduced game that includes only strategies that can be a best response in the original game.

The thinking process is based on common knowledge of rationality of all players. An event E is said to be common knowledge if (i) everyone knows E, (ii) everyone knows that everyone knows E.

Empirical studies

With more than $10,000$ players, in the first round of the play, the mean is 34, mode is 50, median is 33. The winner is 23 since it is closest to 2/3 of the mean. The next mode is 1 while good number of players choose 100, 99, and 33 as well.

After learning the winner is 23, a subset of players were chosen to play the game second time. The new mean is 6, mode is 1, median is 2, while the winner is 4. The new outcomes are now close to the Nash equilibrium.

With more rounds of play, the outcome converges to the Nash equilibrium of "all players choose 1".

Subject pool	*Sample size*	*Mean*
Caltech board	73	49.4
80 year olds	33	37.0
High school students	52	32.5
Economics PhDs	16	27.4
Portfolio managers	26	24.3
Caltech students	24	21.5
Game theorists	136	19.1

Outcomes from various groups on the guessing two-thirds game

Voter participation

Two candidates, A and B, compete in an election. Of the n citizens, k support candidate A and m $(= n - k)$ support candidate B. Each citizen decides whether to vote, at a cost, for the candidate she supports, or to abstain.

- A citizen who abstains receives the payoff of 2 if the candidate she supports wins, 1 if this candidate ties for first place, and 0 if this candidate loses.

- A citizen who votes receives the payoffs $2 - c$, $1 - c$ and $-c$ in these three cases, where the cost c satisfies $0 < c < 1$.

The payoff values to a player under win, tie or lose when he plays "Vote" or "Abstain" are summarized as follows.

	win	tie	lose
Vote	$2-c$	$1-c$	$-c$
Abstain	2	1	0

(a) $k = m = 1$: Suppose player 1 supports A and player 2 supports B.

- If both vote, there is a tie. The payoffs for both are $1 - c$.
- If player 1 votes while player 2 abstains, player 1 has payoff $2-c$ while player 2 has zero payoff. Similar results are obtained if they swap their role.
- If both abstain, there is a tie and no cost incurred, so the payoffs for both are 1.

The bi-matrix game is depicted as follows.

I \ II	Vote	Abstain
Vote	$(1-c, 1-c)$	$(2-c, 0)$
Abstain	$(0, 2-c)$	$(1, 1)$

Note that the Vote strategy of one player always changes "tie" to "win" when the opponent chooses Abstain and changes "lose" to "tie" when the opponent chooses Vote. Therefore, Vote always guarantees a better payoff than Abstain, irrespective to the opponent's strategy. This game resembles the Prisoner's Dilemma, where Vote is the dominant strategy for both players.

The payoff (Abstain, Abstain) Pareto dominates (Vote, Vote).

Consider the general case, where $k + m > 2$.

1. $k = m > 1$

   When the number of supporting voters for both candidates are the same, there is only one Nash equilibrium with strategy profile: Vote for all voters.

   To show that Vote for all voters is a Nash equilibrium, it suffices to show that the best response of any player is Vote if all other players choose Vote. This is because Abstain chosen by this player leads to "lose" while Vote leads to tie, so Vote is the best response.

   In all other strategy profiles where some voters choose Abstain, any of these strategy profiles cannot be a Nash equilibrium. It suffices to show that some voters can be better off if he chooses to deviate unilaterally.

(a) tie
A voter who has chosen Abstain becomes better off if he changes from Abstain to Vote. He turns “tie” into “win”.

(b) one candidate wins by one vote
A voter who did not vote for the losing candidate becomes better off if he changes from Abstain to Vote. He turns “lose” into “tie”.

(c) one candidate wins by two votes
A voter who votes for the winning candidate becomes better off if he changes to Abstain and avoids the cost of voting. The outcome of the voting is not changed.

2. $k < m$

(a) With unequal number of supporting voters among the two candidates, all Vote is not a Nash equilibrium any more. This is because a supporter of the losing candidate can be better off by changing Vote to Abstain since his candidate remains losing. He saves the cost of voting.

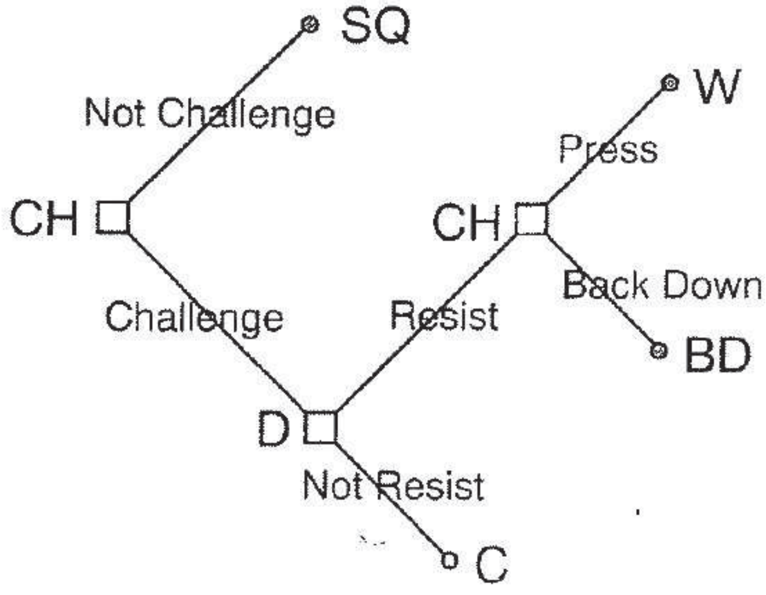
(b) Those strategy profiles with some voters choosing Abstain remain not to be a Nash equilibrium. Similar to the above analysis, it can be shown that some voters can be better off if he chooses to deviate unilaterally.

Game of Chicken — Cuban Missile Crisis

In October 1962, the US and USSR came close to a nuclear confrontation. President Kennedy estimated the probability of a nuclear war to be between $\frac{1}{3}$ to $\frac{1}{2}$. Political issues include:

- The USSR attempted to place offensive missile in Cuba.
- The US chose to respond to the Soviet missile emplacement with a blockade of Cuba.
- The Soviet Union decided to withdraw the missile (chicken out).

We may treat the crisis as a contest between a challenger (USSR) and a defender (US).



All Three Choices
in Deterrence Game

- Is it worthwhile for the challenger (CH) to attempt to change the status quo (SQ)? In the actual historical event, the Soviet was embarassed to withdraw missiles in Cuba. She might be better off not to challenge (changing the status quo).
- How did the other side (defender D) respond to the challenge with a threat? If D does not resist, then the game closes (C).
- Why did one side or the other back down (BD) or the crisis ended in a war (W)?

To model the historical event, the bimatrix of the game should take the following form:

US \ Soviet	no missiles placed in Cuba	placing missiles in Cuba
no blockade of Cuba	(4, 3)	(1, 4)
blockade of Cuba	(3, 2)	(2, 1)

- The challenger (Soviet) did not receive the highest payoff of 4 under the status quo $(4,3)$, so she had the temptation to move into military conflict $(1,4)$.
- $(4,3)$ is not a Nash equilibrium.
- After several sequential moves from the starting position is $(4,3)$, both players come to settle at $(4,3)$.

- The starting position is $(4, 3)$.

- When it comes to the Soviet to choose to move or stay, she chose to move to $(1, 4)$ in order to receive higher payoff of 4. However, US responded by moving to $(2, 1)$, then followed by the Soviet to move to $(3, 2)$. This ends up at lower payoff of 2 instead of 3 for the Soviet. If the Soviet were smart enough, they should not choose to move from $(4, 3)$ to $(1, 4)$.

- At the very end, US relaxed the blockade of Cuba and came back to the payoff of 4 (status quo).

- When it comes to the turn that US makes the move, US should not choose to change the status quo $(4, 3)$ to avoid the military expense of placing blockade in Cuba.

Six Day War (Yom Kippur War) $\mathbf{(1973)}$

- Israel fought against Egypt and Syria and gained upper hand within a few days.

- Soviet Union seriously considered intervening on behalf of Egypt and Syria, but they were aware of the US option of intervention. However, the US was faced with the Watergate scandal at home.

How did USSR and US rank the different outcomes, and was each aware of the other's preferences?

Payoff values as viewed by the Soviets

		Soviets	
		Seek diplomatic solution C	Supply Egypt and Syria with military aid N
U.S.	Cooperate with the Soviet initiative (nonintervention) C	(3, 3)	(2, 4)
	Frustrate the Soviet initiative (intervention) N	(4, 1)	(1, 2)

Frustrated by the Watergate scandal, Soviets thought that the US would rank the payoff to herself of outcome (C, N) to be ahead of that of outcome (N, N).

- Strategy N is a dominant strategy for the Soviets but US has no dominant strategy. Here, (C, N) is a Nash equilibrium.

Nixon's immediate goal is to convince the Soviets that the correct model is the Prisoner's Dilemma. If we assume simultaneous moves of the two players, then strategy N is a dominant strategy for both US and Soviets.

			Soviets	
			Seek diplomatic solution C	Supply Egypt and Syria with military aid N
U.S.	Cooperate with the Soviet initiative (nonintervention)	C	(3, 3)	(1, 4)
	Frustrate the Soviet initiative (intervention)	N	(4, 1)	(2, 2)

Payoff to US of outcome (N, N) is ahead of that of (C, N). Now, (N, N) is a Nash equilibrium. The game resembles the Prisoner's dilemma.

