**MATH4321 – Game Theory**

**Topic Four – Coalitions and bargaining**

4.1 Power indexes in coalitions

- Weighted voting games
- Shapley-Shubik index and Banzhaf index
- Probabilistic characterization of power indexes

4.2 Bargaining games

- Pareto-optimal boundary and status quo payoff point
- Nash model with security point
- Threat strategies
- Sequential bargaining

## 4.1 Power indexes in coalitions

Weighted majority voting game is characterized by a voting vector

$$[q; w_1, w_2, \cdots, w_n]$$

where there are $n$ voters, $w_i$ is the voting weight of player $i$; $N = \{1, 2, \cdots, n\}$ be the set of all $n$ voters; $q$ is the quota (minimum number of votes required to pass a bill).

Let $S$ be a typical coalition of players, which is a subset of $N$. A coalition wins a bill (called winning) whenever

$$\sum_{i \in S} w_i \geq q.$$

It is natural to require the quota to observe $q > \frac{1}{2} \sum_{i \in N} w_i$ so that "complement of a winning coalition would be losing". As a result, there will be no occurrence that two disjoint coalitions are both winning.

The power of a player in a coalition game examines his ability to form winning coalitions with other players.

## Examples

1. $[51; 28, 24, 24, 24]$; the first voter is much stronger than the last 3 since he needs only one other to pass an issue, while the other three must all combine in order to win.

2. $[51; 26, 26, 26, 22]$, the last player seems powerless since any winning coalition containing him can just as well win without him (a dummy).

3. In the equal-vote game $[q; 1, 1, \cdots, 1]$, each player has equal power.

4. $[51; 40, 30, 20, 10]$ and $[51; 30, 25, 25, 20]$ are identical in terms of voting power, since the same set of coalitions are winning in both voting vectors. Similarly, voting vectors $[3; 2, 2, 1], [8; 7, 5, 3]$ and $[51; 49, 48, 3]$ are identical to $[2; 1, 1, 1]$ in terms of voting power, since they give rise to the same collection of winning coalitions (any two players can form a winning coalition in all these weighted voting vectors).

5. If we add to the game $[3; 2, 1, 1]$ the rule that player 2 can cast an additional vote in the case of 2 to 2 tie, then it is effectively $[3; 2, 2, 1]$.

   Player 3 gets a free ride since $[3; 2, 1, 1]$ is equivalent to $[2; 1, 1, 1]$, which gives equal power to all players.

   If player 1 can cast the tie breaker, then it becomes $[3; 3, 1, 1]$ and he is the dictator. He forms a winning coalition by himself.

6. In the game $[50(n-1)+1; 100, 100, \cdots, 100, 1]$, the last player has the same power as the others when $n$ is odd; the game is similar to one in which all players have the same weights. For example, when $n = 5$, we have $[201; 100, 100, 100, 100, 1]$. Any 3 of the 5 players can form a winning coalition.

## Dummy players

Any winning coalition that contains such an impotent voter could win just as well without him.

## Examples

- Player 4 in $[51; 26, 26, 26, 22]$.
- Player $n$ in $[50(n-1)+1; 100; 100, \cdots, 100, 1]$ is a dummy when $n$ is even. For example, take $n = 4$, we have $[151; 100, 100, 100, 1]$. Obviously, the last player is a dummy.
- In $[10; 5, 5, 5, 2, 1, 1]$, the $4^{\text{th}}$ player with 2 votes is a dummy. The $5^{\text{th}}$ and $6^{\text{th}}$ players with only one vote are sure to be dummies. The collection of dummies remains to be a dummy collection. This is because one cannot turn a losing coalition into a winning coalition by adding a dummy one at a time.

**Example**

Consider $[16; 12, 6, 6, 4, 3]$, player 5 with 3 votes is a dummy since no subset of the numbers $12, 6, 6, 4$ sums to $13, 14$ or 15. Therefore, player 5 could never be pivotal in the sense that by adding his vote a coalition would just reach or surpass the quota of 16.

**Example**

If we add the $8^{\text{th}}$ player with one vote into $[15; 5, 5, 5, 5, 2, 1, 1]$ so that the new game becomes $[15; 5, 5, 5, 5, 2, 1, 1, 1]$, the $5^{\text{th}}$ player in the new voting game is not a dummy since sum of votes of some coalition may assume the value of 13.

## Notion of power

- The index should indicate one's relative influence, in some numerical way, to bring about the *passage or defeat of some bill*.
- The index depends critically on the number of players involved, on one's fraction of the total weight, and upon how the remainder of the weight is distributed.
- A winning coalition is said to be *minimal winning* if no proper subset of it is winning. Technically, the one who is the 'last' to join a minimal winning coalition is particularly influential.
- A voter $i$ is a dummy if every winning coalition that contains him is also winning without him, that is, he is in no minimal winning coalition. A dummy has ZERO power.

## Veto power and dictator

A player or coalition is said to have *veto power* if no coalition is able to win a ballot without his or their consent. A subset $S$ of voters is a blocking coalition or has *veto power* if and only if its complement $N - S$ is not winning.

A player $i$ is a dictator if he forms a winning coalition $\{i\}$ by himself.

- If the dictator says "yes", then the bill is passed. If the dictator says "no", then the bill is not passed (any coalition without the dictator is losing).

- If a dictator exists, then all other players are dummies.

If a coalition (may has only one player) has veto power and it is winning, then it is a dictating coalition.

**Example**

Player 1 has veto power in $[51; 50, 49, 1]$ and $[3; 2, 1, 1]$ but not a dictator. In the last case, if he is the chairman with additional power to break ties, then the game becomes $[3; 3, 1, 1]$ and now he becomes a dictator.

**Example**

The ability of an individual to break tie votes in the equal-vote game

$$\begin{cases} \left[\frac{n}{2} + 1; 1, 1, \cdots, 1\right] & \text{when } n \text{ is even} \\ \left[\frac{n+1}{2}; 1, 1, \cdots, 1\right] & \text{when } n \text{ is odd} \end{cases}$$

adds power when $n$ is even and adds nothing when $n$ is odd. Actually, when $n$ is odd, tie votes will not occur.

**Properties on dummies**

A collection of dummies can never turn a losing coalition into a winning coalition.

In other words, it is not possible that $S \cup \{D_1, \cdots, D_m\}$ is winning but $S$ is losing. If otherwise, since the dummies can be successively deleted while the coalition remains to be winning, this gives $S$ to be winning. This leads to a contradiction.

*Corollary*

If both "$d$" and "$\ell$" are dummies, then the coalition $\{d, \ell]$ is dummy.

**Theorem**

In a weighted voting game, let "$d$" and "$\ell$" be two voters with votes $x_d$ and $x_\ell$, respectively. Suppose "$d$" is a dummy and $x_\ell \leq x_d$, then "$\ell$" is also a dummy.

*Proof*

Assume the contrary. Suppose "$\ell$" is not a dummy, then there exists a coalition $S$ that does not contain the dummy "$d$" such that $S$ is losing but $S \cup \{\ell\}$ is winning. Now, $n(S) < q$ while $n(S \cup \{\ell\}) \geq q$. Since $x_\ell \leq x_d$, so $n(S \cup \{d\}) \geq q$, contradicting that "$d$" is a dummy.

**Corollary**

If the coalition $\{d, \ell\}$ is dummy, then both "$d$" and "$\ell$" are dummies. This is obvious since $n(\{d, \ell\}) > \max(x_d, x_\ell)$.

**Shapley-Shubik power index**

1. One looks at all possible orderings of the $n$ players, and consider this as all of the potential ways of building up toward a winning coalition. For each one of these permutations, some unique player joins and thereby turns a losing coalition into a winning one, and this voter is called the *pivot*.

2. In the sequence of players $x_1, x_2, \cdots, x_{i-1}, x_i, \cdots, x_n$, $\{x_1, x_2, \cdots, x_i\}$ is a winning coalition but $\{x_1, x_2, \cdots, x_{i-1}\}$ is losing, then $x_i$ is in the *pivotal position*.

3. The expected frequency with which a voter is the pivot, over all possible orderings of the voters, is taken to be a good indication of his voting power.

**Example – 4-player weighted voting game**

The 24 permutations of the four players $1, 2, 3$ and $4$ in the weighted majority game $[51; 40, 30, 20, 10]$ are listed below. The "*" indicates which player is pivotal in the corresponding ordering.

| | | | |
|---|---|---|---|
| $1\ 2^*\ 3\ 4$ | $2\ 1^*\ 3\ 4$ | $3\ 1^*\ 2\ 4$ | $4\ 1\ 2^*\ 3$ |
| $1\ 2^*\ 4\ 3$ | $2\ 1^*\ 4\ 3$ | $3\ 1^*\ 4\ 2$ | $4\ 1\ 3^*\ 2$ |
| $1\ 3^*\ 2\ 4$ | $2\ 3\ 1^*\ 4$ | $3\ 2\ 1^*\ 4$ | $4\ 2\ 1^*\ 3$ |
| $1\ 3^*\ 4\ 2$ | $2\ 3\ 4^*\ 1$ | $3\ 2\ 4^*\ 1$ | $4\ 2\ 3^*\ 1$ |
| $1\ 4\ 2^*\ 3$ | $2\ 4\ 1^*\ 3$ | $3\ 4\ 1^*\ 2$ | $4\ 3\ 1^*\ 2$ |
| $1\ 4\ 3^*\ 2$ | $2\ 4\ 3^*\ 1$ | $3\ 4\ 2^*\ 1$ | $4\ 3\ 2^*\ 1$ |

*For Player 1* winning coalitions consisting of 2 players.

winning coalitions consisting of 3 players.

Shapley-Shubik power index for the $i^{\text{th}}$ player is

$$\phi_i = \frac{\text{number of sequences in which player } i \text{ is a pivot}}{n!}$$

and we write $\boldsymbol{\phi} = (\phi_1, \cdots, \phi_n)$.

Here, we assume that each of the $n!$ alignments is *equiprobable*.

The power index can be expressed as

$$\phi_i = \sum \frac{(s-1)!(n-s)!}{n!} \qquad \left(\text{with } \sum_{i \in N} \phi_i = 1\right)$$

where $s = |S| =$ number of voters in set $S$. The summation is taken over all winning coalitions $S$ for which $S - \{i\}$ is losing.

*Counting permutations for which a player is pivotal in achieving winning coalitions*

- Player 1 is pivotal in three 3-player coalitions (namely $\{1,2,3\}$, $\{1,2,4\}$, $\{1,3,4\}$ and $s=3$) and in two 2-player coalitions (namely $\{1,2\}$, $\{1,3\}$ and $s=2$). We have

$$\phi_1 = 3\frac{(3-1)!(4-3)!}{4!} + 2\frac{(2-1)!(4-2)!}{4!} = \frac{10}{24}.$$

  Note that player 1 is pivotal in $\{1,2,3\}$ while $\{1,2,3\}$ is not a minimal winning coalition. This is because player 3 can be deleted from $\{1,2,3\}$ and $\{1,2\}$ remains to be winning.

- For player 2, she is pivotal in two 3-player coalitions $\{2,3,4\}$ and $\{1,2,4\}$ and one 2-player coalition $\{1,2\}$. Therefore, we obtain

$$\phi_2 = 2\frac{(3-1)!(4-3)!}{4!} + \frac{(2-1)!(4-2)!}{4!} = \frac{6}{24}.$$

- It is easy to check that player 3 and player 2 have equal power. There are only 2 coalitions where player 4 is pivotal. The Shapley-Shubik indexes are found to be $\phi = \frac{(10,6,6,2)}{24}$.

**Banzhaf index**

- Consider all significant combinations of “yes” or “no” votes, rather than permutations of the players as in the Shapley-Shubik index.
- A player is said to be marginal, or a swing or critical, in a given combination of “yes” and “no” if he can change the outcome.
- Let $b_i$ be the number of voting combinations in which voter $i$ is marginal; then $\beta_i = \dfrac{b_i}{\sum b_i}$.

Assuming that all voting combinations are equally probable.

The game is $[51; 40, 30, 20, 10]$. For the second case, if Player 1 changes from $Y$ to $N$, then the outcome changes from "Pass" to "Fail".

Computation of the Banzhaf Index

| Players | | | | Pass/Fail | | Marginal | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | $P$ | $F$ | 1 | 2 | 3 | 4 |
| $Y$ | $Y$ | $Y$ | $Y$ | $P$ | | | | | |
| $Y$ | $Y$ | $Y$ | $N$ | $P$ | | $X$ | | | |
| $Y$ | $Y$ | $N$ | $Y$ | $P$ | | $X$ | $X$ | | |
| $Y$ | $N$ | $Y$ | $Y$ | $P$ | | $X$ | | $X$ | |
| $N$ | $Y$ | $Y$ | $Y$ | $P$ | | | $X$ | $X$ | $X$ |
| $Y$ | $Y$ | $N$ | $N$ | $P$ | | $X$ | $X$ | | |
| $Y$ | $N$ | $Y$ | $N$ | $P$ | | $X$ | | $X$ | |

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $N$ | $Y$ | $Y$ | $N$ | $F$ | $X$ | | | $X$ |
| $Y$ | $N$ | $N$ | $Y$ | $F$ | | $X$ | $X$ | |
| $N$ | $Y$ | $N$ | $Y$ | $F$ | $X$ | | $X$ | |
| $N$ | $N$ | $Y$ | $Y$ | $F$ | $X$ | $X$ | | |
| $Y$ | $N$ | $N$ | $N$ | $F$ | | $X$ | $X$ | |
| $N$ | $Y$ | $N$ | $N$ | $F$ | $X$ | | | |
| $N$ | $N$ | $Y$ | $N$ | $F$ | $X$ | | | |
| $N$ | $N$ | $N$ | $Y$ | $F$ | | | | |
| $N$ | $N$ | $N$ | $N$ | $F$ | | | | |

$$24 \times \boldsymbol{\beta} = (10, 6, 6, 2)$$

Looking at $YYNN$ (pass) and $NYNN$ (fail), Player 1 can serve as the defector who gives the swing from Pass to Fail in the first case and Fail to Pass in the second case. We expect that the number of swings of winning into losing effected by a particular player is the same as the number of swings of effecting losing into winning by the same player.