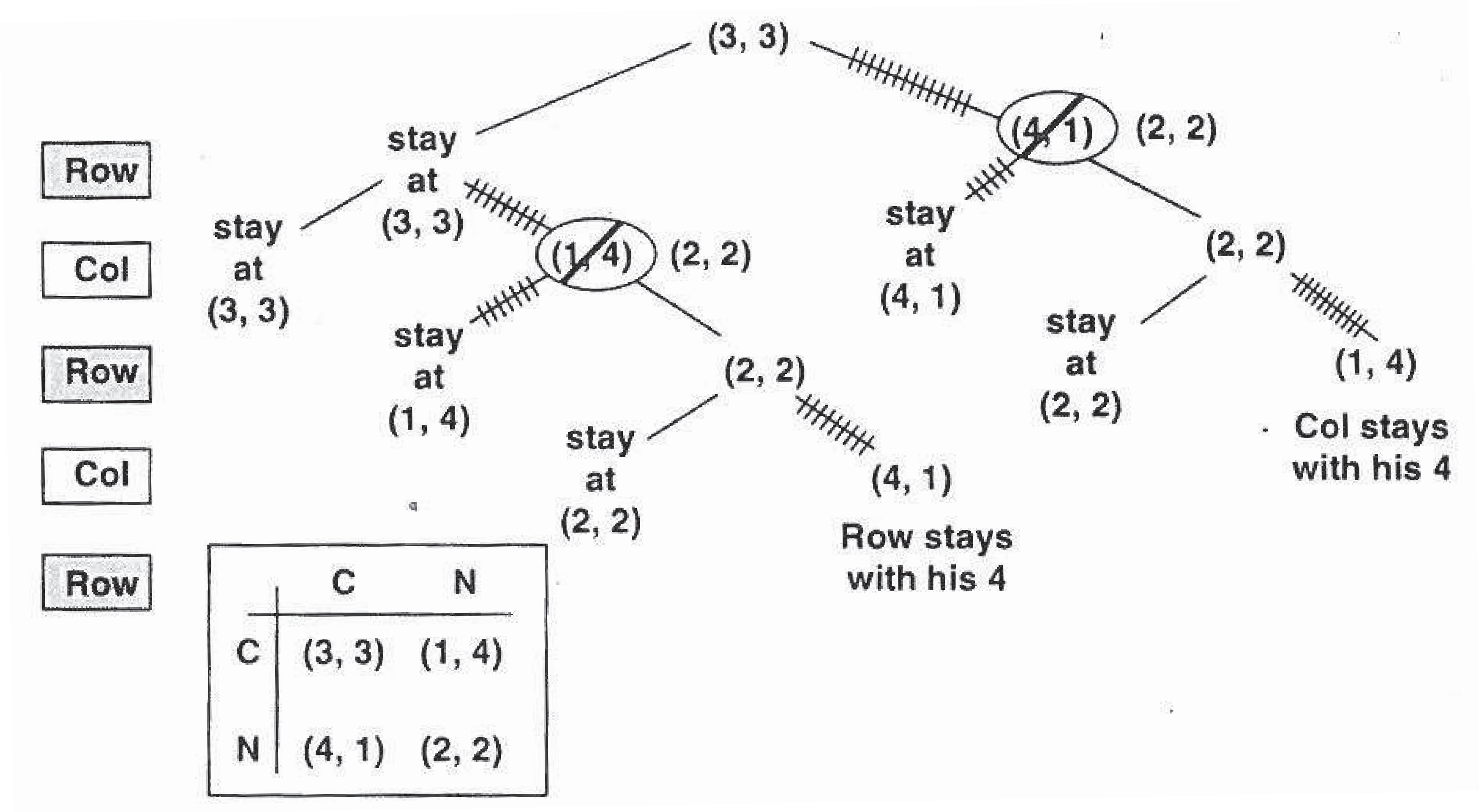
"Deliberate overreaction" by the US — placed the US forces on worldwide alert.

History told us that the two Great Powers ended up at the (3, 3) outcome. This is not a Nash equilibrium, but this is the starting position taken up by both countries since the game should begin with (C, C). The question is whether or not either side should change its status quo of C (cooperation) to N (non-cooperation).

The optimal strategies would change if the two players do not have to choose simultaneously.

- Both players make an initial simultaneous choice of C (initial position of the game).
- Allowing Row to move first, Row has the choice of staying or changing his strategy.
- Column has the same choices as did Row in Step 2.

Remark: Analysis is analogous if Column moves first.



Both players choose to stay at $(3,3)$. An initial move from $(3,3)$ to $(4,1)$ for the Row player leads to $(2,2)$ eventually. A similar argument can be applied to the Column player when he starts at $(3,3)$.

2.2 Two-person nonzero sum games under mixed strategies

In a two-person nonzero sum game, each player has his own payoff matrix:

$$A = \begin{pmatrix} a_{11} & ... & a_{1m} \\ . & & . \\ . & & . \\ . & & . \\ a_{n1} & ... & a_{nm} \end{pmatrix} \text{ and } B = \begin{pmatrix} b_{11} & ... & b_{1m} \\ . & & . \\ . & & . \\ . & & . \\ b_{n1} & ... & b_{nm} \end{pmatrix}.$$

A mixed strategy for Player I is $X = (x_1, ..., x_n) \in S_n$ with $x_i \geq 0$ representing the probability that Player I uses row i, and $x_1 + x_2 + ... + x_n = 1$. Similar definition for $Y = (y_1, ..., y_m) \in S_m$ for a mixed strategy for Player II, and $y_1 + y_2 + ... + y_m = 1$.

The expected payoffs are

$$E_{\mathrm{I}}(X,\ Y) = XAY^T \text{ for Player I,}$$
$$E_{\mathrm{II}}(X,\ Y) = XBY^T = YB^TX^T \text{ for Player II.}$$

Definition - Nash equilibrium

A pair of mixed strategies ($X^* \in S_n,\ Y^* \in S_m$) is a Nash equilibrium if the following pair of conditions on best respenses both hold:

$$E_{\mathrm{I}}(X,\ Y^*) \leq E_{\mathrm{I}}(X^*,\ Y^*) \text{ for every mixed } X \in S_n,$$
$$E_{\mathrm{II}}(X^*,\ Y) \leq E_{\mathrm{II}}(X^*,\ Y^*) \text{ for every mixed } Y \in S_m.$$

We write $v_{\mathrm{I}} = E_{\mathrm{I}}(X^*,\ Y^*)$ and $v_{\mathrm{II}} = E_{\mathrm{II}}(X^*,\ Y^*)$ as the expected payoff to each player under the mixed Nash equilibrium $(X^*,\ Y^*)$.

Neither player can gain any expected payoff if either one chooses to deviate unilaterally from playing his part of the Nash equilibrium, assuming that the other player is implementing his part of the Nash equilibrium.

Each strategy in a Nash equilibrium is a best response strategy against the opponent's Nash strategy. *Nash equilibrium corresponds to a "social norm"* (if every one adheres to it, no individual wishes to deviate from it).

Nash's Existence Theorem

Any n-player game with finite strategy sets A_i for all players has a Nash equilibrium in mixed strategies.

- In games for which players have opposing interests, like the Sum of Fingers game, there will be no pure-strategy Nash equilibrium but a mixed strategy Nash equilibrium always exists.

	odd	even
odd	$(1,-1)$	$(-1,1)$
even	$(-1,1)$	$(1,-1)$

- Allowing for mixed strategies enriches both what players can choose and what they can believe about the choices of other players.

Best response strategy

A mixed strategy $\widehat{X} \in S_n$ is a best response strategy to a given mixed strategy $Y^0 \in S_m$ for Player II if

$$E_{\mathrm{I}}(\widehat{X},\ Y^0) = \max_{X \in S_n} E_{\mathrm{I}}(X,\ Y^0).$$

In a Nash equilibrium $(X^*,\ Y^*)$, X^* maximizes $E_{\mathrm{I}}(X,\ Y^*)$ over all $X \in S_n$ and Y^* maximizes $E_{\mathrm{II}}(X^*,\ Y)$ over all $Y \in S_m$. In other words, X^* is a best response to Y^* and Y^* is a best response to X^*.

In a game model, unlike a standard procedure of maximizing a function, a Nash equilibrium does not maximize the individual payoff. A typical example is the Prisoner's dilemma. One player can achieve the maximum payoff provided that the other player agrees to play that strategy accordingly (compromising her right to achieve higher payoff). It is unlikely that the maximization of Player I's payoff occurs at X when Player II agrees to play Y; while at the same time this pair (X, Y) also maximizes Player II's payoff.

Remarks

1. If $B = -A$, a bi-matrix game is a zero sum two-person game. A Nash equilibrium is the same as a saddle point in mixed s-trategies. To see this, $(X^*,\ Y^*)$ is a Nash equilibrium for both players if and only if

$$X^*AY^{*T} \geq XAY^{*T},\ \forall X \in S_n;$$

$$X^*(-A)Y^{*T} \geq X^*(-A)Y^T,\ \forall Y \in S_m$$

$$\Leftrightarrow \quad E(X,\ Y^*) \leq E(X^*,\ Y^*) \leq E(X^*,\ Y),\ \forall X \in S_n \text{ and } Y \in S_m.$$

The last pair of inequalities is equivalent to $(X^*,\ Y^*)$ being a saddle point of the zero sum game.

2. A Nash equilibrium in pure strategies will be a row i^* and column j^* satisfying

$$a_{ij^*} \leq a_{i^*j^*} \text{ and } b_{i^*j} \leq b_{i^*j^*},\ i = 1, 2, ..., n \text{ and } j = 1, 2, ..., m.$$

That is, $a_{i^*j^*}$ is the largest in column j^* and $b_{i^*j^*}$ is the largest in row i^* simultaneously.

- The payoff of a player in taking any pure strategy cannot be greater than the payoff obtained from the mixed strategy in the Nash equilibrium when the other player stays with the Nash equilibrium strategy. This is easily seen since maximization over all i (pure strategies) should give lower payoff or at most the same payoff when compared with the maximization over all $X \in S_m$ (mixed strategies). That is, $E_I(k, Y^*) \le E_I(X^*, Y^*)$, for all k.

- If a particular pure strategy has a payoff that is lower than that of the mixed strategy Nash equilibrium, then it should be ruled out in the mixed strategy. That is, $E_I(k, Y^*) < E_I(X^*, Y^*)$, then $x_k = 0$. Equivalently, $x_k > 0$ implies $E_I(k, Y^*) = E_I(X^*, Y^*)$.

- All pure strategies used in the mixed strategy Nash equilibrium should have the same payoff in order that payoff of the mixed strategy and any weighted average of the payoffs of the pure strategies in the mixed strategy Nash equilibrium would be the same. This is essentially the statement of the Equality of Payoff Theorem.

Theorem – Equality of Payoff

Let $X^* = (x_1, ..., x_n)$ and $Y^* = (y_1, ..., y_m)$ be a mixed Nash equilibrium. Suppose $x_k > 0$ for some k, then $E_{\mathrm{I}}(k, Y^*) = E_{\mathrm{I}}(X^*, Y^*)$.

Proof

(i) Since Player I cannot be better off by deviating from X^*, obviously, $E_{\mathrm{I}}(i, Y^*) \le E_{\mathrm{I}}(X^*, Y^*) = v_{\mathrm{I}}$, for any i.

(ii) We prove by contradiction. Suppose $x_k > 0$ and $E_{\mathrm{I}}(k, Y) < v_{\mathrm{I}}$, then

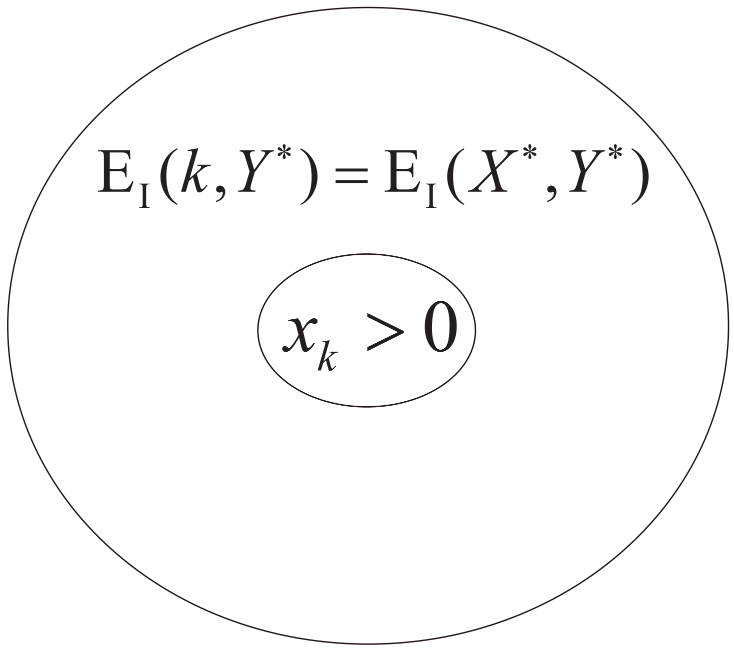
$$x_i v_{\mathrm{I}} \ge x_i E_{\mathrm{I}}(i, Y^*),\ \ i \ne k \text{ and } x_k v_{\mathrm{I}} > x_k E_{\mathrm{I}}(k, Y^*).$$

We then have

$$\sum_{i=1}^{n} x_i v_{\mathrm{I}} = v_{\mathrm{I}} > \sum_{i=1}^{n} x_i E_{\mathrm{I}}(i, Y^*) = E_{\mathrm{I}}(X^*, Y^*) = v_I \text{ which is simply } v_{\mathrm{I}}.$$

A contradiction is encountered.

Suppose there exists $E_{\mathrm{I}}(l, Y^*) < E_{\mathrm{I}}(X^*, Y^*)$, where the l^{th} strategy of Row is an under performer, then we must have $x_l = 0$, otherwise we will encounter the above contradiction. In other words, the pure strategy corresponds to l must be ruled out in the mixed strategy Nash equilibrium.



Battle of Sexes (cooperation with compromise) revisited

We explore whether a mixed Nash equilibrium exists in the game of "Battle of Sexes". Note that (Boxing, Boxing) and (Ballet, Ballet) are Pareto-efficient pure Nash equilibriums.

			Wife: q Boxing	Wife: $1-q$ Ballet
Husband	p	Boxing	**(3,2)**	**(1,1)**
	$1-p$	Ballet	**(0,0)**	**(2,3)**

Under the mixed strategy (p, q), where $X = (p, 1-p)$ and $Y = (q, 1-q)$, the expected value of the husband is given by $\pi_1(p, q) = XAY^T$:

$$\begin{aligned}\pi_1(p,q) &= pE_{\mathrm{I}}(1,Y) + (1-p)E_{\mathrm{I}}(2,Y)\\ &= p[3q + (1-q)] + (1-p)[2(1-q)]\\ &= p(2q+1) + (1-p)(2-2q)\end{aligned}$$

while that of the wife is

$$\pi_2(p,q) = q(2p) + (1-q)[p + 3(1-p)] = q(2p) + (1-q)(3-2p).$$

Husband's best response function

- If $q > \frac{1}{4}$, consider the husband's payoffs: expected payoff of Boxing $= E_{\mathrm{I}}(1, Y) = 3q + 1 - q = 2q + 1$ is higher than that of Ballet $= E_{\mathrm{I}}(2, Y) = 2(1 - q)$ since $2q + 1 > 2 - 2q \Leftrightarrow q > \frac{1}{4}$. Since $E_{\mathrm{I}}(1, Y) > E_{\mathrm{I}}(2, Y)$, so husband's unique best response is $X = (1, 0)$, that is, $p = 1$.

- If $q < \frac{1}{4}$, then $E_{\mathrm{I}}(1, Y) < E_{\mathrm{I}}(2, Y)$. The husband's unique best response is $X = (0, 1)$, that is, $p = 0$.

- If $q = \frac{1}{4}$, then $E_{\mathrm{I}}(1, Y) = E_{\mathrm{I}}(2, Y)$. All mixed strategies played by the husband yield the same expected payoff, that is, $0 \leq p \leq 1$. In this case, $\pi_1(p, q)$ is independent of p. As a check, $\frac{\mathrm{d}\pi_1}{\mathrm{d}p} = 4q - 1$ and setting it be zero gives $q = \frac{1}{4}$.

Husband's best response function is

$$B_1(q) = \begin{cases} p = 0 & \text{if } q < \frac{1}{4}, \\ p \in [0, 1] & \text{if } q = \frac{1}{4}, \\ p = 1 & \text{if } q > \frac{1}{4}. \end{cases}$$

Wife's best response function

In a similar manner, we can show that

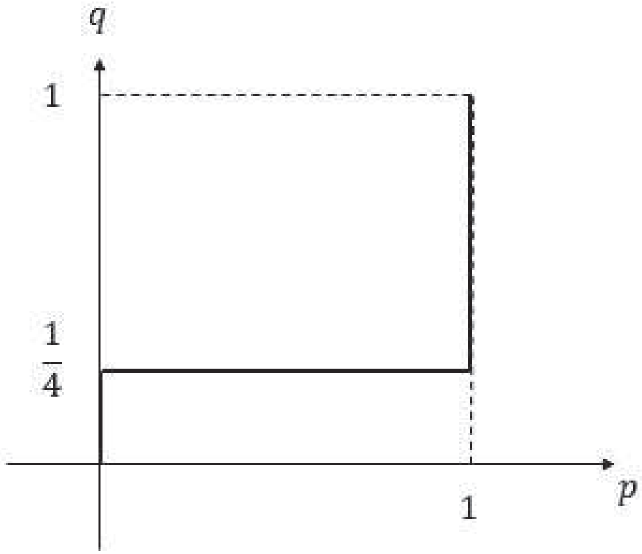
wife's expected payoff of Boxing $>$ wife's expected payoff of Ballet

$$\begin{aligned} &\Leftrightarrow \quad 2p = E_{\mathrm{II}}(X,1) > E_{\mathrm{II}}(X,2) = p+3(1-p) = 3-2p \\ &\Leftrightarrow \quad p > \frac{3}{4}. \end{aligned}$$

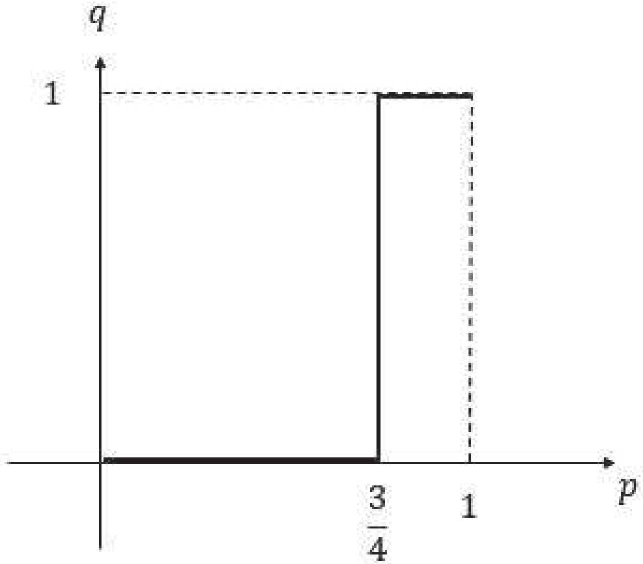
Wife's best response function is

$$B_2(p) = \begin{cases} q = 0 & \text{if } p < \frac{3}{4}, \\ q \in [0,1] & \text{if } p = \frac{3}{4}, \\ q = 1 & \text{if } p > \frac{3}{4}. \end{cases}$$

Suppose the husband chooses the mixed strategy as characterized by the probability vector $(\frac{3}{4}, \frac{1}{4})$ of choosing Boxing and Ballet while the wife chooses the probability vector $(\frac{1}{4}, \frac{3}{4})$ in her mixed strategy, then each mixed strategy is in the respective player's best response function. Hence, $\left\{(\frac{3}{4}, \frac{1}{4}), (\frac{1}{4}, \frac{3}{4})\right\}$ is a mixed strategy Nash equilibrium.