## Example

Players with the same number of votes are considered alike. Such symmetry can save us writing out all $n!$ orderings. For example, consider the weighted majority game

$$[5; 3, 2, 1, 1, 1, 1].$$

Since the "1" players are all alike, we need to write out only $6 \cdot 5 = 6!/4! = 30$ distinct orderings (instead of $6! = 720$):

| | | | | | |
|---|---|---|---|---|---|
| $3\underline{2}1111$ | $2\underline{3}1111$ | $21\underline{3}111$ | $211\underline{3}11$ | $211\underline{1}31$ | $211\underline{1}13$ |
| $31\underline{2}111$ | $13\underline{2}111$ | $12\underline{3}111$ | $121\underline{3}11$ | $121\underline{1}31$ | $121\underline{1}13$ |
| $31\underline{1}211$ | $13\underline{1}211$ | $11\underline{3}211$ | $112\underline{3}11$ | $112\underline{1}31$ | $112\underline{1}13$ |
| $31\underline{1}121$ | $13\underline{1}121$ | $11\underline{3}121$ | $111\underline{3}21$ | $111\underline{2}31$ | $111\underline{2}13$ |
| $31\underline{1}112$ | $13\underline{1}112$ | $11\underline{3}112$ | $111\underline{3}12$ | $1111\underline{3}2$ | $1111\underline{2}3$ |

Notice that the 1's pivot 12/30 of the time, but since there are four of them, each 1 pivots only 3/30 of the time. We get

$$\begin{aligned}\text{Shapley-Shubik index} &= \phi = \left(\frac{12}{30}, \frac{6}{30}, \frac{3}{30}, \frac{3}{30}, \frac{3}{30}, \frac{3}{30}\right) \\ &= (0.4, 0.2, 0.1, 0.1, 0.1, 0.1).\end{aligned}$$

Power as measured by the Shapley-Shubik index in a weighted voting game is *not* proportional to the number of votes cast. For instance, the first player with $3/9 = 33\frac{1}{3}\%$ of the votes has 40% of the power.

Use the same game $[5; 3, 2, 1, 1, 1, 1]$ for the computation of the *Banzhaf index*

| Types of winning coalitions with | | Number of ways this can occur | Number of swings for 3 | 2 | 1 |
|---|---|---|---|---|---|
| 5 votes: | $\underline{32}$ | 1 | 1 | 1 | |
| | $\underline{311}$ | $6 = {}_4C_2$ | 6 | | 12 |
| | $\underline{2111}$ | $4 = {}_4C_3$ | | 4 | 12 |
| 6 votes: | $\underline{32}1$ | $4 = {}_4C_1$ | 4 | 4 | |
| | $\underline{3}111$ | $4 = {}_4C_3$ | 4 | | |
| | $\underline{2}1111$ | $1 = {}_4C_4$ | | 1 | |
| 7 votes: | $\underline{3}211$ | $6 = {}_4C_2$ | 6 | | |
| | $\underline{3}1111$ | $1 = {}_4C_4$ | 1 | | |
| | | | 22 | 10 | 24 |

We do not need to include those winning coalitions of 8 or 9 votes, since not even the player with 3 votes can be marginal to them.

$$\beta = \left(\frac{22}{56}, \frac{10}{56}, \frac{6}{56}, \frac{6}{56}, \frac{6}{56}, \frac{6}{56}\right)$$
$$\approx (0.392, 0.178, 0.107, 0.107, 0.107, 0.107).$$

*Remark*

It suffices to consider the swings only in winning coalitions in the calculation of the Banzhaf index. A defector that turns a winning coalition into a losing coalition also gives the symmetric swing that turns a losing coalition into a winning coalition.

The numbers in the second column are derived from the theory of combinations. For instance, the number of ways that you could choose 311 from 321111 is ${}_4C_2 = 6$. In other words, there are 6 ways of choosing 2 players with one vote from 4 players with one vote.

Comparing this with $\phi$, we see that the two indices turn out to be quite close in this case, with $\beta$ giving slightly less power to the two large players and slightly more to the small players.

## United Nations Security Council: power indexes calculations of "big" and "small" countries

1. Big "five" – permanent member each has veto power; ten "small" countries whose (non-permanent) membership rotates.

2. It takes 9 votes, the "big five" plus at least 4 others to carry an issue.

   For simplicity, we assume no "abstain" votes. The game is $[39; 7, 7, 7, 7, 7, 1, 1, \cdots, 1]$. Why? Let $x$ be the weight of any of the permanent member and $q$ be the quota. Then

$$4x + 10 < q \quad \text{and} \quad q = 5x + 4$$

   so that $4x + 10 < 5x + 4$ giving $x > 6$. Taking $x = 7$, we then have $q = 39$.

3. A "small" country $i$ can be pivotal in a winning coalition if and only if $S$ contains exactly 9 countries including the big "five". There are ${}_9C_3$ such different $S$ that contain $i$ since the remaining 3 "small" countries are chosen from 9 "small" countries (other than country $i$ itself). For each such $S$, the corresponding coefficient in the Shapley-Shubik formula for this 15-person game is $\frac{(9-1)!(15-9)!}{15!}$. Hence, $\phi_s = {}_9C_3 \times \frac{8!6!}{15!} \approx 0.001863$. Since sum of the Shapley-Shubik indexes of all 15 countries is one, any "big-five" has index $\phi_b = \frac{1-10\phi_S}{5} = 0.1963$.

4. Old Security Council before 1963, which was

$$[27; 5, 5, 5, 5, 5, 1, 1, 1, 1, 1, 1].$$

What is the corresponding yes-no voting system?

Answer for $\phi$ : $\phi_b = \frac{1}{5} \cdot \frac{76}{77}; \phi_S = \frac{1}{6} \cdot \frac{1}{77}.$

*Remark – Direct computation of the power index of the big countries*

To compute $\phi_b$ directly, we observe that a particular big country (say, China) can be pivotal in 9-player coalitions, 10-player coalitions, ..., 15-player coalitions since she is holding the veto power. For example, in a 9-player coalition, 4 big countries and 4 out of 10 small countries are ahead of the pivotal position held by China, leaving 6 small countries behind. Repeating the same argument for 10-player coalitions,...,15-player coalitions, we then have

$$
\begin{aligned}
\phi_b = {} & {}_{10}C_4 \frac{(9-1)!(15-9)!}{15!} + {}_{10}C_5 \frac{(10-1)!(15-10)!}{15!} \\
& + {}_{10}C_6 \frac{(11-1)!(15-11)!}{15!} + {}_{10}C_7 \frac{(12-1)!(15-12)!}{15!} \\
& + {}_{10}C_8 \frac{(13-1)!(15-13)!}{15!} + {}_{10}C_9 \frac{(14-1)!(15-14)!}{15!} \\
& + \frac{(15-1)!0!}{15!}.
\end{aligned}
$$

## Canadian Constitutional Amendment

Investigate the voting powers exhibited in a 10-person game between the provinces in Canada, and to compare the results with the provincial populations.

The winning coalitions and those with veto power can be described as follows. In order for passage, approval is required of

(a) any province that has (or ever had) more than 25% of the population,

(b) at least two of the four Atlantic provinces, and

(c) at least two of the four western provinces that currently contain together at least 50% of the total western population.

## Veto power

Recall that a blocking coalition (holding veto power) is a subset of players whose complement is not winning and itself is not winning. Using the current population figures, the veto power is held by

(i) Ontario ($O$)

(ii) Quebec ($Q$),

(iii) any three of the four Atlantic ($A$) provinces [New Brunswick ($NB$), Nova Scotia ($NS$), Prince Edward Island ($PEI$), and Newfoundland ($N$)],

(iv) British Columbia ($BC$) plus any one of the three prairie ($P$) provinces [Alberta ($AL$), Saskatchewan ($S$), and Manitoba ($M$)], or the three prairie provinces taken together.

We list all possible winning coalitions. Note that any of these winning coalitions must contain Quebec and Ontario. The number of Atlantic provinces can be 2, 3 or 4. When $BC$ is included, the number of prairie provinces can be 1, 2 or 3. Without $BC$, the number of prairie provinces must be 3.

| Type | S | s | No. of such S | |
|---|---|---|---|---|
| 1 | 1P , 2A, BC, Q, O | 6 | 18 | |
| 2 | 2P , 2A, BC, Q, O | 7 | 18 | |
| 3 | 3P , 2A, Q, O | 7 | 6 | 36 |
| 4 | 1P , 3A, BC, Q, O | 7 | 12 | |
| 5 | 3P , 2A, BC, Q, O | 8 | 6 | |
| 6 | 2P , 3A, BC, Q, O | 8 | 12 | 25 |
| 7 | 3P , 3A, Q, O | 8 | 4 | |
| 8 | 1P , 4A, BC, Q, O | 8 | 3 | |
| 9 | 3P , 3A, BC, Q, O | 9 | 4 | |
| 10 | 2P , 4A, BC, Q, O | 9 | 3 | 8 |
| 11 | 3P , 4A, Q, O | 9 | 1 | |
| 12 | 3P , 4A, BC, Q, O | 10 | 1 | |
| | | Total: | 88 | |

*Ontario's Shapley-Shubik index*

$$\varphi_O = \frac{18(5!4!) + 36(6!3!) + 25(7!2!) + 8(8!1!) + 1(9!0!)}{10!} = \frac{53}{168}$$

- There are 18 winning coalitions that contain 6 provinces. In order that Ontario serves as the pivotal player, 5 provinces are in front of her and 4 provinces are behind her. This explains why there are altogether 18(5!4!) permutations in these 6-province winning coalitions.

- Ontario and Quebec are equivalent in terms of influential power (though their populations are different).

*British Columbia*

Listing of all winning coalitions that upon deleting British Columbia the corresponding coalition becomes losing. These are the winning coalitions that British Columbia can serve as the pivotal player.

| Type | $S$ | $s$ | No. of such $S$ |
|---|---|---|---|
| 1 | 1P, 2A, BC, Q, O | 6 | ${}_3C_1 \times {}_4C_2 = 18$ |
| 2 | 1P, 3A, BC, Q, O | 7 | ${}_3C_1 \times {}_4C_3 = 12$ |
| 3 | 1P, 4A, BC, Q, O | 8 | ${}_3C_1 \times {}_4C_4 = 3$ |
| 4 | 2P, 2A, BC, Q, O | 7 | ${}_3C_2 \times {}_4C_2 = 18$ |
| 5 | 2P, 3A, BC, Q, O | 8 | ${}_3C_2 \times {}_4C_3 = 12$ |
| 6 | 2P, 4A, BC, Q, O | 9 | ${}_3C_2 \times {}_4C_4 = 3$ |

- Note that we exclude those coalitions with 3 prairie provinces since the deletion of British Columbia does not cause the coalition to become losing.

$$\phi_{BC} = \frac{18(5!4!) + 30(6!3!) + 15(7!2!) + 3(8!1!)}{10!}.$$

*Atlantic provinces*

We consider winning coalitions that contain a particular Atlantic province and one of the three other Atlantic provinces.

| Type | $S$ | $s$ | No. of such $S$ |
|---|---|---|---|
| 1 | $A_{sp}, 1A, 1P, BC, Q, O$ | 6 | ${}_3C_1 \times {}_3C_1 = 9$ |
| 2 | $A_{sp}, 1A, 2P, BC, Q, O$ | 7 | ${}_3C_1 \times {}_3C_2 = 9$ |
| 3 | $A_{sp}, 1A, 3P, BC, Q, O$ | 8 | ${}_3C_1 = 3$ |
| 4 | $A_{sp}, 1A, 3P, Q, O$ | 7 | ${}_3C_1 = 3$ |

$$\phi_{A_{sp}} = \frac{9(5!4!) + 12(6!3!) + 3(7!2!)}{10!}.$$

*Prairie provinces*

We consider winning coalitions that contain

(i) a particular prairie province and British Columbia

(ii) a particular prairie province and two other prairie provinces

| Type | *S* | *s* | No. of such *S* |
|---|---|---|---|
| 1 | $P_{sp}$, 2A, BC, Q, O | 6 | 6 |
| 2 | $P_{sp}$, 3A, BC, Q, O | 7 | 4 |
| 3 | $P_{sp}$, 4A, BC, Q, O | 8 | 1 |
| 4 | $P_{sp}$, 2P, 2A, Q, O | 7 | 6 |
| 5 | $P_{sp}$, 2P, 3A, Q, O | 8 | 4 |
| 6 | $P_{sp}$, 2P, 4A, Q, O | 9 | 1 |

$$\phi_{P_{sp}} = \frac{6(5!4!) + 10(6!3!) + 5(7!2!) + 8!1!}{10!}.$$

## Shapley-Shubik Index Provinces

| Province | φ (in %) | % Population | φ/Population |
|---|---|---|---|
| *BC* | 12.50 | 9.38 | 1.334 |
| *AL* | 4.17 | 7.33 | 0.570 |
| *S* | 4.17 | 4.79 | 0.872 |
| *M* | 4.17 | 4.82 | 0.865 |
| (4 Western) | (25.01) | (26.32) | (0.952) |
| *O* | 31.55 | 34.85 | 0.905 |
| *Q* | 31.55 | 28.94 | 1.092 |
| *NB* | 2.98 | 3.09 | 0.965 |
| *NS* | 2.98 | 3.79 | 0.786 |
| *PEI* | 2.98 | 0.54 | 5.53 |
| *N* | 2.98 | 2.47 | 1.208 |
| (4 Atlantic) | (11.92) | (9.89) | (1.206) |

- British Columbia has a higher index value per capita compared to other Western provinces.

## Probabilistic characterization of power indexes

*What is the probability that my vote will make a difference, that is, that a proposal will pass if I vote for it, but fail if I vote against it?*

- The answers depend on both the decision rule of the voting game and the probabilities that various members will vote for or against a proposal.
- If we are interested in general theoretical questions of power, we cannot reasonably assume particular knowledge about individual players or proposals. We should only make assumptions about voting probabilities which do not discriminate among the players.

**Homogeneity Assumption.** *Every proposal to come before the decision-making body has a certain probability $p$ of appealing to each member of the body. The homogeneity is among members: they all have the same probability $p$ of voting for a given proposal. However, $p$ varies from proposal to proposal, giving rise to the random nature of voting probability.*

The homogeneity assumption does not assume that members will all vote the same way, but it does say something about their similar criteria for evaluating proposals. For instance, some bills that came before a legislature seem to have a high probability of appealing to all members, and pass by large margins: those have high $p$. Others are overwhelmingly defeated (low $p$) or controversial ($p$ near $1/2$).

*Remark* For the Shapley-Shubik index, we further assume the common $p$ to be uniformly distributed between 0 and 1.

Shapley-Shubik index focuses on the *order* in which a winning coalition forms, and defines the power of a player to be proportional to the number of orderings in which she is pivotal.

**Theorem 1.** *The Shapley-Shubik index $\phi$ gives the probability that an individual voter can make a difference under the homogeneity assumption about voting probabilities (together with the assumption of uniform distribution of all these random voting probabilities).*

*Remark* Player $i$ is pivotal if a coalition $S_i$ exists such that

$$\sum_{j \in S_i} w_j < q \quad \text{and} \quad w_i + \sum_{j \in S_i} w_j \geq q.$$

Note that $S_i$ is losing while $S_i \cup \{i\}$ is winning. We write $s_i = n(S_i)$, where $n(S_i)$ is the number of players in $S_i$.

*Proof of Theorem 1*

We randomize the probabilities $p_1, \ldots, p_N$ and invoke the conditional independence assumption. Given the realization of $\boldsymbol{p}_i = (p_1, \ldots, p_{i-1}, p_{i+1}, \ldots, p_N)$, the conditional probability that player $i$'s vote will make a difference is given by

$$\pi_i(\boldsymbol{p}_i) = \sum_{S_i} \prod_{j \in S_i} p_j \prod_{j \notin S_i} (1 - p_j),$$

where $p_j$ is the voting probability of player $j$. The sum is taken over all such coalitions where player $i$ is pivotal. Note that $\pi_i(\boldsymbol{p}_i)$ does not involves $p_i$ since voter $i$ says "yes" (steps in to make difference), and it also depends on the voting probabilities of $N-1$ other voters. Under the homogeneity assumption where all players share the same $p$, the expected frequency where player $i$ is pivotal is obtained by integrating over the probability distribution:

$$E[\pi_i(p)] = \int_0^1 \pi_i(p) f(p)\, dp,$$

where $f(p)$ is the density function of the common voting probability $p$.

Density function $f_X(x)$ of a uniform distribution over $[a,b]$ is given by

$$f_X(x) = \begin{cases} \frac{1}{b-a} & a < x < b \\ 0 & \text{otherwise} \end{cases}.$$

Under the homogeneity assumption, a number $p$ is selected from the uniform distribution on $[0,1]$ and $p_j$ is set equal to $p$ for all $j$. In this case, $f(p) = 1$ since $a = 0$ and $b = 1$ so that

$$E[\pi_i(p)] = \int_0^1 \pi_i(p)\, dp \quad \text{where} \quad \pi_i(p) = \sum_{S_i} p^{s_i}(1-p)^{N-s_i-1}, \quad s_i = n(S_i).$$

Lastly, making use of the Beta integral:

$$\frac{s_i!(N-s_i-1)!}{N!} = \int_0^1 p^{s_i}(1-p)^{N-s_i-1}\, dp,$$

we obtain

$$E[\pi_i(p)] = \sum_{S_i} \frac{s_i!(N-s_i-1)!}{N!} = \phi_i = \text{ Shapley-Shubik index for player } i.$$

The Beta integral links the probability of being pivotal under the homogeneity assumption of voting probabilities with the expected frequency of being pivotal in various orderings of voters. As we count the occurrences of these orderings with equal probability, homogeneity of voting probabilities is implicitly assumed.

*Proof of the Beta integral formula*

$$\begin{aligned} I_{m,n} &= \int_0^1 p^m (1-p)^n \; dp \\ &= \left[ -\frac{m}{n+1} p^{m-1} (1-p)^{n+1} \right]_0^1 + \frac{m}{n+1} I_{m-1,n+1} \\ &= \frac{m(m-1)}{(n+1)(n+2)} I_{m-2,n+2} = \frac{m!}{(n+1)(n+2)\dots(n+m)} I_{0,n+m} \\ &= \frac{m!n!}{(m+n+1)!}. \end{aligned}$$

**Example**

$$[3; 2, 1, 1]$$
$$A \; B \; C$$

- Each voter will vote for a proposal with probability $p$. What is the probability that $A$'s vote will make a difference between approval and rejection?
- If both $B$ and $C$ vote against the proposal, $A$'s vote will *not* make a difference, since the proposal will fail regardless of what he does.
- If $B$ or $C$ or both vote for the proposal, $A$'s vote will decide between approval and rejection.

The conditional probability at given values of $p_B$ and $p_C$ that $A$'s vote will make a difference is given by

$$\pi_A(p_B, p_C) = \underbrace{p_B(1 - p_C)}_{B \text{ for, } C \text{ against}} + \underbrace{(1 - p_B)p_C}_{B \text{ against, } C \text{ for}} + \underbrace{p_B p_C}_{\text{both for}}.$$

Setting the homogeneity assumption, the conditional probability $\pi_A(p_A, p_B)$ is simplified to

$$\pi_A(p) = 2p - p^2,$$

where $p$ is the homogeneous voting probability among the two voters other than $A$.

Similarly, $B$'s vote will make a difference only if $A$ votes for, and $C$ votes against. If they both voted for, the proposal would pass regardless of what $B$ did.

$$\pi_B(p) = \underbrace{p(1 - p)}_{A \text{ for, } C \text{ against}} = p - p^2.$$

By symmetry, we also have $\pi_C(p) = p - p^2$.

- Shapley-Shubik index: voting probabilities are chosen by players from a common uniform distribution on the unit interval.

We average the probability of making a difference $\pi_A(p)$ over all $p$ between 0 and 1, where $p$ is uniformly distributed in $[0,1]$.

$$\text{for } A: \quad \int_0^1 \pi_A(p)\,dp = \int_0^1 (2p - p^2)\,dp = \frac{2}{2} - \frac{1}{3} = \frac{2}{3} = \phi_A$$

$$\text{for } B: \quad \int_0^1 \pi_B(p)\,dp = \int_0^1 (p - p^2)\,dp = \frac{1}{2} - \frac{1}{3} = \frac{1}{6} = \phi_B$$

$$\text{for } C: \quad \int_0^1 \pi_C(p)\,dp = \int_0^1 (p - p^2)\,dp = \frac{1}{2} - \frac{1}{3} = \frac{1}{6} = \phi_C$$

**Independence Assumption.** *Every proposal has a probability* $p_i$ *of appealing to the* $i^{th}$ *member. Each of the* $p_i$ *is chosen independently from the interval* $[0, 1]$*. Here how one member feels about the proposal has nothing to do with how any other member feels.*

Banzhaf index ignores the question of ordering and looks only at the final coalition which forms in support of some proposal. The power of a player is defined to be proportional to the number of such coalitions. If the voters in some political situation behave completely independently, then $\beta$ is the most appropriate index.

**Theorem 2.**

*The absolute Banzhaf index* $\beta'$ *gives the probability that an individual voter can make a difference under the independence assumption (together with mean value of voting probability equals* $1/2$*) about voting probabilities.*

The absolute Banzhaf index $\beta_i'$ can be interpreted as assuming that voting probabilities are selected randomly and independently from a distribution with mean 1/2 without regard for the forms of those distributions.

Each player can be thought of as having mean voting probability 1/2 for any given proposal, so we can think of all coalitions to be equally likely to form. For example, suppose there are 5 players where players 1, 3 and 5 say "yes" and players 2 and 4 say "no". By independence of the voting probabilities, the probability of forming such coalition is

$$\begin{aligned}
&\int_0^1 \int_0^1 \int_0^1 \int_0^1 \int_0^1 p_1(1-p_2)p_3(1-p_4)p_5\ f_1(p_1)\cdots f_5(p_5)\mathrm{d}p_1\cdots\mathrm{d}p_5 \\
= \ &\int_0^1 p_1 f_1(p_1)\ dp_1 \int_0^1 (1-p_2)f_2(p_2)\ dp_2 \int_0^1 p_3 f_3(p_3)\ dp_3 \\
&\int_0^1 (1-p_4)f_4(p_4)\ dp_4 \int_0^1 p_5 f_5(p_5)\ dp_5 = \left(\frac{1}{2}\right)^5.
\end{aligned}$$

Note that we assume the means of the voting probabilities to be $\frac{1}{2}$, so

$$\int_0^1 p_1 f_1(p_1)\ \mathrm{d}p_1 = \frac{1}{2},\ \int_0^1 (1-p_2)f_2(p_2)\ \mathrm{d}p_2 = \frac{1}{2},\ \text{etc.}$$

*Proof of Theorem 2*

Under the independence assumption, the voting probabilities are selected independently from distributions (not necessarily uniform) on $[0,1]$ with $E[p_j] = 1/2$, $j = 1, 2, \ldots, N$. Since $p_j$ are independent, the joint density of the voter's random probabilities (other than voter $i$) is

$$f_i(\boldsymbol{p}_i) = \prod_{j \neq i} f_j(p_j)$$

where $f_j(p_j)$ is the marginal density for $p_j$. The probability that player $i$ can make a swing from losing to winning is given by

$$E[\pi_i(\boldsymbol{p}_i)] = \sum_{S_i} \int_0^1 \cdots \int_0^1 \prod_{j \in S_i} p_j \prod_{j \notin S_i} (1 - p_j) \prod_{j \neq i} f_j(p_j)\, dp_1 \cdots dp_N,$$

where we integrate the conditional probability $\pi_i(\boldsymbol{p}_i)$ over the underlying joint density of the voting probabilities $\boldsymbol{p}_i$.

Since $p_j f_j(p_j)$ or $(1-p_j)f_j(p_j)$, $j = 1, 2, \ldots, N$, $j \neq i$, are separable due to independence assumption, we can write the $(N-1)-$fold integral into a product of one-dimensional integrals as follows:

$$\begin{aligned} E[\pi_i(\boldsymbol{p}_i)] &= \sum_{S_i} \prod_{j \in S_i} \int_0^1 p_j f_j(p_j)\, dp_j \prod_{j \notin S_i} \int_0^1 (1-p_j) f_j(p_j)\, dp_j \\ &= \pi_i\left(\frac{1}{2}, \ldots, \frac{1}{2}\right) = \sum_{S_i} \frac{1}{2^{N-1}} = \frac{\eta_i}{2^{N-1}} = \beta_i' \\ &= \text{absolute Banzhaf index for player } i, \end{aligned}$$

where $\eta_i$ is the number of swings for player $i$. Note that we have used the assumption that the mean of each of the voting probability is 1/2.

Apparently, we set all values of $p_j$, $j = 1, 2, \ldots, N$, $j \neq i$, in $\pi_i(\boldsymbol{p}_i)$ to be $\frac{1}{2}$ so that

$$\pi_i\left(\frac{1}{2}, \ldots, \frac{1}{2}\right) = \frac{\eta_i}{2^{N-1}}.$$

Since $\sum_{i=1}^{N} \eta_i \neq 2^{N-1}$ in general, so the sum of the absolute Banzhaf indexes for all players is not equal to 1.

*What is the probability that the group decision agrees with the player's decision on a proposal? The answer to player $i$'s question of individual-group agreement, under the independence assumption about voting probabilities, is given by $(1 + \beta_i')/2$.*

Theorem 2 says that $\beta_i'$ gives the probability that player $i$'s vote will make the difference between approval and rejection. Since his vote makes the difference, in this situation the group decision always agrees with his.

With probability $1 - \beta_i'$, player $i$'s vote will *not* make a difference. In these coalitions, the passage or failure of a bill depends on the votes of other players. Under this case, player $i$ has equal probability to say "yes" or "no", the group will still agree with him half the time. Hence, the probability that the group decision agrees with player $i$'s voting choice is

$$(\beta_i')(1) + (1 - \beta_i')\left(\frac{1}{2}\right) = \frac{1 + \beta_i'}{2}.$$

**Example**

Consider the weighted voting game: $[51; 40, 30, 20, 10]$. We list all the marginal cases where joining of a player changes losing to winning.

| Players | | | | marginal (losing to winning) | | | |
|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 1 | 2 | 3 | 4 |
| N | Y | Y | N | × | | | × |
| Y | N | N | Y | | × | × | |
| N | Y | N | Y | × | | × | |
| N | N | Y | Y | × | × | | |
| Y | N | N | N | | × | × | |
| N | Y | N | N | × | | | |
| N | N | Y | N | × | | | |

Note that $\eta_1 = 5$, $\eta_2 = 3$, $\eta_3 = 3$, $\eta_4 = 1$, so $\sum_{i=1}^{4} \eta_i = 12$.

*First step:* For the given player $i$, determine all $S_i$'s. Each $S_i$ is a losing coalition without player $i$, but it becomes winning with player $i$ joining.

For player 1, we have $\eta_1 = 5$, where the 5 marginal coalitions are

$$S_1^{(1)} = \{2,3\}, S_1^{(2)} = \{2,4\}, S_1^{(3)} = \{3,4\}, S_1^{(4)} = \{2\}, S_1^{(5)} = \{3\}.$$

The conditional probability that player 1 makes a difference:

$$\begin{aligned}\pi_1(p_2,p_3,p_4) = {} & p_2p_3(1-p_4) + p_2(1-p_3)p_4 + (1-p_2)p_3p_4 \\ & + p_2(1-p_3)(1-p_4) + (1-p_2)p_3(1-p_4).\end{aligned}$$

By setting $p_1 = p_2 = p_3 = p$ in $\pi_1(p_1,p_2,p_3)$ and $p$ is uniformly distributed, the Shapley-Shubik index is given by

$$\begin{aligned}\phi_1 &= \int_0^1 \pi_1(p,p,p)\ \mathrm{d}p \\ &= \int_0^1 [3p^2(1-p) + 2p(1-p)^2]\ \mathrm{d}p \\ &= 3\frac{2!}{4!} + 2\frac{2!}{4!} = \frac{5}{12}.\end{aligned}$$

Under the independence assumption and expected voting probabilities all equal $\frac{1}{2}$, the *absolute* Banzhaf index of Player 1 is given by

$$
\begin{aligned}
E[\pi_1(p_2, p_3, p_4)] = & \int_0^1 p_2 f_2(p_2)\ dp_2 \int_0^1 p_3 f_3(p_3)\ dp_3 \int_0^1 (1-p_4) f_4(p_4)\ dp_4 \\
& + \int_0^1 p_2 f_2(p_2)\ dp_2 \int_0^1 (1-p_3) f_3(p_3)\ dp_3 \int_0^1 p_4 f_4(p_4)\ dp_4 \\
& + \int_0^1 (1-p_2) f_2(p_2)\ dp_2 \int_0^1 p_3 f_3(p_3)\ dp_3 \int_0^1 p_4 f_4(p_4)\ dp_4 \\
& + \int_0^1 p_2 f_2(p_2)\ dp_2 \int_0^1 (1-p_3) f_3(p_3)\ dp_3 \int_0^1 (1-p_4) f_4(p_4)\ dp_4 \\
& + \int_0^1 (1-p_2) f_2(p_2)\ dp_2 \int_0^1 p_3 f_3(p_3)\ dp_3 \int_0^1 (1-p_4) f_4(p_4)\ dp_4 \\
= & \ \pi_1(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}) = \frac{5}{2^3} = \beta_1'.
\end{aligned}
$$

Similarly, we obtain $\beta_2' = \frac{3}{8},\ \beta_3' = \frac{3}{8},\ \beta_4' = \frac{1}{8}$.