Husband's best response correspondence



Wife's best response correspondence

The pure and mixed Nash equilibriums can be found by the intersection points of the best response correspondences of the two players. The two best response correspondences intersect at 3 points in the p-q diagram:

(i) $(0,0)$ that corresponds to the outcome (Ballet, Ballet);

(ii) $(\frac{3}{4},\frac{1}{4})$ that corresponds to $75-25$ chance of (Boxing, Ballet) for the husband and $25-75$ chance of (Boxing, Ballet) for the wife, a reasonable result since (Boxing, Ballet) has higher outcome than that of (Baller, Boxing);

(iii) $(1,1)$ that corresponds to the outcome (Boxing, Boxing).

Finding all Nash equilibriums for a two-person 2×2 nonzero sum game

Let $X = (x, 1 - x)$ and $Y = (y, 1 - y)$, we write

$$\begin{aligned} f(x,y) &= E_{\mathrm{I}}(X,\ Y) = XAY^T = (x, 1-x)\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} y \\ 1-y \end{pmatrix} \\ &= xE_{\mathrm{I}}(1, Y) + (1-x)E_{\mathrm{I}}(2, Y), \\ g(x,y) &= E_{\mathrm{II}}(X,\ Y) = XBY^T = (x, 1-x)\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}\begin{pmatrix} y \\ 1-y \end{pmatrix} \\ &= yE_{\mathrm{II}}(X, 1) + (1-y)E_{\mathrm{II}}(X, 2). \end{aligned}$$

Note that $E_I(X, Y)$ is a weighted average (according to different choices of x) of $E_I(1, Y)$ and $E_I(2, Y)$. Also, $\dfrac{\partial f}{\partial x} = 0 \Leftrightarrow E_I(1, Y) = E_I(2, Y)$, which means that Player I is indifferent to different choices of $X = (x, 1 - x)$.

Write R_{I} and R_{II} as the best response correspondence of Player I and Player II, respectively.

- A point $(x^*, y) \in R_{\text{I}}$ means that x^* is the point in $[0, 1]$ where $f(x, y)$ is maximized at x^* for y fixed. That is, $X^* = (x^*, 1 - x^*)$ is a best response to $Y = (y, 1 - y)$.
- A point (x^*, y^*) in both R_{I} and R_{II} means that $X^* = (x^*, 1 - x^*)$ and $Y^* = (y^*, 1 - y^*)$, as best responses to each other, is a Nash equilibrium. Note that $x^* = B_1(y^*)$ and $y^* = B_2(x^*)$.

Recall $f(x, y) = xE_{\text{I}}(1,\ Y) + (1 - x)E_{\text{I}}(2,\ Y)$ so that

$$\begin{aligned}
\max_{0 \le x \le 1} f(x, y) &= \max_{0 \le x \le 1} x[E_{\text{I}}(1,\ Y) - E_{\text{I}}(2,\ Y)] + E_{\text{I}}(2,\ Y) \\
&= \begin{cases} E_{\text{I}}(2,\ Y) \text{ at } x = 0 \text{ if } E_{\text{I}}(1,\ Y) < E_{\text{I}}(2,\ Y); \\ E_{\text{I}}(1,\ Y) \text{ at } x = 1 \text{ if } E_{\text{I}}(1,\ Y) > E_{\text{I}}(2,\ Y); \\ E_{\text{I}}(2,\ Y) \text{ at any } 0 \le x \le 1 \text{ if } E_{\text{I}}(1,\ Y) = E_{\text{I}}(2,\ Y). \end{cases}
\end{aligned}$$

There are 3 possible cases:

(i) $x = 0$ if $E_I(1, Y) < E_I(2, Y)$;

(ii) $x = 1$ if $E_I(1, Y) > E_I(2, Y)$;

(iii) x takes any values in $[0, 1]$ if $E_I(1, Y) = E_I(2, Y)$.

That is, player I does not play strategy 1 if $E_{\mathrm{I}}(1,\ Y) < E_{\mathrm{I}}(2,\ Y)$. These results are consistent with the equality of payoff theorem.

The sign of $E_I(1,Y) - E_I(2,Y)$ depends on $Y = (y, 1-y)$. When y is chosen such that $E_{\mathrm{I}}(1,Y) = E_{\mathrm{I}}(2,Y)$, then $f(x,y)$ is independent of x so that $\dfrac{\partial f}{\partial x} = 0$. Solving $E_{\mathrm{I}}(1,Y) = E_{\mathrm{I}}(2,Y)$, we obtain

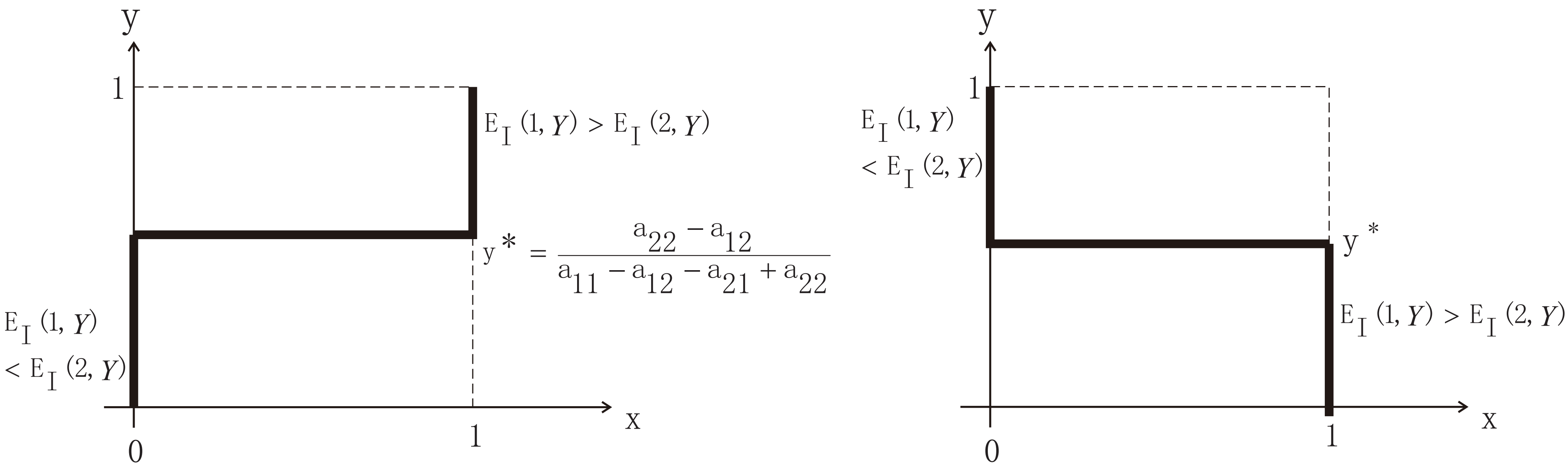
$$
\begin{aligned}
& (a_{11}, a_{12}) \begin{pmatrix} y \\ 1-y \end{pmatrix} = (a_{21}, a_{22}) \begin{pmatrix} y \\ 1-y \end{pmatrix} \\
\Leftrightarrow \quad & a_{11}y + a_{12}(1-y) = a_{21}y + a_{22}(1-y)
\end{aligned}
$$

giving

$$
y^* = \frac{a_{22} - a_{12}}{a_{11} - a_{12} - a_{21} + a_{22}}.
$$

When Player II plays $Y^* = (y^*, 1-y^*)$, Player I obtains the same payoff from each of his pure strategies.