By normalizing the sum of the Banzhaf indexes to be one, the *relative* Banzhaf index is

$$\beta = (\frac{5}{12}\ \frac{3}{12}\ \frac{3}{12}\ \frac{1}{12}).$$

*Individual - group agreement for player 1*

Out of $2^4 = 16$ cases, there are $2\eta_1 = 2 \times 5 = 10$ cases where Player 1 is marginal. In the remaining 6 cases (out of 16 cases), Player 1 does not make a difference. In half of these 6 cases, player 1 and group agree.

| players 1 | 2 | 3 | 4 | pass/fail | |
|---|---|---|---|---|---|
| $\boldsymbol{Y}$ | Y | Y | Y | P | with Y for players 2,3&4 gives "Pass" already, |
| $\boldsymbol{N}$ | Y | Y | Y | P | player 1 has equal probability to say Y or N |
| $\boldsymbol{Y}$ | N | N | Y | F | with N for players 2&3 gives "Fail" already, |
| $\boldsymbol{N}$ | N | N | Y | F | player 1 has equal probability to say Y or N |
| $\boldsymbol{Y}$ | N | N | N | F | with N for players 2,3&4 gives "Fail" already, |
| $\boldsymbol{N}$ | N | N | N | F | player 1 has equal probability to say Y or N |

We assume equal chance of getting $Y\ Y\ Y\ Y$ and $N\ Y\ Y\ Y$, and similar assumption for other pairs. Probability of player 1-group agreement = $\frac{1}{2} \times (1 - \frac{5}{8}) + 1 \times \frac{5}{8} = \frac{13}{16}$.

**Example**

Look again at $\underset{A\ \ B\ \ C}{[3; 2, 1, 1]}$. What is the probability that, under the independence assumption, the group decision will agree with $A$'s preference?

*Independence assumption*

Assume that all players vote with mean probability of 1/2 for or against a proposal. Recall $\pi_A(p) = 2p - p^2, \pi_B(p) = \pi_C(p) = p - p^2$ (see p.41). We obtain the absolute Banzhaf indexes as follow:

$$\pi_A\left(\frac{1}{2}\right) = 2\left(\frac{1}{2}\right) - \left(\frac{1}{2}\right)^2 = \frac{3}{4} = \beta'_A$$
$$\pi_B\left(\frac{1}{2}\right) = \pi_C\left(\frac{1}{2}\right) = \frac{1}{4} = \beta'_B = \beta'_C,$$

Finally, the relative Banzhaf indexes are $\beta_A = \frac{3}{5}$, $\beta_B = \frac{1}{5}$, $\beta_C = \frac{1}{5}$.

- With probability 1/2, $A$ will support a proposal. It will then pass *unless* $B$ and $C$ both oppose it, which will happen with probability 1/4.

  If $A$ opposes the proposal (probability 1/2), it will always fail.

  The probability of agreement with $A$ is thus

$$\frac{1}{2}\left(1-\frac{1}{4}\right)+\frac{1}{2}(1)=\frac{7}{8}=\frac{1+\frac{3}{4}}{2}=\frac{1+\beta'_A}{2}.$$

- Similarly, if $B$ supports a proposal (probability 1/2), it will pass if and only if $A$ supports it (probability 1/2).

- If $B$ opposes the proposal (probability 1/2), it will fail unless both $A$ and $C$ support it (probability 1/4):

$$\frac{1}{2}\left(\frac{1}{2}\right)+\frac{1}{2}\left(1-\frac{1}{4}\right)=\frac{5}{8}=\frac{1+\frac{1}{4}}{2}=\frac{1+\beta'_B}{2}.$$

**Example**

Consider $\underset{A\ \ B\ \ C\ D}{[5; 3, 2, 1, 1]}$. Let $\rho_i(p)$ be the probability that the group decision agrees with player $i$'s decision, given that all players (including $i$) vote for a proposal with probability $p$. Note that $A$ has veto power.

*Remark*

In the calculation procedure, it is convenient to set $p_A = p_B = p_C = p_D = p$. This is because under the independence assumption and common mean of probabilities of 1/2, we may set $p = 1/2$ apparently in the calculation of $E[\rho_i(p_A, p_B, p_C, p_D)]$.

We consider the two separate scenarios: "yes" or "no" for a particular voter, and examine the probability of forming coalition that the individual's choice agrees with the outcome of the coalition.

(a) It can be shown easily that

$$\rho_A(p) = \underset{A\text{ yes}}{p}\ [\underset{B\text{ yes}}{p} + \underset{B\text{ no, } C+D\text{ yes}}{(1-p)\,p^2}] + \underset{A\text{ no}}{(1-p)(1)} = 1 - p + p^2 + p^3 - p^4.$$

$$\rho_B(p) = \underset{B\text{ yes}}{p}\ \underset{A\text{ yes}}{(p)} + \underset{B\text{ no, not all of } A,\,C,\,D\text{ yes}}{(1-p)\,(1-p^3)} = 1 - p + p^2 - p^3 + p^4$$

$$\rho_C(p) = \underset{C\text{ yes}}{p}\ [\underset{A\text{ yes}}{p}\ (\underset{B\text{ yes}}{p} + \underset{B\text{ no, } D\text{ yes}}{(1-p)\,p})] + \underset{C\text{ no}}{(1-p)}\,[\underset{A\text{ no}}{(1-p)} + p\,[\underset{A\text{ yes, } B\text{ no}}{(1-p)}]]$$

$$= 1 - p - p^2 + 3p^3 - p^4.$$

(b) Now calculate $\rho_A(1/2), \rho_B(1/2)$, and $\rho_C(1/2)$ and show that these are $(1+\beta'_A)/2, (1+\beta'_B)/2$, and $(1+\beta'_C)/2$, thus verifying Theorem 3 for this case.

## Example – combination of homogeneity and independence

Consider the majority-minority voting system with 7 voters, where 5 of them are in the majority group and the remaining 2 voters are in the minority group. The passage of a bill requires at least 4 votes from all voters and at least 1 vote from the minority group. Suppose the 5 members in the majority group vote as a homogeneous group and the 2 members in the minority group vote as another homogeneous group. The two groups vote independently.

(a) Compute the probability that a majority player's vote decides the passage of a bill.

(b) Compute the probability that a minority player's vote decides the passage of a bill.

*Solution*

Under the homogeneity assumption, we let $p$ and $q$ denote the homogeneous voting probability of the majority group and minority group, respectively.

(a) Consider a particular majority member, her vote can decide the passage of a bill if

   (i) 1 minority member and 2 other majority members say "yes" and other members say "no";

   (ii) 2 minority members and 1 other majority member say "yes" and other members say "no".

$$
\begin{aligned}
& P[\text{majority player's vote can decide the passage}|p,q] \\
= \; & C_1^2 C_2^4 q(1-q)p^2(1-p)^2 + C_1^4 q^2 p(1-p)^3 \\
= \; & 12q(1-q)p^2(1-p)^2 + 4q^2p(1-p)^3.
\end{aligned}
$$

Assuming independence of the random probabilities $p$ and $q$, and both of them follow the uniform distribution under homogeneity assumption, we obtain

$$
\begin{aligned}
& P[\text{majority player's vote can decide the passage}] \\
= \; & \int_0^1 \int_0^1 \left[ 12q(1-q)p^2(1-p)^2 + 4q^2p(1-p)^3 \right] dpdq \\
= \; & 12 \int_0^1 p^2(1-p)^2 \; dp \int_0^1 q(1-q) \; dq + 4 \int_0^1 p(1-p)^3 \; dp \int_0^1 q^2 \; dq \\
= \; & 12 \frac{2!2!}{5!} \frac{1}{3!} + 4 \frac{1!3!}{5!} \frac{1}{3} = \frac{1}{5}.
\end{aligned}
$$

(b) Consider a particular minority member, her vote can decide the passage of a bill if

(i) 3 or more majority members say "yes" and other members say "no";

(ii) 2 majority members and the other minority member say "yes" and other members say "no".

Using similar assumptions on $p$ and $q$, we obtain

$$
\begin{aligned}
& P[\text{minority player's vote can decide the passage}] \\
= & \int_0^1 \int_0^1 \left[ \sum_{k=3}^{5} C_k^5 p^k (1-p)^{5-k}(1-q) + C_2^5 p^2 (1-p)^3 q \right] dp dq \\
= & 10 \left[ \int_0^1 p^3 (1-p)^2 \ dp \int_0^1 (1-q) \ dq + \int_0^1 p^2 (1-p)^3 \ dp \int_0^1 q \ dq \right] \\
& + 5 \int_0^1 p^4 (1-p) \ dp \int_0^1 (1-q) \ dq + \int_0^1 p^5 \ dp \int_0^1 (1-q) \ dq \\
= & 10 \left( \frac{3!2!}{6!} \cdot \frac{1}{2} + \frac{2!3!}{6!} \cdot \frac{1}{2} \right) + 5 \frac{4!}{6!} \cdot \frac{1}{2} + \frac{1}{6} \cdot \frac{1}{2} = \frac{1}{3}.
\end{aligned}
$$

## Example

Consider the voting game: $[2; 1, 1, 1]$.

Let $p_A, p_B$ and $p_C$ be the probabilities that $A, B$ and $C$ will vote for a proposal. Assuming independence of the random voting probabilities, we calculate the probabilities of a player's vote making a difference:

$$\begin{aligned}
\pi_A &= p_B(1-p_C)+(1-p_B)p_C, \\
\pi_B &= p_A(1-p_C)+(1-p_A)p_C, \\
\pi_C &= p_A(1-p_B)+(1-p_A)p_B.
\end{aligned}$$

- If the $p_i$s are all independent with mean 1/2 ($\beta'$) or all equal ($\phi$) as they vary between 0 and 1, then the players have equal power.

Suppose $B$ and $C$ are homogeneous ($p_B = p_C$), but $A$ is independent. Then the answers to the question of individual effect are

$$\text{for } A: \quad \int_0^1 2p_B(1-p_B)\,dp_B = \frac{1}{3} = \beta'_A$$

$$\begin{aligned}\text{for } B \text{ or } C: \quad & \left(\int_0^1 p_A f_A(p_A)\ dp_A\right)\left(\int_0^1 (1-p_B) f_B(p_B)\,dp_B\right) \\ & + \left(\int_0^1 (1-p_A) f_A(p_A)\,dp_A\right)\left(\int_0^1 p_B f_B(p_B)\,dp_B\right) \\ = \ & \frac{1}{2}\cdot\frac{1}{2} + \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{2} = \beta'_B = \beta'_C.\end{aligned}$$

With the pair sharing homogeneity in voting probabilities, $B$ and $C$ both have more power than $A$. In particular, we could normalize $(1/3, 1/2, 1/2)$ to $(1/4, 3/8, 3/8)$ and compare that to $(1/3, 1/3, 1/3)$.

## Canadian Constitutional Amendment Scheme revisited

| $B_1$ | $\otimes$ | $B_2$ | $\otimes$ | $M_{4,2}$ | $\otimes$ | $[3;2,1,1,1]$ |
|---|---|---|---|---|---|---|
| Quebec | | Ontario | | Atlantic | | British Columbia and Central. |

Intersection of 4 weighted voting systems.

| | Percentage of power | | | |
|---|---|---|---|---|
| Province | Shapley–Shubik index | Banzhaf index | Percentage of population | |
| Ontario | 31.55 | 21.7 | 34.85 | average 31.90 (Ontario, Quebec, British Columbia) |
| Quebec | 31.55 | 21.7 | 28.94 | |
| British Columbia | 12.50 | 16.3 | 9.38 | |
| Central | | | | |
| Alberta | 4.17 | 5.45 | 7.33 | average 5.65 (Central) |
| Saskatchewan | 4.17 | 5.45 | 4.79 | |
| Manitoba | 4.17 | 5.45 | 4.82 | |
| Atlantic | | | | |
| New Brunswick | 2.98 | 5.94 | 3.09 | average 2.47 (Atlantic) |
| Nova Scotia | 2.98 | 5.94 | 3.79 | |
| Prince Edward Island | 2.98 | 5.94 | 0.54 | |
| Newfoundland | 2.98 | 5.94 | 2.47 | |

*Observations*

- British Columbia enjoys higher power relative to her population due to the designed voting system.
- As the Shapley-Shubik index calculations are based on homogeneity of the voters, the scheme "produces a distribution of power that matches the distribution of population surprisingly well".
- Based on the Banzhaf analysis, the scheme would seriously under-represent Ontario and Quebec (both with veto power) and seriously over-represent British Columbia and the Atlantic provinces.
- It is disquieting that the two power indexes actually give different orders for the power of the players. $\phi$ says the Central Provinces are more powerful than the Atlantic provinces, and $\beta$ says the opposite.

Which index is more applicable?

- Use $\phi$ if we believe there is a certain kind of homogeneity among the provinces.
- Use $\beta$ if we believe there are more likely to act independently of each other.

*Actual behavior*

- Quebec and British Columbia would likely to behave independently.
- The four Atlantic provinces would more likely to satisfy the homogeneity assumption.

*Hybrid approach*

If a group of provinces is homogeneous, assign the members of that group the same $p$, which varies between 0 and 1 (independent of the $p$ assigned to other provinces or groups of provinces).

The conditional probability that Quebec's vote will make a difference is given by

$$\begin{aligned}\pi_Q(p_O, p_A, p_B, p_C) \;=\; & p_O[6p_A^2(1-p_A)^2 + 4p_A^3(1-p_A) + p_A^4] \\ & O \text{ yes} \qquad\qquad\qquad 2 \text{ or more } A\text{'s yes} \\ & \cdot\{p_B[3p_C(1-p_C)^2 + 3p_C^2(1-p_C) + p_C^3] + (1-p_B)p_C^3\} \\ & B \text{ yes} \quad 1 \text{ or } 2C\text{'s yes or} \qquad\qquad 3C\text{'s yes}\end{aligned}$$

We now compute the expectation of $\pi_Q$ as $p_O, p_A, p_B$, and $p_C$ vary independently between 0 and 1. Note that the joint density function of $p_C$, $p_B$, $p_A$ and $p_D$ reduces to 1 since it is the product of the marginal functions of $p_C$, $p_B$, $p_A$ and $p_D$ (due to independence assumption) and each of these marginal density functions equals 1 since they are uniform density functions over $[0,1]$. Technically, that involves a "fourfold multiple integral."

$$E[\pi_Q] = \int_0^1\int_0^1\int_0^1\int_0^1 \pi_Q f_C(p_C) f_B(p_B) f_A(p_A) f_O(p_O)\, dp_C\, dp_B\, dp_A\, dp_O.$$

We obtain

$$E[\pi_Q] = E[\pi_O] = 24/160, E[\pi_C] = 8/160, E[\pi_B] = 12/160, E[\pi_A] = 5/160.$$

There are $3C$'s and $4A$'s, the $\pi$'s sum to 104/160, so we normalize by multiplying the factor 160/104. The final power indexes under this scenarios are tabulated below under "$A_s$ homogeneous and $C_s$ homogeneous".

Table 2

| Provinces | All homogeneous ($\phi$) | As homogeneous Cs and B homogeneous | As homogeneous Cs homogeneous | All independent ($\beta$) | Average % of population |
|---|---|---|---|---|---|
| Quebec or Ontario | 31.55 | 26.09 | 23.08 | 21.78 | 31.90 |
| British Columbia | 12.50 | 13.04 | 11.54 | 16.34 | 9.38 |
| Central province | 4.17 | 4.35 | 7.69 | 5.45 | 5.65 |
| Atlantic province | 2.98 | 5.43 | 4.81 | 5.94 | 2.47 |

The power indexes computed under various hybrid homogeneity-independence assumptions must lie between the corresponding Shapley-Shubik and Banzhaf indexes.

*Quebec as independent*

Quebec seems often to consider itself an island of French culture in the sea of English Canada. Treat all 9 other provinces as homogeneous among themselves, and Quebec as independent.

| | |
|---|---|
| Quebec: 38.69 | British Columbia: 11.61 |
| Ontario: 25.84 | Central provinces: 3.87 |
| | Atlantic provinces: 3.07 |

Quebec's veto gives it considerable power. Alternatively, by staying homogeneous with other provinces, Ontario loses her power when compared to Quebec.

*British Columbia's possible homogeneity with the Central provinces*

Such homogeneity gives Quebec and Ontario more power (jump from 23.08 to 26.09). A higher level of homogeneity of other players gives more influential power to the province with veto power.

## 4.2 Bargaining games

### Characterization of non-cooperative payoff set

Consider the two-player nonzero sum game

| | $\mathrm{II}_1$ | $\mathrm{II}_2$ |
|---|---|---|
| $\mathrm{I}_1$ | $(2,1)$ | $(-1,-1)$ |
| $\mathrm{I}_2$ | $(-1,-1)$ | $(1,2)$ |

The payoff matrices of I and II are

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}.$$

Suppose I and II play their respective mixed strategies:

$$X = \begin{pmatrix} x & 1-x \end{pmatrix} \text{ and } Y = \begin{pmatrix} y & 1-y \end{pmatrix},$$

their expected payoffs are

$$E_{\mathrm{I}}(x,y) = (x \ \ 1-x)A\begin{pmatrix} y \\ 1-y \end{pmatrix}, \quad E_{\mathrm{II}}(x,y) = (x \ \ 1-x)B\begin{pmatrix} y \\ 1-y \end{pmatrix}.$$

We would like to generate all possible pairs of payoffs under non-cooperation between the two players using the following Maple commands. That is, they choose $X$ and $Y$ without prior agreement on their combination of strategies.

```
> with(plots):with(plottools):with(LinearAlgebra):
> A:=Matrix([[2,-1],[-1,1]]);B:=Matrix([[1,-1],[-1,2]]);
> f:=(x,y)->expand(Transpose(<x,1-x>).A.<y,1-y>);
> g:=(x,y)->expand(Transpose(<x,1-x>).B.<y,1-y>);
> points:={seq(seq([f(x,y),g(x,y)],x=0..1,0.05),y=0..1,0.05)}:
> pure:=([[2,1],[-1,-1],[-1,-1],[1,2]]);
> pp:=pointplot(points);
> pq:=polygon(pure,color=yellow);
> display(pq,pp,title=''Payoffs with and without cooperation'');
```

The horizontal axis (abscissa) is the payoff to player I, and the vertical axis (ordinate) is the payoff to player II. Any point in the parabolic region is achievable for some $0 \le x \le 1$, $0 \le y \le 1$.

When the two players happen to choose $x = y$ (not under cooperation though), the resulting payoff point $(E_1, E_2)$ lies on the parabola. To verify that, we consider

$$E_{\rm I}(x) = (x \;\; 1-x)\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} x \\ 1-x \end{pmatrix},$$

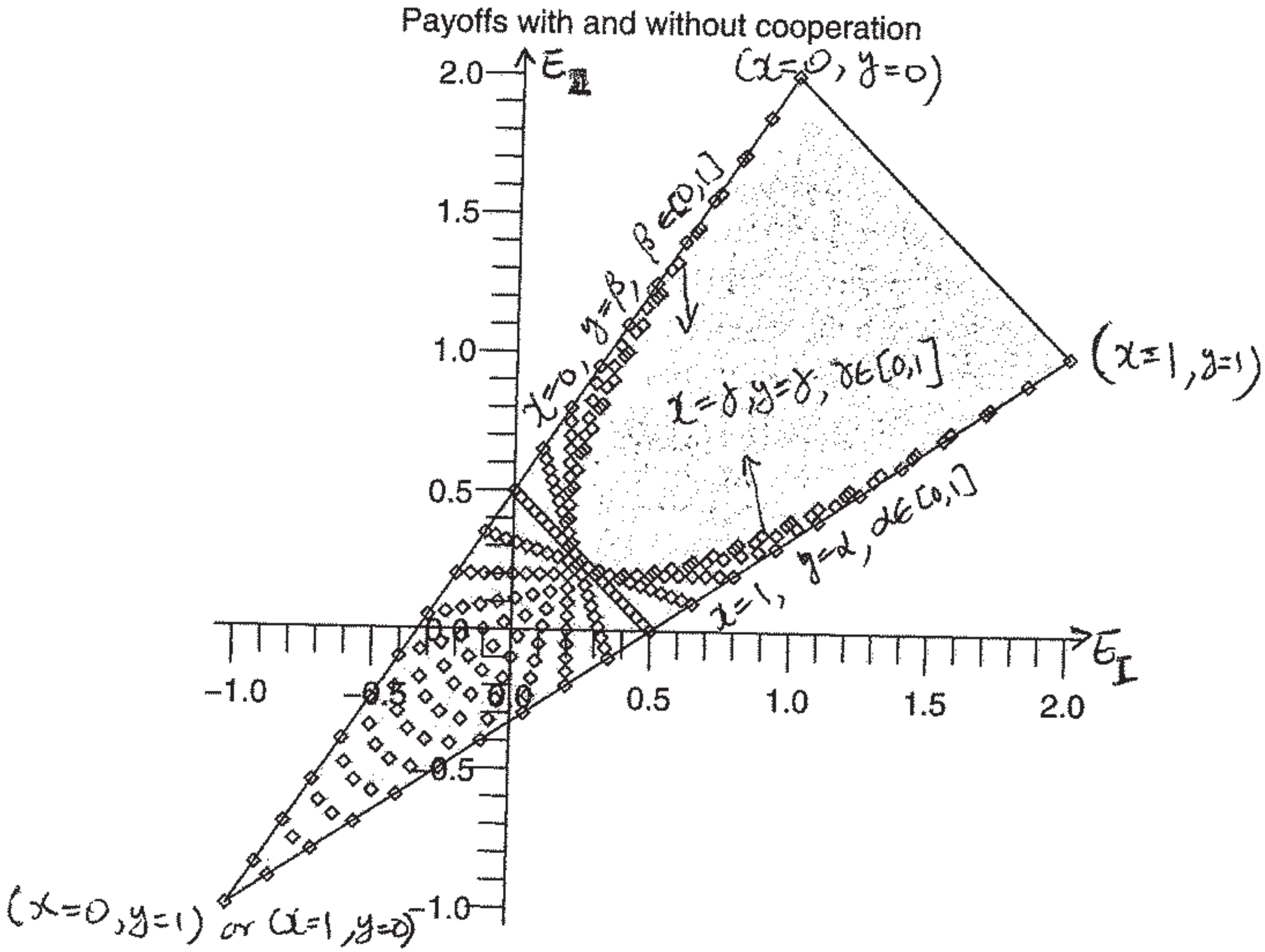
$$E_{\rm II}(x,x) = (x \;\; 1-x)\begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} x \\ 1-x \end{pmatrix},$$

$$E_{\rm I}(x,x) - E_{\rm II}(x,x) = (x \;\; 1-x)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ 1-x \end{pmatrix} = 2x-1,$$

$$E_{\rm I}(x,x) + E_{\rm II}(x,x) = (x \;\; 1-x)\begin{pmatrix} 3 & -2 \\ -2 & 3 \end{pmatrix}\begin{pmatrix} x \\ 1-x \end{pmatrix} = 10x^2 - 10x + 3.$$

Eliminating $x$, we obtain the relation:

$$5(E_1 - E_2)^2 - 2(E_1 + E_2) + 1 = 0.$$




The non-cooperative payoff set is bounded by

(i) line joining $(-1,-1)$ and $(1,2)$; (I plays $\mathrm{I}_2$ purely and II plays mixed)

(ii) line joining $(-1,-1)$ and $(2,1)$; (I plays $\mathrm{I}_1$ purely and II plays mixed)

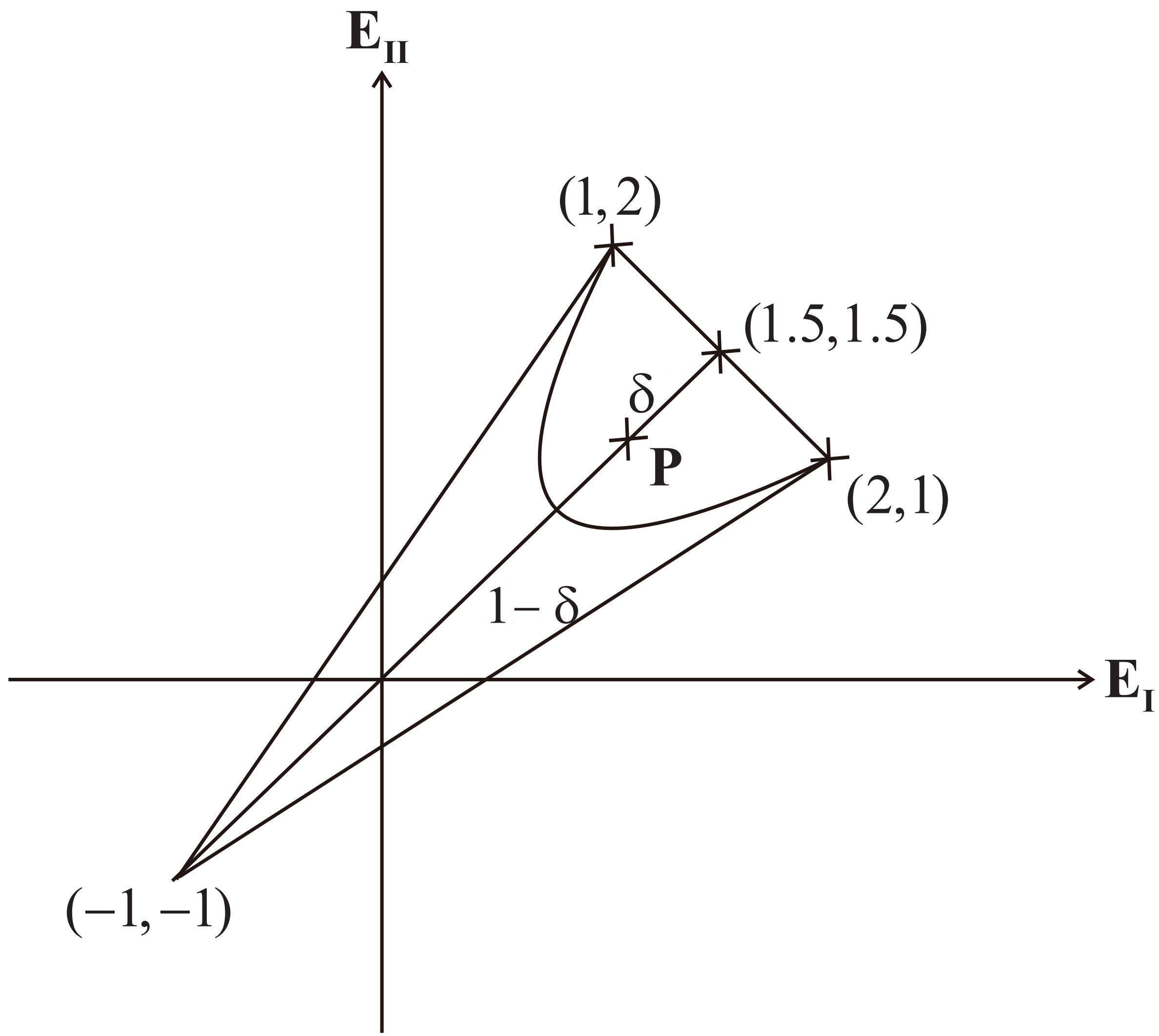
(iii) parabola: $5(E_1 - E_2)^2 - 2(E_1 + E_2) + 1 = 0$.

In order to achieve payoffs beyond the bounding parabola, the players have to come to an agreement as to which combination of strategies each player will use and the proportion of time that the strategies will be used.