Best response correspondence of Player I

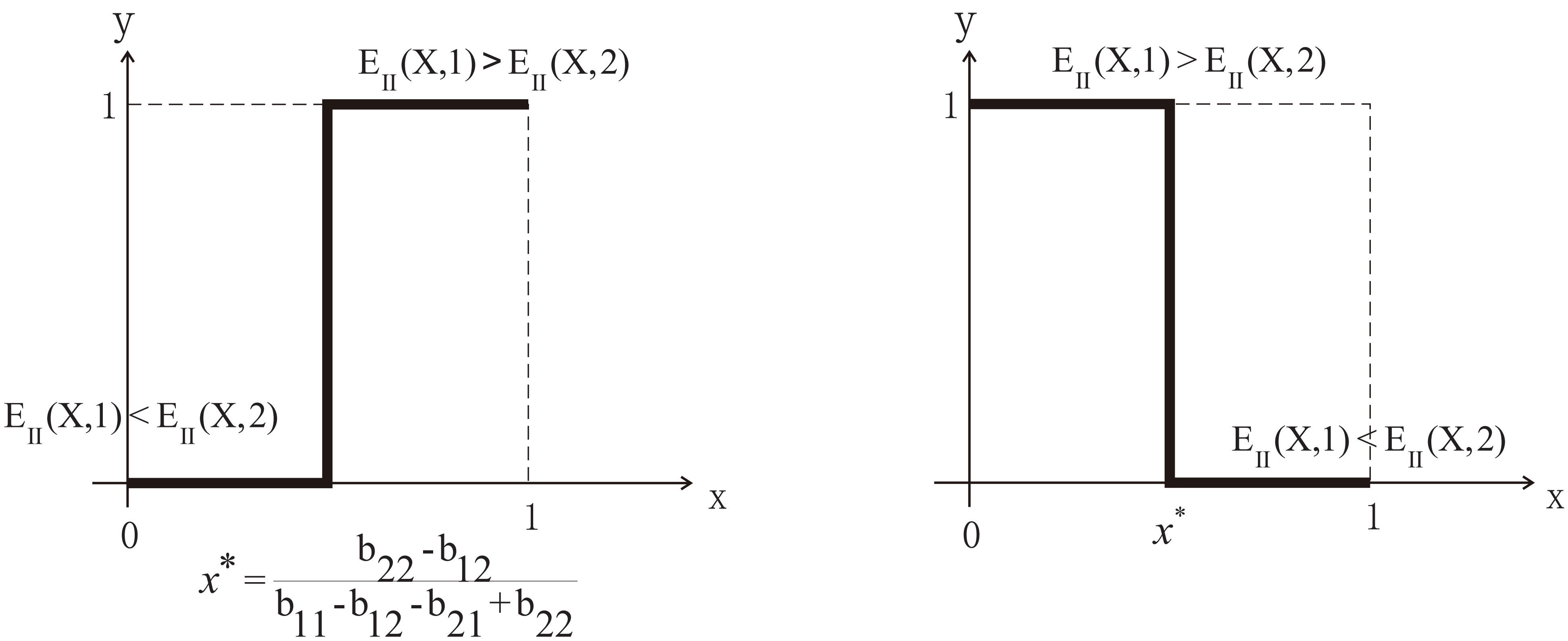


The best response correspondences of Player I can take two different forms, depending on the relative magnitude of $E_{\mathrm{I}}(1, y)$ and $E_{\mathrm{I}}(2, y)$ at a given $Y = (y, 1 - y)$, $y \neq y^*$. In the left figure, we have $E_{\mathrm{I}}(1, y) < E_{\mathrm{I}}(2, y)$ for $y < y^*$. Therefore, the best response of Player I to this value of y is $x^* = 0$.

Best response correspondence of Player II

Recall that x^* is determined by $E_{\mathrm{II}}(X,1) = E_{\mathrm{II}}(X,2)$. This gives

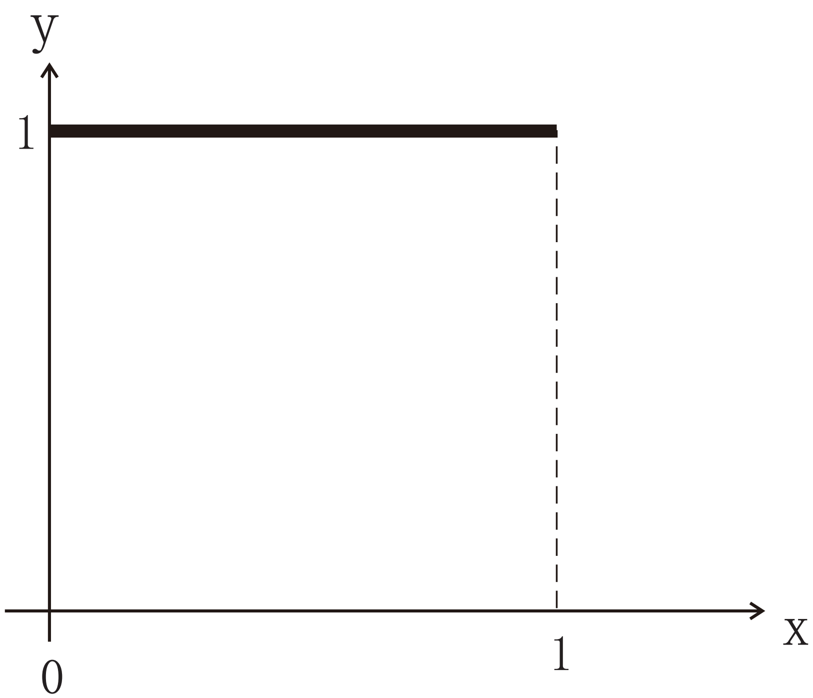
$$x^* = \frac{b_{22} - b_{21}}{b_{11} - b_{12} - b_{21} + b_{22}}.$$



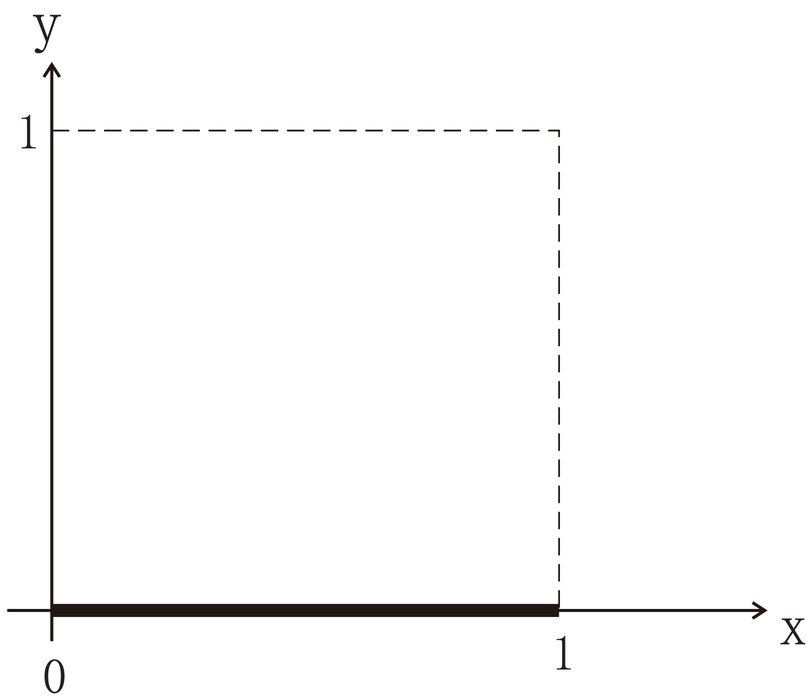
In the left figure, we have $E_{\mathrm{II}}(x,1) < E_{\mathrm{II}}(x,2)$ for $x < x^*$. Therefore, the best response of Player II to this value of x is $y^* = 0$.

It may occur that $x^* \leq 0$ or $x^* \geq 1$. This gives rise to the degenerated case of pure strategy $y^* = 1$ or $y^* = 0$ for Player **II**.

For $x^* \leq 0$, we have the following cases:

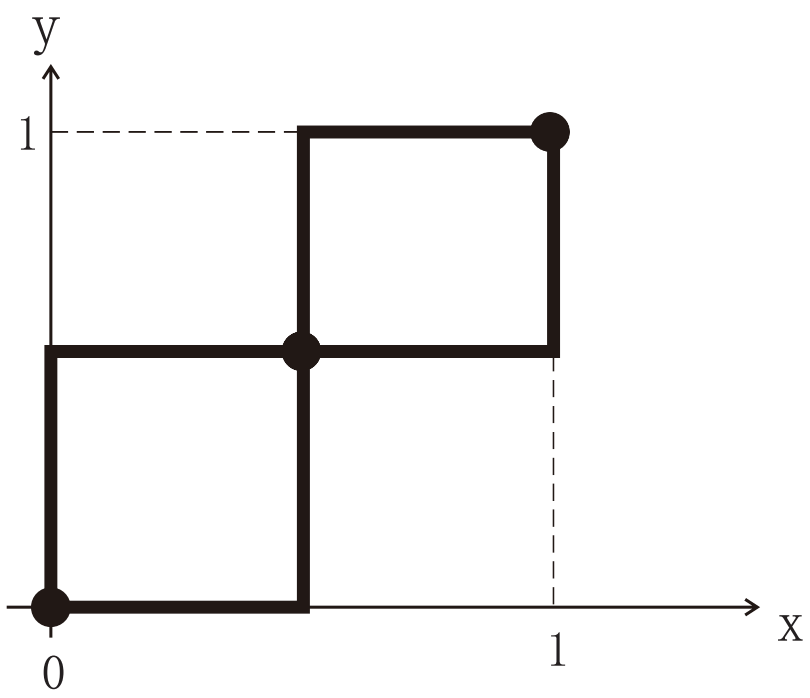


When $E_{\mathrm{II}}(X, 1) > E_{\mathrm{II}}(X, 2)$ and $x^* \leq 0, y^* = 1$ for all $x \in [0, 1]$

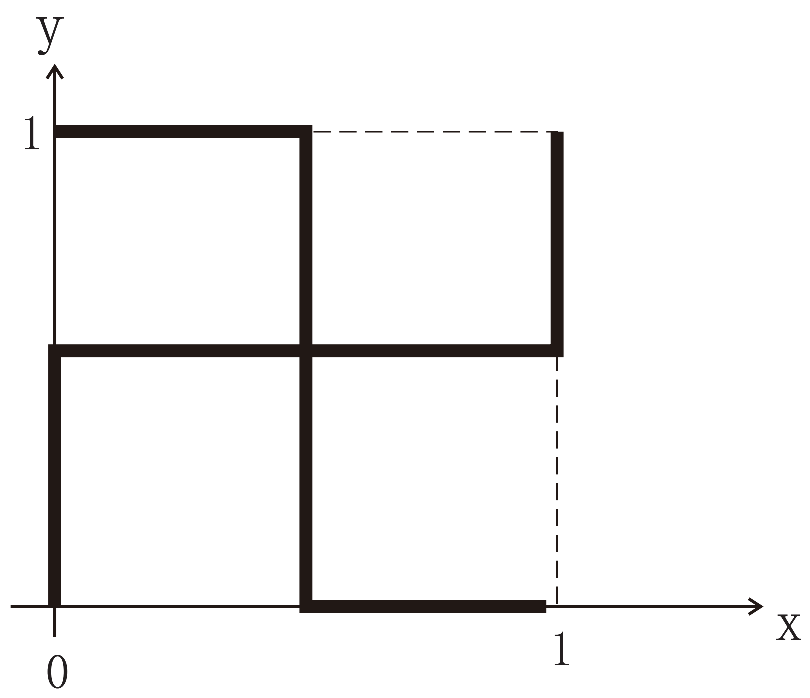


When $E_{\mathrm{II}}(X, 1) < E_{\mathrm{II}}(X, 2)$ and $x^* \leq 0, y^* = 0$ for all $x \in [0, 1]$.