The triangle is the *convex hull* of (smallest convex set containing) the pure payoff pairs. We cannot achieve payoff pair that lies outside the convex hull.

The line segment joining $(1,2)$ and $(2,1)$ is the Pareto-optimal boundary of the feasible region (convex hull) since no player can improve his payoff without lowering the payoff of the other player.

*Definition of convex set and convex hull*
A point set is said to be convex if for any pair of points chosen in the set line segment joining the pair lies completely inside the set. The convex hull of a given set $A$ is the smallest convex set that contains $A$.




The payoff points beyond the parabola can be achieved by some linear combination of pure strategies: $(-1,-1)$, $(1,2)$ and $(2,1)$.

1. To achieve the payoff point $(E_1, E_2) = (1.5, 1.5)$, the players cooperate to choose

   - 50% of time playing ($x = 0$, $y = 0$)
   - 50% of time playing ($x = 1$, $y = 1$)

   Note that $x = 0$ and $y = 0$ is the agreed combination of strategies, not interpreted as I happens to adopt $x = 0$ and II happens to adopt $y = 0$.

2. The point $P$ divides the line segment joining $(1.5, 1.5)$ and $(-1, -1)$ into $(\delta, 1 - \delta)$ portion. To achieve the payoff point $P$, both players cooperate to play

   - $\delta$ portion of the time of 50% on ($x = 0$, $y = 0$) and 50% on ($x = 1$, $y = 1$), equivalent to $\delta/2$ portion of time on ($x = 0$, $y = 0$) and another $\delta/2$ portion of time on ($x = 1$, $y = 1$).
   - $1 - \delta$ portion of time on ($x = 0$, $y = 1$) or ($x = 1$, $y = 0$).

Bargaining games are cooperative games in which the players bargain to improve both of their payoffs.

**Example**

| | $II_1$ | $II_2$ | $II_3$ |
|---|---|---|---|
| $I_1$ | $(1,4)$ | $(-2,1)$ | $(1,2)$ |
| $I_2$ | $(0,-2)$ | $(3,1)$ | $(\frac{1}{2},\frac{1}{2})$ |

The vertices of the polygon on p.77 are the pure payoffs directly from the matrix. The solid lines connect the pure payoffs. The top dotted line joining $(1,4)$ and $(3,1)$ extends the region of payoffs to those payoffs that could be achieved if both players cooperate. These payoffs on the dotted line are obtained by agreeing to play $(1,4)$ and $(3,1)$, each of them with varying fixed proportions of time.

Suppose that player I always chooses row 2 as the pure strategy and player II plays the mixed strategy $Y = (y_1, y_2, y_3)$, where $y_i \geq 0$, $y_1 + y_2 + y_3 = 1$.

The expected payoff to I is then

$$E_1(2, Y) = 0y_1 + 3y_2 + \frac{1}{2}y_3,$$

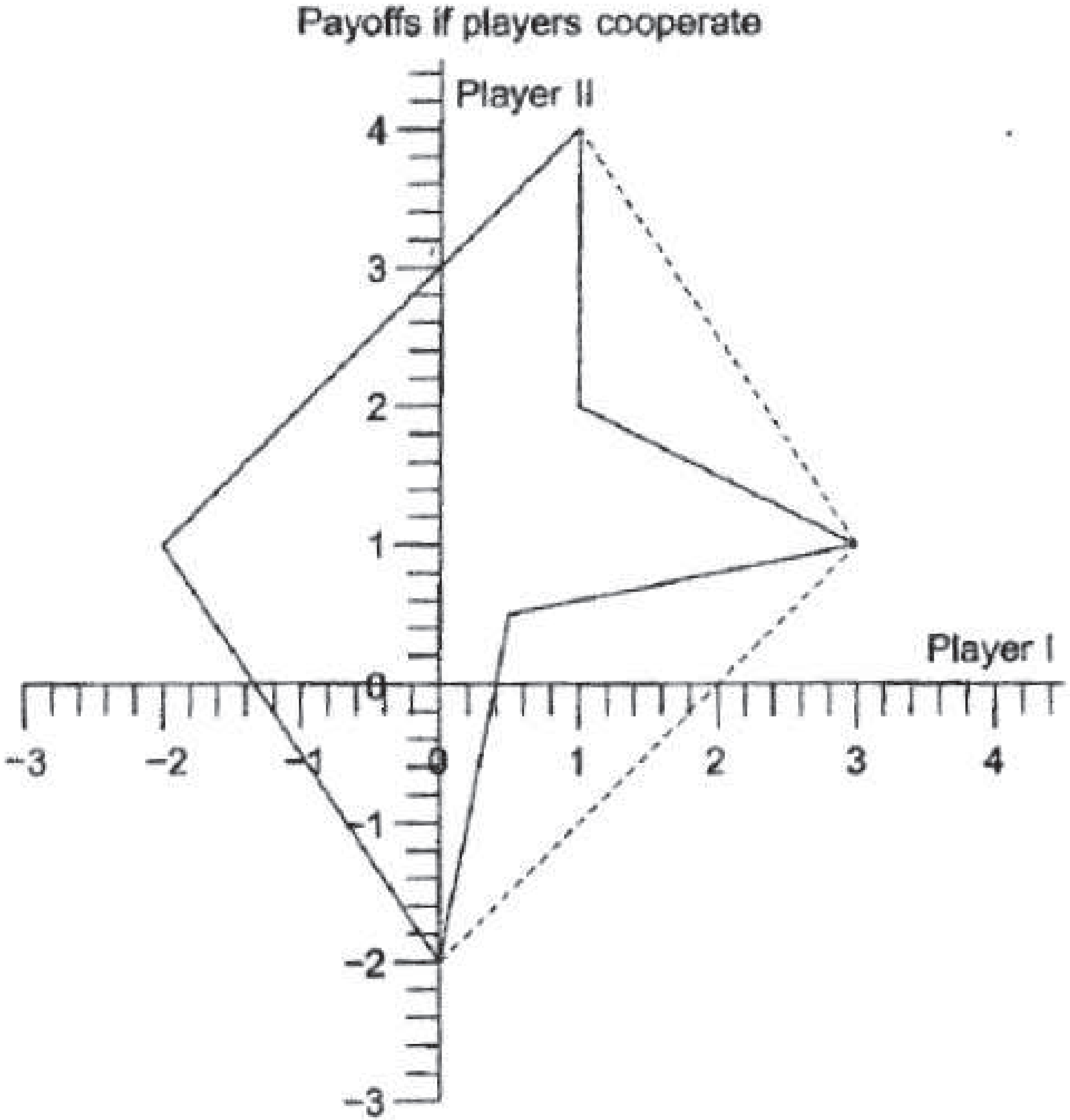
and the expected payoff to II is

$$E_2(2, Y) = -2y_1 + 1y_2 + \frac{1}{2}y_3.$$

Hence, the players' payoffs are

$$(E_1, E_2) = y_1(0, -2) + y_2(3, 1) + y_3(\frac{1}{2}, \frac{1}{2}),$$

as a linear combination of the 3 points $(0, -2)$, $(3, 1)$ and $(\frac{1}{2}, \frac{1}{2})$.




Achievable payoffs with cooperation.

*Convex hull formed by the pure payoff points*

- The feasible set is the convex hull of all the payoff points corresponding to pure strategies of the players.

The triangle bounded by the lower dotted line in the figure and the lines connecting $(0,-2)$ with $(\frac{1}{2},\frac{1}{2})$ and $(\frac{1}{2},\frac{1}{2})$ with $(3,1)$ is the convex hull of these three points.

Any point in the convex hull of all the payoff points is achievable if the players agree to cooperate. The entire four-sided region is called the feasible set for the game problem.

The objective of player I is to obtain a payoff as far to the right as possible in the figure, and the objective of player II is to obtain a payoff as far up as possible in the same figure.

Player I's ideal payoff is at the point $(3, 1)$, but that is attainable only if II agrees to play $II_2$. Why would he do that? Similarly, II would do best at $(1, 4)$, which will happen only if I plays $I_1$, and why would she do that?

There is an incentive for the players to reach a compromise agreement in which they would agree to play in such a way so as to obtain a payoff along the line connecting $(1, 4)$ and $(3, 1)$.

## Pareto-optimal boundary

That portion of the boundary is known as the *Pareto-optimal boundary* because it is the edge of the feasible set and has the property that if either player tries to do better (say, player I tries to move further right), then the other player will do worse (player II must move down to remain feasible).

The Pareto-optimal boundary of the feasible set is the set of payoff points in which no player can improve his payoff without the other player decreasing her payoff.

There is an incentive for the players to cooperate and try to reach an agreement that will benefit both players. The result will always be a payoff pair occurring on the Pareto-optimal boundary of the feasible set (see Nash's Theorem later).

**Status quo payoff point**

In any bargaining problem, there is always the possibility that negotiations will fail. Hence, each player must know what the possible worst payoff would be if there were no bargaining.

The *status quo payoff point*, or *safety point*, or *security point* in a two-person game is the pair of payoffs $(u^*, v^*)$ that each player can achieve if there is no cooperation between the players.

*Determination of the security point for each player*

We take the security point to be the values that each player can guarantee receiving no matter what. This means that we take it to be the value of the zero sum game for each player.

Consider the payoff matrix for player I:

$$A = \begin{pmatrix} 1 & -2 & 1 \\ 0 & 3 & \frac{1}{2} \end{pmatrix}.$$

Recall value$(A) = \max_{X \in S_n} \min_{Y \in S_m} XAY^T$. We find that $v(A) = \frac{1}{2}$ and the corresponding strategies are $Y = (\frac{5}{6}, \frac{1}{6}, 0)$ for player II and $X = (\frac{1}{2}, \frac{1}{2})$ for player I.

Next, we consider the payoff matrix B for player II and want to find the value of the game from player II's perspective. We need to work with $B^T$, where

$$\text{value}(B^T) = \max_{Y \in S_m} \min_{X \in S_n} YB^TX^T.$$

Now

$$B^T = \begin{pmatrix} 4 & -2 \\ 1 & 1 \\ 2 & \frac{1}{2} \end{pmatrix}.$$

For this matrix $v(B^T) = 1$, it happens that we have a saddle point at row 2 and column 2 of $B^T$. Note that $v(B^T) = \max\left(\min(4, -2), \min(1, 1), \min(2, \frac{1}{2})\right) = \max(-2, 1, \frac{1}{2}) = 1$.

The status quo point for this game is $(E_1, E_2) = (\frac{1}{2}, 1)$ since that is the guaranteed payoff to each player without cooperation or negotiation. Any bargaining must begin with the guaranteed payoff pair $(\frac{1}{2}, 1)$. This cuts off the feasible set as shown in the figure.

The new feasible set consists of the points in the figure and to the right of the lines emanating from the security point $(\frac{1}{2}, 1)$.

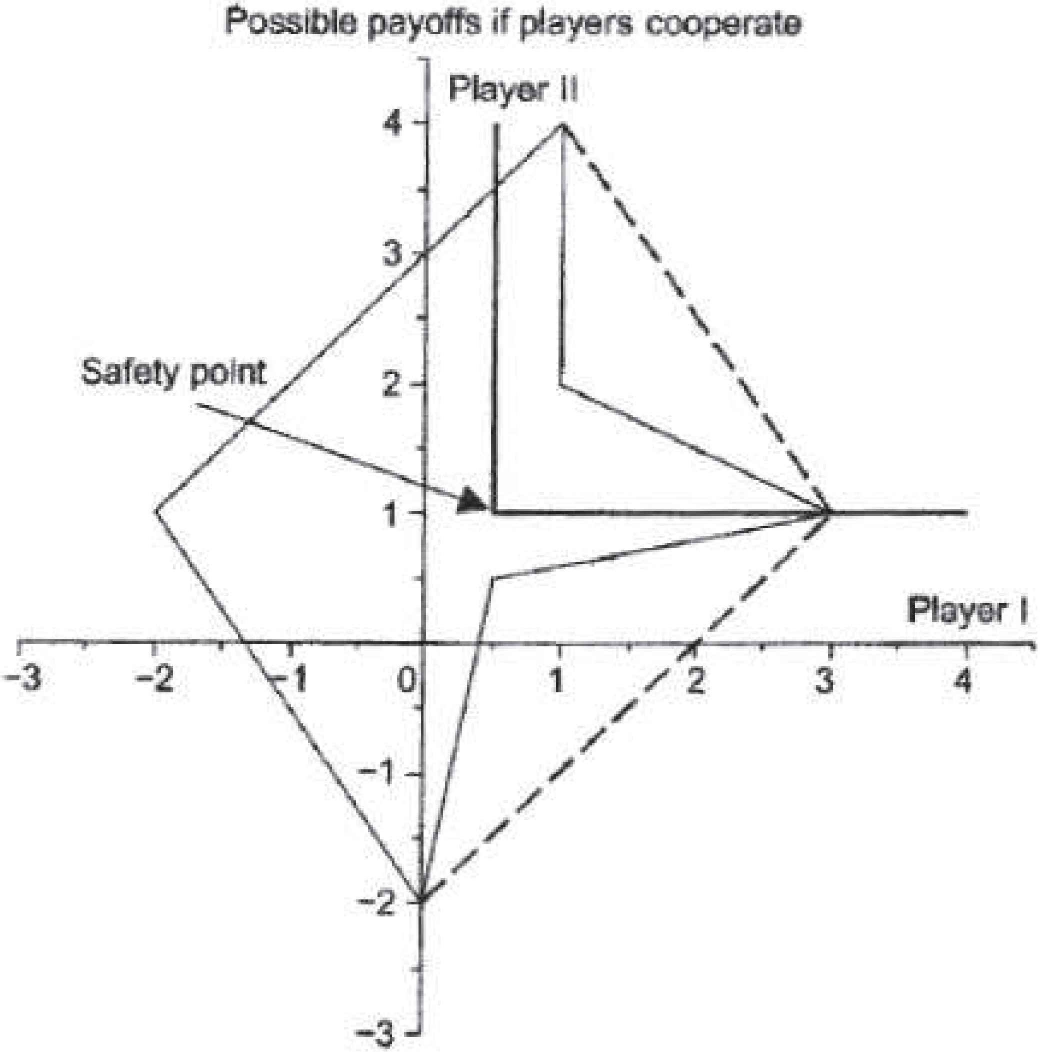
The Pareto-optimal boundary is the line connecting $(1,4)$ and $(3,1)$ because no player can get a higher payoff on this line without forcing the other player to get a smaller payoff.

A point in the set cannot go to the right and stay in the set without also going down; a point in the set cannot go up and stay in the set without also going to the left.

*Finding the cooperative, negotiated best payoff for each player*

How does cooperation help?

If they agree to play 50% of $(1,4)$ and 50% of $(3,1)$, they will get $\frac{1}{2}(1,4)+\frac{1}{2}(3,1)=(2,\frac{5}{2})$. So player I obtains $2>\frac{1}{2}$ and player II obtains $\frac{5}{2}>1$, an improvement for each player over individual safety level. Hence, they have good incentive to cooperate.




The reduced feasible set; safety at ($\frac{1}{2}$, 1).

**Example**

The bimatrix is

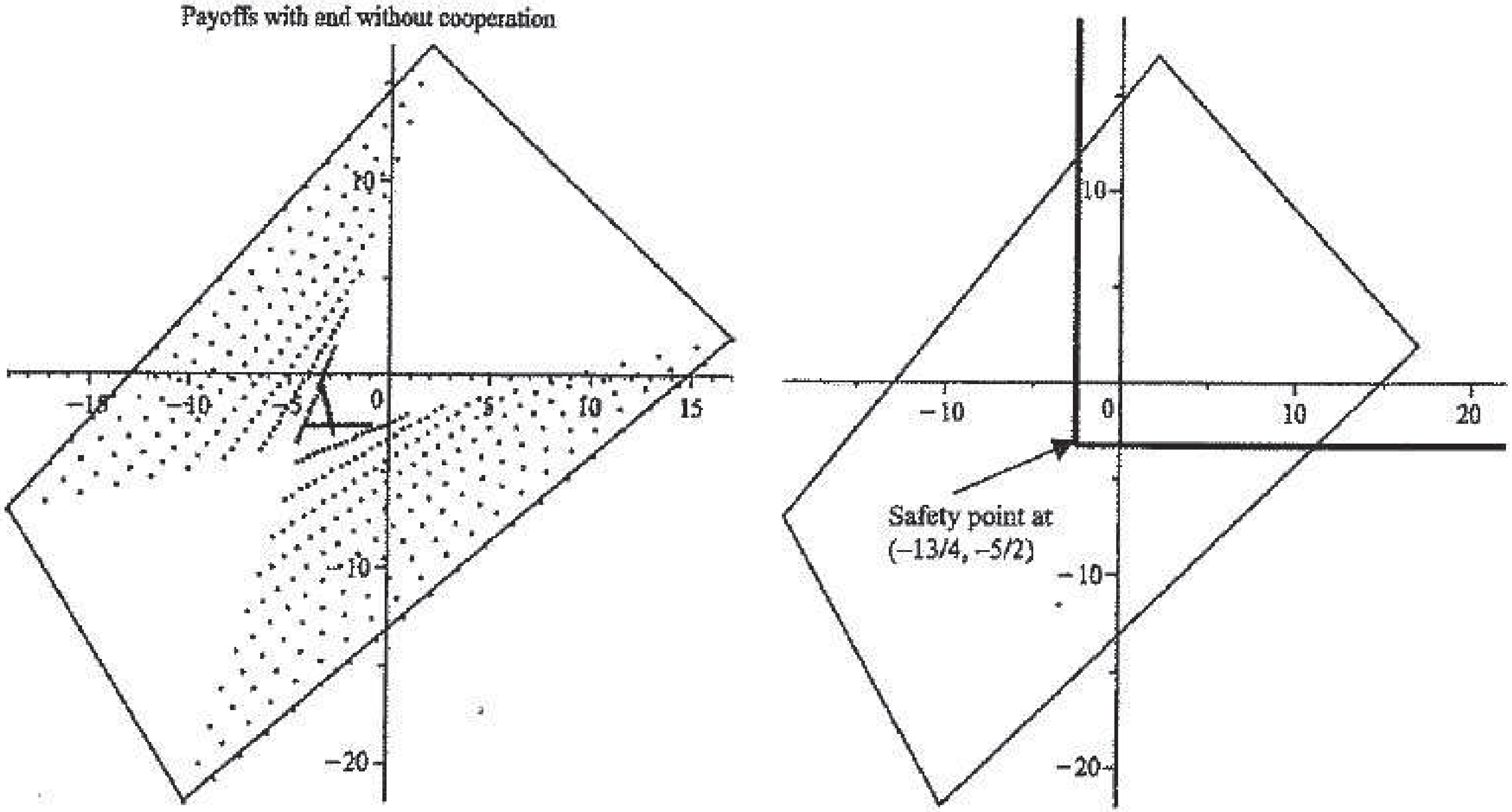
| | $II_1$ | $II_2$ |
|---|---|---|
| $I_1$ | $(2, 17)$ | $(-10, -22)$ |
| $I_2$ | $(-19, -7)$ | $(17, 2)$ |

Recall that the safety levels are the guaranteed amounts each player can obtain by using their individual maximin strategies. The safety level is given by the point

$$(\text{value}(A), \text{value}(B^T)) = (-\frac{13}{4}, -\frac{5}{2}),$$

and the strategies that will give these values are $X_A = (\frac{3}{4}, \frac{1}{4})$, $Y_A = (\frac{9}{16}, \frac{7}{16})$, and $X_B = (\frac{1}{2}, \frac{1}{2})$, $Y_B(\frac{3}{16}, \frac{13}{16})$.

Negotiations start from the safety point. The next figure shows the safety point and the associated feasible payoff pairs above and to the right of the dark lines.




Achievable payoff pairs with cooperation; safety point = $(\frac{13}{4}, -\frac{5}{2})$.

The 4-sided polygon in the figure is the convex hull of the pure payoffs, namely, the feasible set, and is the set of all possible negotiated payoffs. The region of dot points is the set of noncooperative payoff pairs if we consider the use of all possible mixed strategies.

It appears that a negotiated set of payoffs will benefit both players and will be on the line farthest to the right, which is the Pareto-optimal boundary. Player I would desire to get $(17, 2)$, while player II would love to get $(2, 17)$. That probably will not occur but they could negotiate a point along the line connecting these two points and compromise on obtaining, say, the midpoint

$$\frac{1}{2}(2, 17) + \frac{1}{2}(17, 2) = (9.5, 9.5).$$

So they could negotiate to get 9.5 each if they agree that each player would use the pure strategies $X = (1, 0) = Y$ half the time and play pure strategies $X = (0, 1) = Y$ exactly half the time. They have an incentive to cooperate.

*Threat possibilities*

Now suppose that player II threatens player I by saying that she will always play strategy $\mathrm{II}_1$ unless I cooperates. Player II's goal is to get the 17 if and when I plays $\mathrm{I}_1$, so I would receive 2. Of course, I does not have to play $\mathrm{I}_1$, but if he doesn't, then I will get $-19$ (highly negative payoff), and II will get $-7$.

So, if I does not cooperate and II carries out her threat, they will both lose, but I will lose much more than II. Therefore, II is in a stronger position than I in this game and can essentially *force* I to cooperate. This also seems to imply that maybe player II should expect to get more than 9.5 to reflect her stronger bargaining position from the start.

This example indicates that there may be a more realistic choice for a safety level than the values of the associated games, taking into account various threat possibilities.

**Nash model with security point**

We start with the security status quo point $(u^*, v^*)$ for a two-player co-operative game with matrices $A$ and $B$. This leads to a feasible set of possible negotiated outcomes depending on the choice of the point $(u^*, v^*)$.

One convenient choice may be $u^* = \text{value}(A)$ and $v^* = \text{value}(B^T)$. Given $(u^*, v^*)$ and feasible set $S$, we seek for a negotiated outcome, call it $(\bar{u}, \bar{v})$. Since this point will depend on $(u^*, v^*)$ and the set $S$, so we may write

$$(\bar{u}, \bar{v}) = f(S, u^*, v^*).$$

The question is how to determine the point $(\bar{u}, \bar{v})$ and an appropriate choice of $f(S, u^*, v^*)$. John Nash proposed the following requirements for the point to be a negotiated solution:

- *Axiom 1.* We must have $\bar{u} \geq u^*$ and $\bar{v} \geq v^*$. Each player must get at least the status quo point.

- *Axiom 2.* The point $(\bar{u}, \bar{v}) \in S$, that is, it must be a feasible point.

- *Axiom 3.* $(\overline{u}, \overline{v})$ *Pareto-optimality*. There is no other point in $S$, where both players receive more.

- *Axiom 4.* If $(\bar{u}, \bar{v}) \in T \subset S$ and $(\bar{u}, \bar{v}) = f(T, u^*, v^*)$ is the solution to the bargaining problem with feasible set $T$, then for the larger feasible set $S$, either $(\bar{u}, \bar{v}) = f(S, u^*, v^*)$ is the bargaining solution for $S$, or the actual bargaining solution for $S$ is in $S - T$. We are assuming that the security point is the same for $T$ and $S$. If we enlarge the set of alternatives from $T$ to $S$, the new negotiated position cannot be one of the old possibilities in $T$ other than $(\bar{u}, \bar{v})$.

  As an one-dimensional analogy, this is similar to finding the maximum value of a function $f(x)$ over different intervals.

- *Axiom 5.* If $T$ is an affine transformation of $S$, $T = aS + b = \varphi(S)$ and $(\bar{u}, \bar{v}) = f(S, u^*, v^*)$ is the bargaining solution of $S$ with security point $(u^*, v^*)$, then $(a\bar{u} + b, a\bar{v} + b) = f(T, au^* + b, av^* + b)$ is the bargaining solution associated with $T$ and security point $(au^* + b, av^* + b)$. This says that the solution will not depend on the scale or units used in measuring payoffs.

- *Axiom 6.* If the game is symmetric with respect to the players, then so is the bargaining solution. In other words, if $(\bar{u}, \bar{v}) = f(S, u^*, v^*)$ and (i) $u^* = v^*$, and (ii) $(u, v) \in S \Rightarrow (v, u) \in S$, then $\bar{u} = \bar{v}$. If the players are essentially interchangeable, they should get the same negotiated payoff.

**Theorem**

Let the set of feasible points for a bargaining game be nonempty and convex, and let $(u^*, v^*) \in S$ be the security point. Consider the nonlinear programming problem

$$\begin{aligned} \text{Maximize } & g(u,v) = (u-u^*)(v-v^*) \\ \text{subject to } & (u,v) \in S,\ u \geq u^*,\ v \geq v^*. \end{aligned}$$

Assume that there is at least one point $(u,v) \in S$ with $u > u^*$, $v > v^*$. Then there exists one and only one point $(\bar{u}, \bar{v}) \in S$ that solves this problem, and this point is the unique solution of the bargaining problem $(\bar{u}, \bar{v}) = f(S, u^*, v^*)$ that satisfies Axioms 1-6. If, in addition, the game satisfies the symmetry assumption, then the conclusion of Axiom 6 tells us that $\bar{u} = \bar{v}$.

## Uniqueness

We prove by contradiction. Suppose the maximum of $g$ occurs at two points: $(u', v')$ and $(u'', v'')$, where

$$g(u', v') = g(u'', v'') = M > 0.$$

(i) If $u' = u''$, then obviously $v' = v''$ since we can cancel the common factor $u' - u^* = u'' - u^*$ in both $g(u', v')$ and $g(u'', v'')$ and obtain $v' = v''$.

(ii) Consider the case $u' < u''$, then $v' > v''$. Let $u = \frac{u'+u''}{2}$ and $v = \frac{v'+v''}{2}$. Obviously, $(u, v) \in S$ since $S$ is convex and $u > u^*$ and $v > v^*$. Consider

$$\begin{aligned} g(u, v) &= \left(\frac{u' + u''}{2} - u^*\right)\left(\frac{v' + v''}{2} - v^*\right) \\ &= M + \frac{(u' - u'')(v'' - v')}{4} > M \text{ since } u'' > u' \text{ and } v' > v''. \end{aligned}$$

This contradicts the fact that $(u', v')$ provides a maximum for $g$.

**Pareto optimality**

We prove by contradiction. Suppose there exists another feasible point $(u', v') \in S$ for which $u' > \bar{u}$ and $v' \geq \bar{v}$ or $v' > \bar{v}$ and $u' \geq \bar{u}$. Let us consider the first possibility. It is obvious that

$$g(u', v') = (u' - u^*)(v' - v^*) > (\bar{u} - u^*)(\bar{v} - v^*) = g(\bar{u}, \bar{v}).$$

This contradicts the fact that $(\bar{u}, \bar{v})$ maximizes $g$ over the feasible set. Hence, $(\bar{u}, \bar{v})$ is Pareto-optimal.

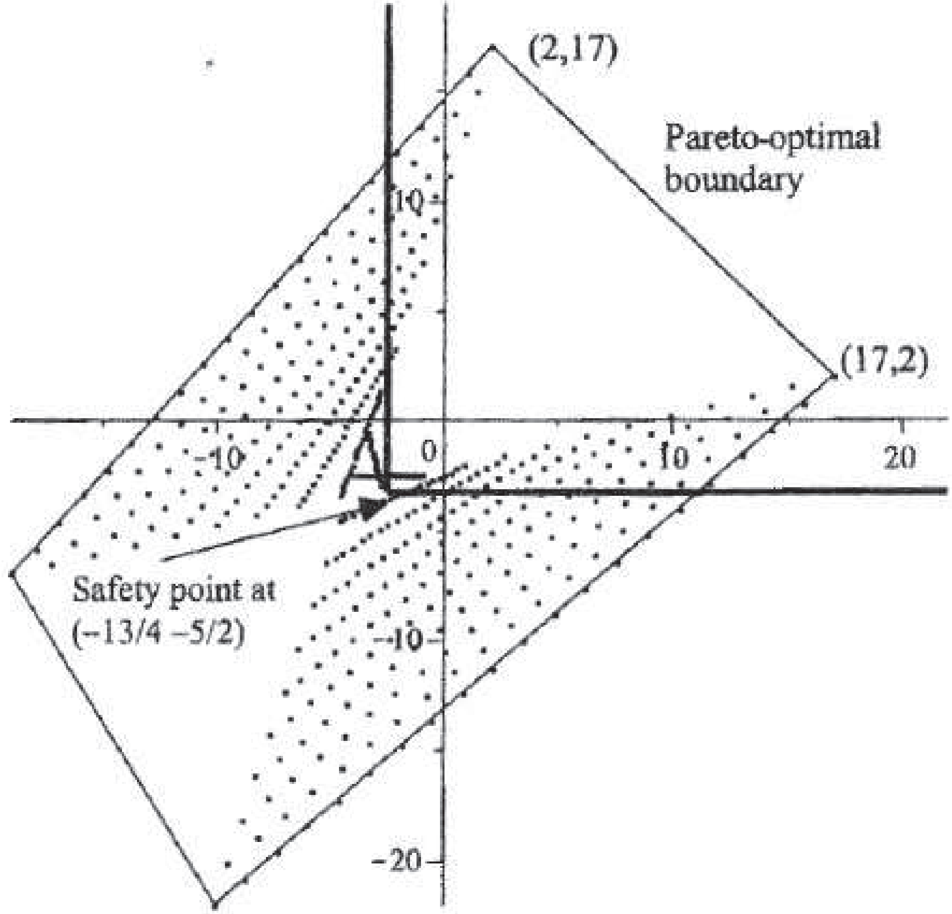
**Example**

We consider the game with bimatrix

| | $\mathrm{II}_1$ | $\mathrm{II}_2$ |
|---|---|---|
| $\mathrm{I}_1$ | $(2, 17)$ | $(-10, -22)$ |
| $\mathrm{I}_2$ | $(-19, -7)$ | $(17, 2)$ |

The safety levels are $u^* = \text{value}(A) = -\frac{13}{4}$, $v^* = \text{value}(B^T) = -\frac{5}{2}$. The safety point and the associated feasible payoff pairs are above and to the right.