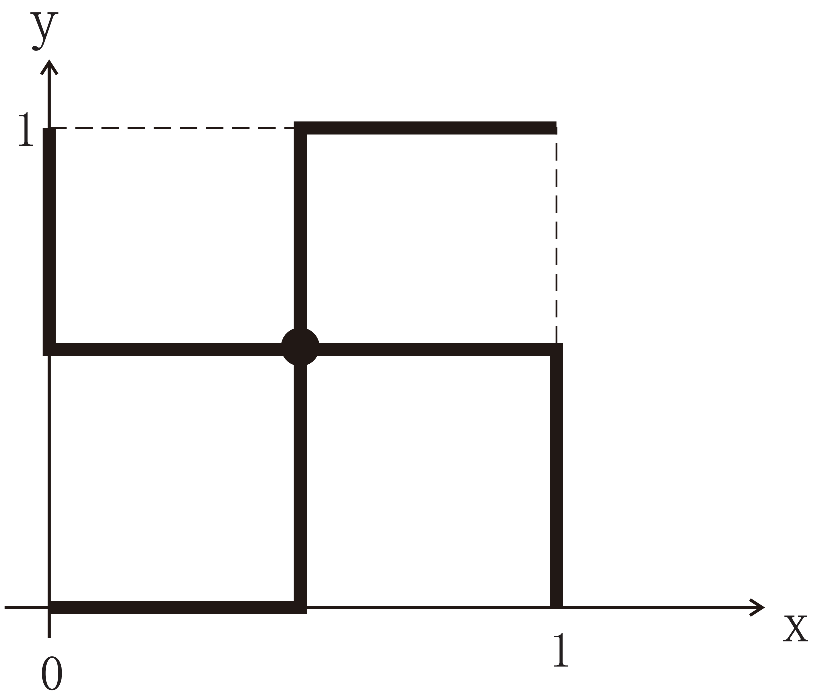
Four possible cases of intersection of the best response correspondences of the two players under $0 < x^* < 1$ and $0 < y^* < 1$



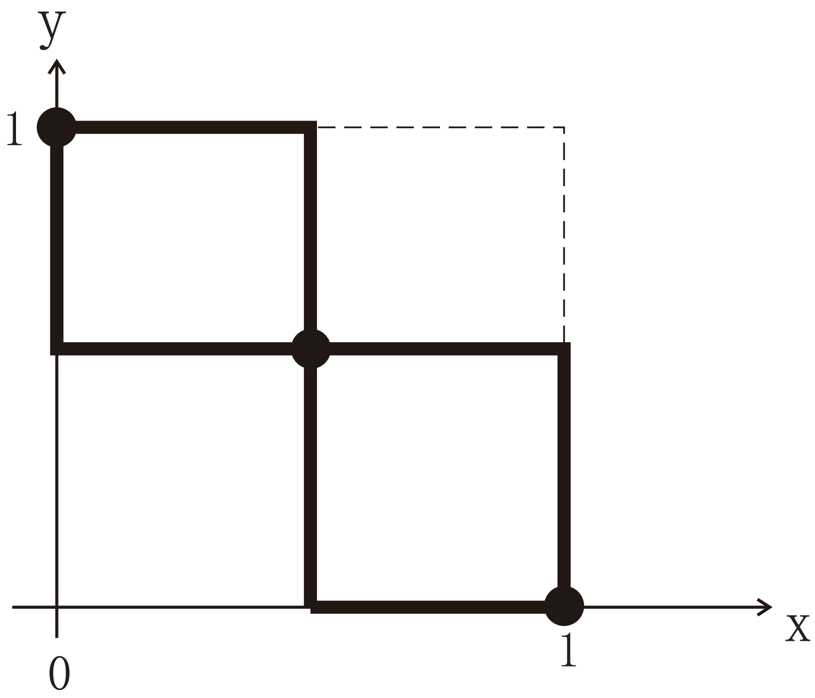
$(0,0)$, $(1,1)$, (x^*, y^*): two pure Nash equilibriums and one mixed Nash equilibrium



(x^*, y^*): one mixed Nash equilibrium



(x^*, y^*): one mixed Nash equilibrium



$(0, 1)$, $(1, 0)$, (x^*, y^*): two pure Nash equilibriums and one mixed Nash equilibrium

Welfare Game

The Welfare Game models a government that wishes to aid a pauper if he searches for work (see the left arrow), and a pauper who searches for work only if he cannot depend on government aid (see the bottom arrow). The payoff to Government under (No Aid, Work) is -1 since this reflects failure of public policy.

			Pauper	
		Work (γ_w)		*Loaf* ($1-\gamma_w$)
	Aid (θ_a)	$3, 2$	$\rightarrow$	$-1, 3$
Government		$\uparrow$		$\downarrow$
	No Aid ($1-\theta_a$)	$-1, 1$	$\leftarrow$	$0, 0$

Payoffs to: (Government, Pauper). Arrows show how a player can increase his payoff.

Neither player has a dominant strategy and no pure strategy Nash equilibrium exists. For example, (Aid, Work) is not a Nash equilibrium since the pauper would respond with Loaf if the government picked Aid.

(Aid, Work) does not Pareto-dominate all other outcomes. Suppose (Aid, Work)'s outcome is changed to $(3, 4)$, then (Aid, Work) Pareto-dominates all other outcomes, and thus it is a Nash equilibrium.

The government's expected payoff

$$
\begin{aligned}
\pi_{\text{gov}} &= \theta_a \pi_{\text{gov}}(\text{aid}, Y) + (1-\theta_a)\pi_{\text{gov}}(\text{no aid}, Y) \\
&= \theta_a(3, -1)\begin{pmatrix} \gamma_w \\ 1-\gamma_w \end{pmatrix} + (1-\theta_a)(-1, 0)\begin{pmatrix} \gamma_w \\ 1-\gamma_w \end{pmatrix} \\
&= \theta_a[3\gamma_w + (-1)(1-\gamma_w)] + (1-\theta_a)[(-1)\gamma_w + 0(1-\gamma_w)] \\
&= \theta_a(5\gamma_w - 1) - \gamma_w.
\end{aligned}
$$

In the mixed extension, the government's action of θ_a lies in $[0,\ 1]$, the pure strategies correspond to the extreme values 0 and 1.

Based on the indifference principle, we find the mixed Nash equilibrium strategy of the pauper by computing the first order condition for the government. This gives

$$0 = \frac{\mathrm{d}\pi_{\text{gov}}}{\mathrm{d}\theta_a} = 5\gamma_w - 1 \text{ so that } \gamma_w = 0.2.$$

The first order condition dictates that at $\gamma_w = 0.2$, the government is indifferent between aid and no aid. Note that

$$
\begin{aligned}
& E_{\text{gov}}(\text{aid}, Y) \geq E_{\text{gov}}(\text{no aid}, Y) \\
\Leftrightarrow \quad & 3\gamma_w + (-1)(1-\gamma_w) \geq (-1)\gamma_w + 0(1-\gamma_w) \Leftrightarrow \gamma_w \geq 0.2.
\end{aligned}
$$

Depending on the pauper's strategy, we consider the following 3 strategies that maximize the government's payoff.

(i) When $\gamma_w < 0.2$ (pauper is less likely to work), then government choose $\theta_a = 0$ and $\pi_{\text{gov}} = -\gamma_w$.

(ii) More likely to work so that $\gamma_w > 0.2$, then government chooses $\theta_a = 1$ and $\pi_{\text{gov}} = 5\gamma_w - 1$.

(iii) On the border line of $\gamma_w = 0.2$, the government is indifferent to any probability of aid. The expected payoff to government is -0.2, independent on θ_a. This explains why $\frac{\mathrm{d}\pi_{\text{gov}}}{\mathrm{d}\theta_a} = 0$ at $\gamma_w = 0.2$.

The strategy played by the government depends on probability of selecting Work by the pauper. God only saves those who want to save themselves.

To obtain the probability of the government choosing Aid, consider

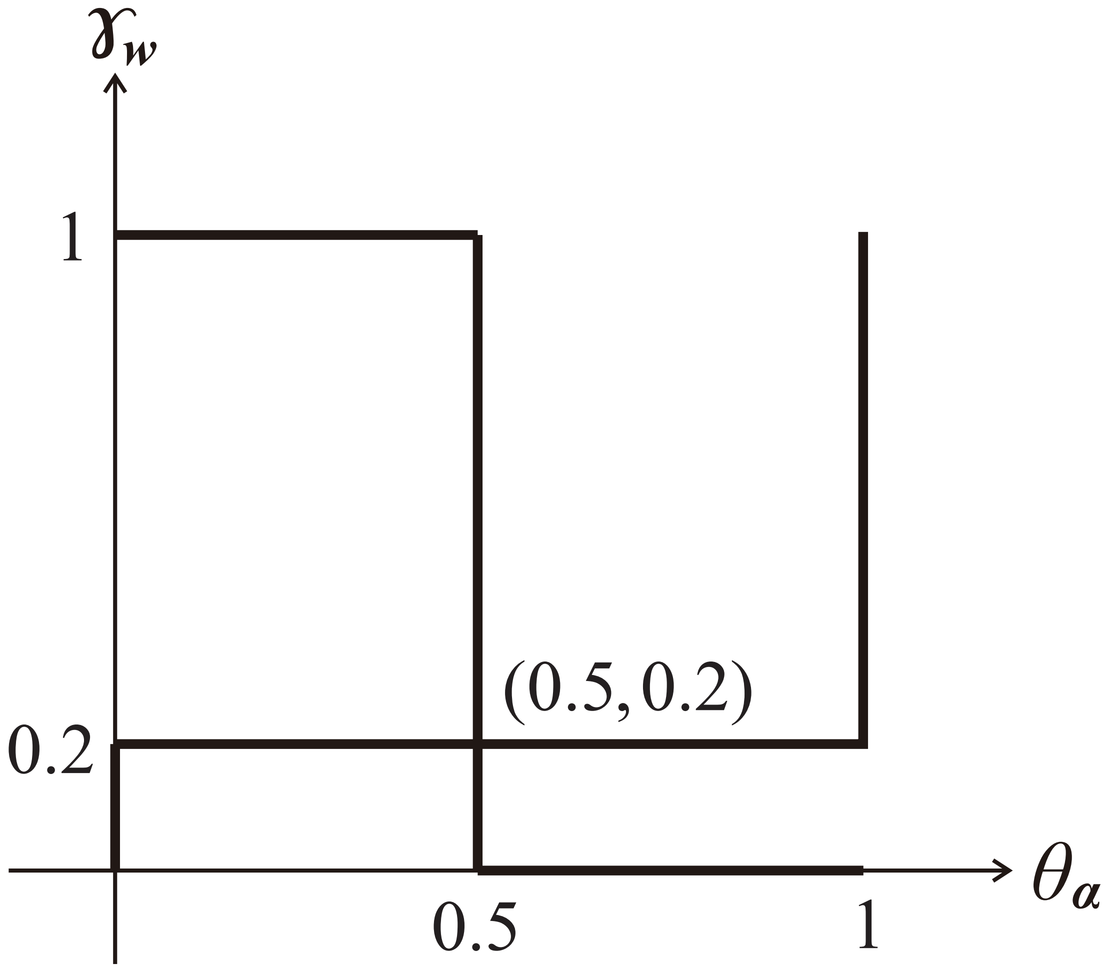
$$\begin{aligned}
\pi_{\text{pauper}} &= \gamma_w \pi_{\text{pauper}}(X, \text{work}) + (1-\gamma_w)\pi_{\text{pauper}}(X, \text{loaf}) \\
&= \gamma_w(\theta_a, 1-\theta_a)\begin{pmatrix} 2 \\ 1 \end{pmatrix} + (1-\gamma_w)(\theta_a, 1-\theta_a)\begin{pmatrix} 3 \\ 0 \end{pmatrix} \\
&= \gamma_w[2\theta_a + (1-\theta_a)] + (1-\gamma_w)[3\theta_a + 0\cdot(1-\theta_a)] \\
&= -\gamma_w(2\theta_a - 1) + 3\theta_a.
\end{aligned}$$

The first order condition is

$$\frac{\mathrm{d}\pi_{\text{pauper}}}{\mathrm{d}\gamma_w} = -(2\theta_a - 1) = 0 \text{ giving } \theta_a = 0.5.$$

Note that $\theta_a = 0.5$ and $\gamma_w = 0.2$ is in the intersection point of the players' best response functions.

In the mixed strategy Nash equilibrium, the government selects Aid with probability 0.5 and the pauper selects Work with probability 0.2. The probability of Work can be interpreted as the percentage of a group of paupers that select Work.



The intersection of the two best response functions gives the mixed Nash equilibrium $(0.5, 0.2)$ for the Welfare game.

Chicken Game

		Jones		
		Continue (θ)		*Swerve* ($1-\theta$)
	Continue (θ)	$-3, -3$	$\rightarrow$	**2, 0**
Smith		$\downarrow$		$\uparrow$
	Swerve ($1-\theta$)	**0, 2**	$\leftarrow$	1, 1

Payoffs to: (Smith, Jones). Arrows show how a player can increase his payoff.

Payoff-equating method

This method is applicable only when solution of *completely* mixed strategies exists; that is, $\theta \in (0, 1)$. Assuming existence of completely mixed strategy equilibrium of the Column player, Jones, we equate the expected payoffs to give

$$
\begin{aligned}
\pi_{Jones}(Swerve) &= (\theta_{Smith}) \cdot (0) + (1 - \theta_{Smith}) \cdot (1) \\
&= \pi_{Jones}(Continue) \\
&= (\theta_{Smith}) \cdot (-3) + (1 - \theta_{Smith}) \cdot (2).
\end{aligned}
$$

We obtain $1 - \theta_{Smith} = 2 - 5\theta_{Smith}$ so $\theta_{Smith} = 0.25$. Under this choice of θ_{Smith}, Jones is indifferent to choose either strategy.

In the symmetric equilibrium, both players use the same probability, so we replace θ_{Smith} by θ. The two players will survive when the event that both choose continue does not occur and the probability $1 - \theta^2 = 0.9375$.

Suppose we change -3 to x, we obtain $\theta = \frac{1}{1-x}$. If $x = -3$, this gives $\theta = 0.25$ as before. If $x = -9$, it gives $\theta = 0.1$ (making good sense). Increasing the loss from crashes reduces the equilibrium probability of continuing down the middle of the road.

If $x = 0.5$, then $\theta = 2$. This does not make sense since probability must be bounded above by one. There will be no completely mixed strategy Nash equilibrium.

This absurdity of probability is a valuable aid to the fallible modeller. With positive value of x, both drivers would choose to continue down the road since crashes provide gain. Obviously, (Continue, Continue) is a dominant strategy equilibrium.

Civic Duty Game

A notorious example in social psychology: murder of Kitty Genovese in New York City

"For more than half an hour, 38 respectable and law-abiding citizens in Queens watched a killer stalk and stab a woman in three separate attacks in Kew Gardens ... Twice the sound of their voices and the sudden glow of their bedroom lights interrupted him and frightened him off. Each time he returned, sought her out, and stabbed her again. Not one person telephoned the police during the assault: one witness called after the woman was dead."

- Higher chance of complete "Ignore" when the number of spectators increases. It is more likely that spectators speculate that there would be somebody who would take the trouble to call the police.

The Civic Duty Game

			Jones	
		Ignore (γ)		*Telephone* ($1-\gamma$)
	Ignore (γ)	0, 0	$\rightarrow$	**10, 7**
Smith		$\downarrow$		$\uparrow$
	Telephone ($1-\gamma$)	**7, 10**	$\leftarrow$	7, 7

Payoffs to:(Row, Column). Arrows show how a player can increase his payoff.

The Civic Duty Game has two asymmetric pure strategy Nash equilibrium (one of the citizens who is more Civic Duty minded calls) and a symmetric mixed strategy Nash equilibrium (each one calls with the same probability that is less than one).

If the players are drawn from a single homogeneous population and there is no way for them to cooperate, then a symmetric equilibrium (everyone uses the same mixed strategy) is more compelling.

N-player mixed strategy equilibrium

Recall that each would like someone to call the police and stopped the crime (payoff = 10). However, neither wishes to make the call himself since the effort subtracts 3.

We determine the Nash equilibrium strategy of other $N-1$ players (with common value of γ) using the equality of payoff of Smith under the two pure strategies of Telephone and Ignore. We obtain

$$\begin{aligned} \pi_{Smith}(Telephone) = 7 &= \pi_{Smith}(Ignore) \\ &= \gamma^{N-1}(0) + (1-\gamma^{N-1})(10), \end{aligned}$$

where γ^{N-1} is the probability that none of $N-1$ players call. This gives $\gamma^{N-1} = 0.3$ or $\gamma^* = 0.3^{\frac{1}{N-1}}$. When $N = 2$, we obtain $\gamma^* = 0.3$ and the probability of Ignore by both is $0.3^2 = 0.09$.

When $N = 2$ or $N = 38$, the expected payoff $\pi_{Smith}(Ignore)$ remains the same since it is always equal to $\pi_{Smith}(Telephone) = 7$. When $N = 38$, $\gamma^* = 0.97$ and $(\gamma^*)^{38} = 0.29$. The probability of Ignore by all is given by $(0.3)^{\frac{N}{N-1}}$, which increases with N.

Example - Expert diagnosis

Statement of the problem

I am relatively ill-informed about my car, computer or body stops working properly. I consult an expert, who makes a diagnosis and recommends an action.

- I am not sure whether the diagnosis is correct - the expert, after all, has an interest in selling his services.
- Should I follow the expert's advice or try to fix the problem myself, put up with it, or consult another expert?

There are two types of problems: major and minor.

- The expert knows, on seeing a problem, whether it is major or minor.
- The consumer knows only the probability of a major problem, r.

Expert's strategies and payoffs

An expert may recommend either a major or a minor repair (regardless of the true nature of the problem). A major repair fixes both a major problem and a minor one.

- The expert obtains the same profit $\pi > 0$ from (i) selling a minor repair with a minor problem (ii) selling a major repair with a major problem. However, he obtains π' (which is greater than π) from selling a major repair to a minor problem.

Two strategies of the expert

- Honest: recommend a minor (major) repair for a minor (major) problem

- Dishonest: always recommend a major repair

Customer's strategies and payoffs

The customer pays E for a major repair and $I < E$ for a minor one.

The cost he effectively bears if he chooses some other remedy is $E' > E$ if his problem is major and $I' > I$ if it is minor.

Furthermore, we assume $E > I'$; otherwise, seeking for other remedy for minor repair is irrelevant. Hence, we have $E' > E > I' > I$.

Two strategies of the customer

- Accept: buy whatever repair the expert recommends
- Reject recommendation of a major repair: buy a minor repair but seek some other remedy if a major repair is recommended

Game matrix payoff

		Consumer	
		Accept (q)	Reject $(1-q)$
Expert	Honest (p)	$\boldsymbol{\pi, -rE-(1-r)I}$	$\boldsymbol{(1-r)\pi, -rE'-(1-r)I}$
	Dishonest $(1-p)$	$\boldsymbol{r\pi+(1-r)\pi', -\mathrm{E}}$	$\boldsymbol{0, -rE'-(1-r)I'}$

(H,A): With probability r, the consumer's problem is major, so he pays E. With probability $1-r$, it is minor, so he pays I. The expected payoff is $-rE-(1-r)I$.

(D,A): The customer's problem is major with probability r, yielding the expert π; otherwise, the expert receives π'. The customer always pays E.

(H,R): The expert earns a payoff of π only if the consumer's problem is minor. Hence, the expected payoff is $(1-r)\pi$. The cost to the customer is E' if the problem is major since he rejects the expert's advice to get a major repair; otherwise, the cost is I since he allows the expert to fix a minor problem. Therefore, the expected payoff is $-rE' - (1-r)I$.

(D,R): Since the dishonest expert always poses the problem as major, so the customer never accepts the expert's advice. The payoff to the expert is always zero. On the other hand, the expected customer's payoff is $-rE' - (1-r)I'$ since he always chooses some other remedy to fix the problem.

Expert's best response function

- We determine the mixed Nash equilibrium strategy of the consumer using the payoff equality property of the expert's expected payoffs. Given the consumer's choice of q, the expert's expected payoff to H is $q\pi+(1-q)(1-r)\pi$ and that to D is $q[r\pi+(1-r)\pi']$. We find q such that the expert is indifferent. This gives

  $$E_{\mathrm{I}}(1,Y) = q\pi + (1-q)(1-r)\pi = q[r\pi + (1-r)\pi'] = E_{\mathrm{I}}(2,Y)$$

  giving $q^* = \frac{\pi}{\pi'}$. Interestingly, the profit parameters of the expert determines the mixed Nash strategy of the customer.

- When $q = 0$ (the customer is always skeptical), $E_{\mathrm{I}}(1,Y) > E_{\mathrm{I}}(2,Y)$, the expert's best response is obviously $p = 1$ (the expert is induced to be always honest). This best response persists when $0 \leq q < \frac{\pi}{\pi'}$. Skepticism induces honesty.

- When $q = 1$ (the customer always accepts), the expert's best response is $p = 0$ (the credulous customer drives the expert to become always dishonest) since $r\pi + (1-r)\pi' > \pi$. This best response persists when $\frac{\pi}{\pi'} < q \leq 1$.

Consumer's best response function

The crucial issue is to compare the cost of a major repair and the expected cost of an alternative strategy. We consider the following characterization:

(i) $E < rE' + (1-r)I'$, the cost of a major repair by an expert is less than the expected cost of an alternative remedy, then the customer's best response is $q = 1$ for any value of p.

(ii) $E > rE' + (1-r)I'$, then the customer is indifferent between A and R if

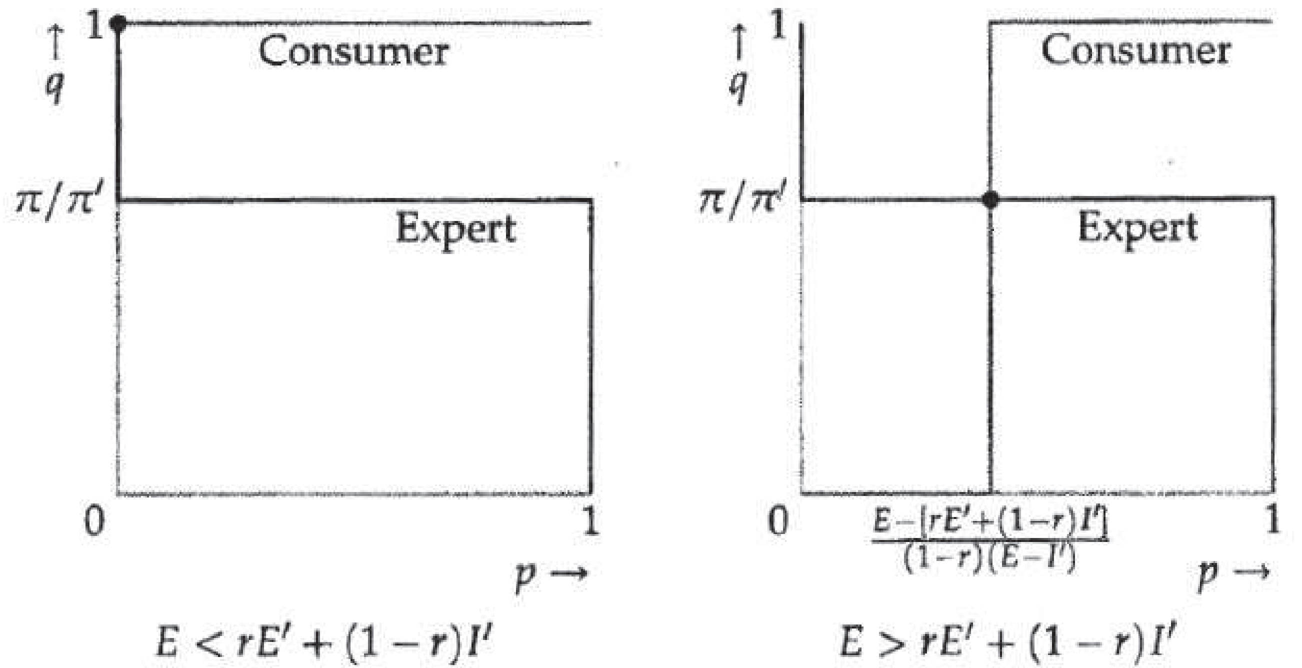
$$\begin{aligned} E_{\text{II}}(X,1) &= -p[rE + (1-r)I] - (1-p)E \\ &= -p[rE' + (1-r)I] - (1-p)[rE' + (1-r)I'] \\ &= E_{\text{II}}(X,2), \end{aligned}$$

giving

$$p = \frac{E - [rE' + (1-r)I']}{(1-r)(E-I')}.$$

Note that the mixed Nash strategy of the expert is determined by the cost parameters of the customer.

In the left figure, when the cost of major repair is lower than the expected cost of seeking for an alternative remedy, the customer should always accept.



The players' best response functions in the game of expert diagnosis. The probability assigned by the expert to H is p and the probability assigned by the consumer to A is q.

When E decreases in value, the vertical line in the consumer's best response function moves to the left until it hits the q-axis eventually.

Summary of Nash equilibrium strategies

1. $E < rE' + (1-r)I'$, (D,A) is the unique pure strategy Nash equilibrium - the dismal outcome that the expert is always dishonest and the customer always accepts his advice.

2. $E > rE' + (1-r)I'$, the unique mixed strategy Nash equilibrium with (p^*, q^*) is given by

$$p^* = \frac{E - [rE' + (1-r)I']}{(1-r)(E-I')} \text{ and } q^* = \frac{\pi}{\pi'}.$$

   Interpretation of mixed strategy Nash equilibrium in steady state: The fraction p^* of experts is honest while the fraction q^* of consumers is credulous (accepting any recommendation). Honest and dishonest experts obtain the same expected payoff, as do credulous and wary consumers.

Safety values

The amount that Player I can be guaranteed to receive is obtained by assuming that Player II is always trying to minimize Player I's payoff. This is the *maxmin* strategy of Player I.

The value of the game with matrix A is the guaranteed amount for Player I. Likewise, Player II can guarantee that he will receive the value of the game with matrix B^T. They are the *safety values* for the two players.

Recall that suppose $(X^A,\ Y^A)$ is a mixed strategy saddle point of a zero sum game with game matrix A, then

$$v^+ = \min_{Y \in S_m} \max_{X \in S_n} XAY^T = \text{value}(A) = \max_{X \in S_n} \min_{Y \in S_m} XAY^T = v^-.$$

Note that $(X^A,\ Y^A)$ is guaranteed to exist (may not be unique) while $v(A)$ is unique.

- Suppose A has the saddle point $(X^A,\ Y^A)$, then X^A is given by the *maxmin* strategy for Player I. Also, the safety value is given by value(A), where

$$\text{value}(A) = \max_{X \in S_n} \min_{Y \in S_m} XAY^T.$$

For any strategy X played by Player I, Player II tries to achieve $\min_{Y \in S_m} XAY^T$. The guaranteed floor payoff to Player I is $\max_{X \in S_n} \min_{Y \in S_m} XAY^T$.

- We interchange the role of the row player and column player and observe $XBY^T = YB^TX^T$. Under B^T, Player II becomes the row player and adopts the *maxmin* strategy to achieve the guaranteed floor value of $v(B^T)$. Suppose B^T has the saddle point $(X^{B^T},\ Y^{B^T})$, then X^{B^T} is the corresponding *maxmin* s-trategy for Player II. The guaranteed payoff to Player II is the safety value(B^T) as given by

$$\text{value}(B^T) = \max_{Y \in S_m} \min_{X \in S_n} YB^TX^T.$$

Note that when comparing value(A) and value(B^T), we swap B^T for A, X for Y and Y for X.

Payoff at a Nash equilibrium is bounded below by the safety value

If $(X^*,\ Y^*)$ is a Nash equilibrium for the bi-matrix game $(A,\ B)$, then

$$\begin{aligned} E_{\mathrm{I}}(X^*,\ Y^*) = X^*AY^{*T} &= \max_{X\in S_n} XAY^{*T} \\ &\geq \min_{Y\in S_m}\max_{X\in S_n} XAY^T \\ &= \max_{X\in S_n}\min_{Y\in S_m} XAY^T = \text{value}(A); \end{aligned}$$

and similarly,

$$\begin{aligned} E_{\mathrm{II}}(X^*,\ Y^*) &= X^*BY^{*T} = Y^*B^TX^{*T} \\ &= \max_{Y\in S_m} YB^TX^{*T} \geq \min_{X\in S_n}\max_{Y\in S_m} YB^TX^T \\ &\qquad\qquad\qquad = \max_{Y\in S_m}\min_{X\in S_n} YB^TX^T = \text{value}(B^T). \end{aligned}$$

If players use their mixed Nash equilibrium strategies, then their expected payoffs are at least their safety levels.