Next, we find the equation of the lines forming the Pareto-optimal boundary. In this example, it is simply $v = -u + 19$, which is the line with negative slope to the right of the safety point. Along this line, both players cannot simultaneously improve their payoffs. If player I moves right and in order to stay in the feasible set, player II must go down.




Pareto-optimal boundary is line connecting (2, 17) and (17, 2).

To find the bargaining solution for this problem, we solve the nonlinear programming problem:

$$\text{Maximize } (u+\frac{13}{4})(v+\frac{5}{2})$$
$$\text{subject to } u \geq -\frac{13}{4},\ v \geq -\frac{5}{2},\ v \leq -u+19.$$

This gives the optimal bargained payoff pair ($\bar{u} = \frac{73}{8} = 9.125,\ \bar{v} = \frac{79}{8} = 9.875$). The maximum of $g$ is $g(\bar{u}, \bar{v}) = 153.14$.

The bargained payoff to player I is $\bar{u} = 9.125$ and the bargained payoff to player II is $\bar{v} = 9.875$.

We do not get the point we expected, namely $(9.5, 9.5)$; that is due to the fact that the security point is not symmetric. Player II has a small advantage.

The solution of the problem occurs just where the level curves, or contours of $g$ are tangent to the boundary of the feasible set. Since the function $g$ has concave up contours and the feasible set is convex, this must occur at exactly one point.

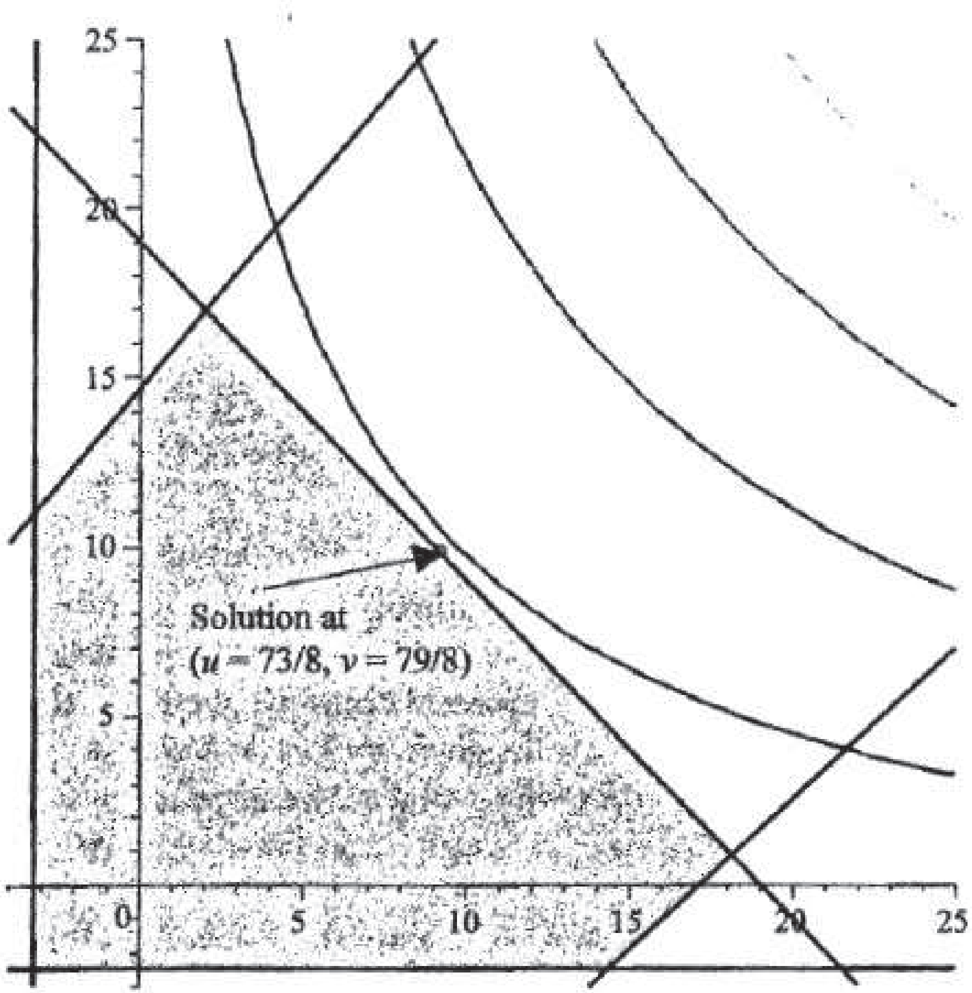
Finally, knowing that the optimal point must occur on the Pareto-optimal boundary means we could solve the nonlinear programming problem by calculus. We want to maximize

$$f(u) = g(u, -u + 19) = (u + \frac{13}{4})(-u + 19 + \frac{5}{2}),$$

on the interval $2 \leq u \leq 17$. This is an elementary calculus maximization problem. Note that the first order condition gives

$$f'(u) = -u + 19 + \frac{5}{2} - u - \frac{13}{2} = 0.$$

This gives $u = 9.125$.




Bargaining solution where curves just touch Pareto boundary at (9.125, 9.875).

The hyperbolic curves are the level curves: $\left(u + \frac{13}{4}\right)\left(v + \frac{5}{2}\right) = k$, for some constant value of $k$.

**Example**

Consider the following bimatrix:

| | $\text{II}_1$ | $\text{II}_2$ |
|---|---|---|
| $\text{I}_1$ | $(1,3)$ | $(-4,-2)$ |
| $\text{I}_2$ | $(-1,-3)$ | $(2,1)$ |

1. *Find the security point.* For the associated matrices

$$A = \begin{pmatrix} 1 & -4 \\ -1 & 2 \end{pmatrix}, \quad B^T = \begin{pmatrix} 3 & -3 \\ -2 & 1 \end{pmatrix}.$$

Recall that value $(A)$ is the value of the zero-sum game with payoff matrix $A$ that provides the guaranteed floor value for player I. We obtain value$(A) = -\frac{1}{4}$, value$(B^T) = -\frac{1}{3}$. Hence, the security point is $(-\frac{1}{4}, -\frac{1}{3})$.

2. *Find the feasible set.* The feasible set, taking into account the security point, is

$$S^* = \left\{(u,v) | u \geq -\frac{1}{4},\ v \geq -\frac{1}{3},\ 0 \leq 10 + 5u - 5v,\ 0 \leq 10 + u + 3v, \right.$$
$$\left. 0 \leq 5 - 4u + 3v,\ 0 \leq 5 - 2u - v \right\}.$$

3. *Set up and solve the nonlinear programming problem.*

$$\text{Maximize } g(u,v) \equiv (u + \frac{1}{4})(v + \frac{1}{3})$$
$$\text{subject to } (u,v) \in S^*.$$

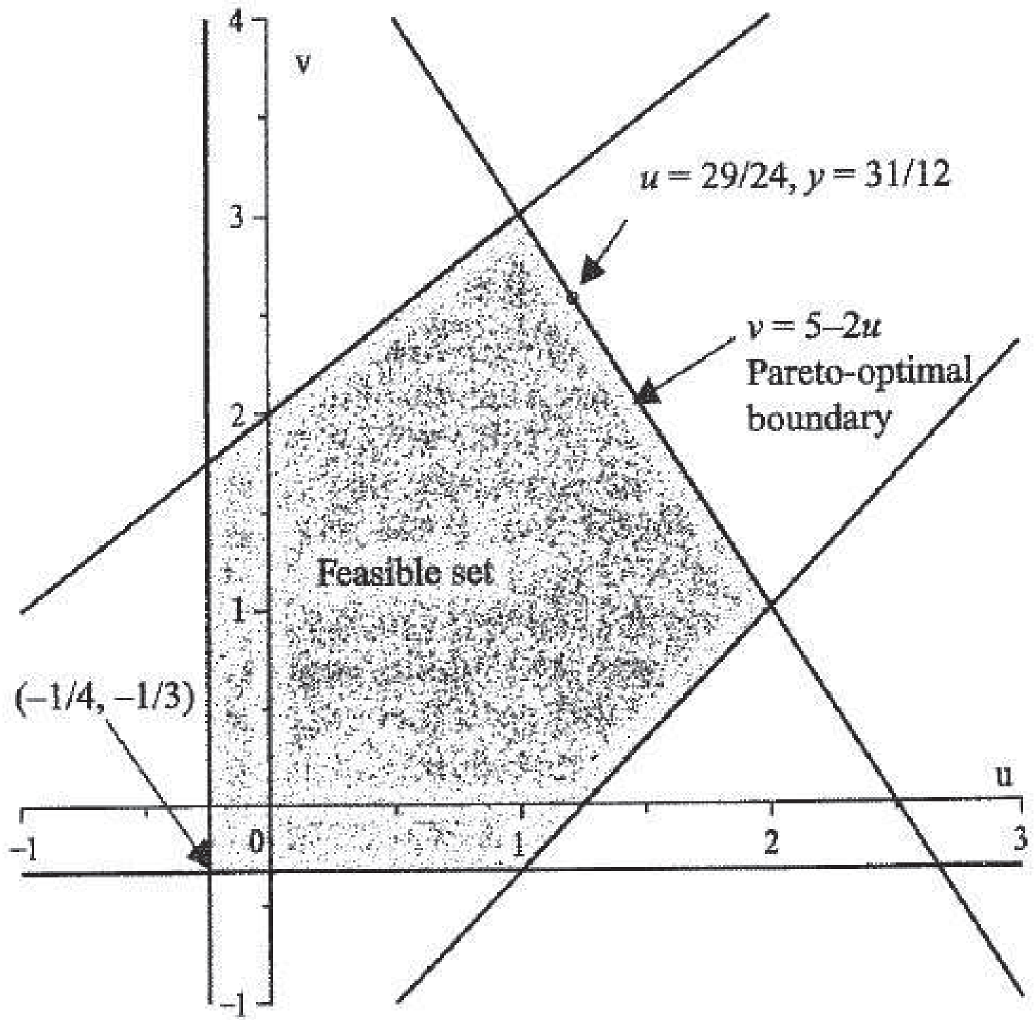
Maple gives the solution $\bar{u} = \frac{29}{24} = 1.208$, $\bar{v} = \frac{31}{12} = 2.583$.

Looking at the figure for $S^*$, the Pareto-optimal boundary is the line $v = -2u+5$, $1 \leq u \leq 2$. The solution with the safety point given by the values of the zero sum games is at point $(\bar{u}, \bar{v}) = (1.208, 2.583)$. With this security point, player I receives the negotiated solution $\bar{u} = 1.208$ and player II the amount $\bar{v} = 2.583$. It seems that player II has slight advantage, where $\bar{v} > \bar{u}$. Looking at the payoffs to the two players in the second row, can player I threaten to play $I_2$ to improve his negotiated solution?

We know the line where the maximum occurs, and here is $v = -2u + 5$. We may substitute into $g$ and use calculus:

$$\begin{aligned} f(u) &= g(u, -2u+5) = (u + \frac{1}{4})(-2u + \frac{16}{3}) \\ &\Rightarrow f'(u) = -4u + \frac{29}{6} = 0 \Rightarrow u = \frac{29}{24}. \end{aligned}$$

This gives the same solution as that obtained by Maple.




Security point $\left(-\frac{1}{4}, -\frac{1}{3}\right)$, Pareto boundary $v = -2u + 5$, solution (1.208, 2.583).

The feasible region is the collection of achievable payoff points with players' cooperation. In this example, it is given by the convex formed by the 4 payoff points under pure strategies.

4. *Find the strategies giving the negotiated solution.* How should the players cooperate in order to achieve the bargained solutions?

The only points in the bimatrix that are of interest are the endpoints of the Pareto-optimal boundary, namely, $(1, 3)$ and $(2, 1)$. So the cooperation must be a linear combination of the strategies yielding these payoffs, Solve

$$(\frac{29}{24}, \frac{31}{12}) = \lambda(1, 3) + (1 - \lambda)(2, 1)$$

to get $\lambda = \frac{19}{24}$. This says that (I, II) *must agree* to play the pure strategy (row 1, column 1) with $\frac{19}{24}$ of the time and another pure strategy (row 2, column 2) $\frac{5}{24}$ of the time.

This is different from playing individual mixed strategy by each player (maximizing the player's own expected payoff without cooperation). Indeed, we cannot find $X$ and $Y$ such that

$$\frac{29}{24} = E_{\rm I}(X, Y) = XAY^T \quad \text{and} \quad \frac{31}{12} = E_{\rm II}(X, Y) = XBY^T.$$

**Example - objective function other than product of $(u-u^*)(v-v^*)$**

Suppose that two persons are given \$1000, which they can split if they can agree on how to split it. If they cannot agree they each get nothing. One player is rich, so her payoff function is

$$u_1(x,y) = \frac{x}{2},\ 0 \leq x+y \leq 1000.$$

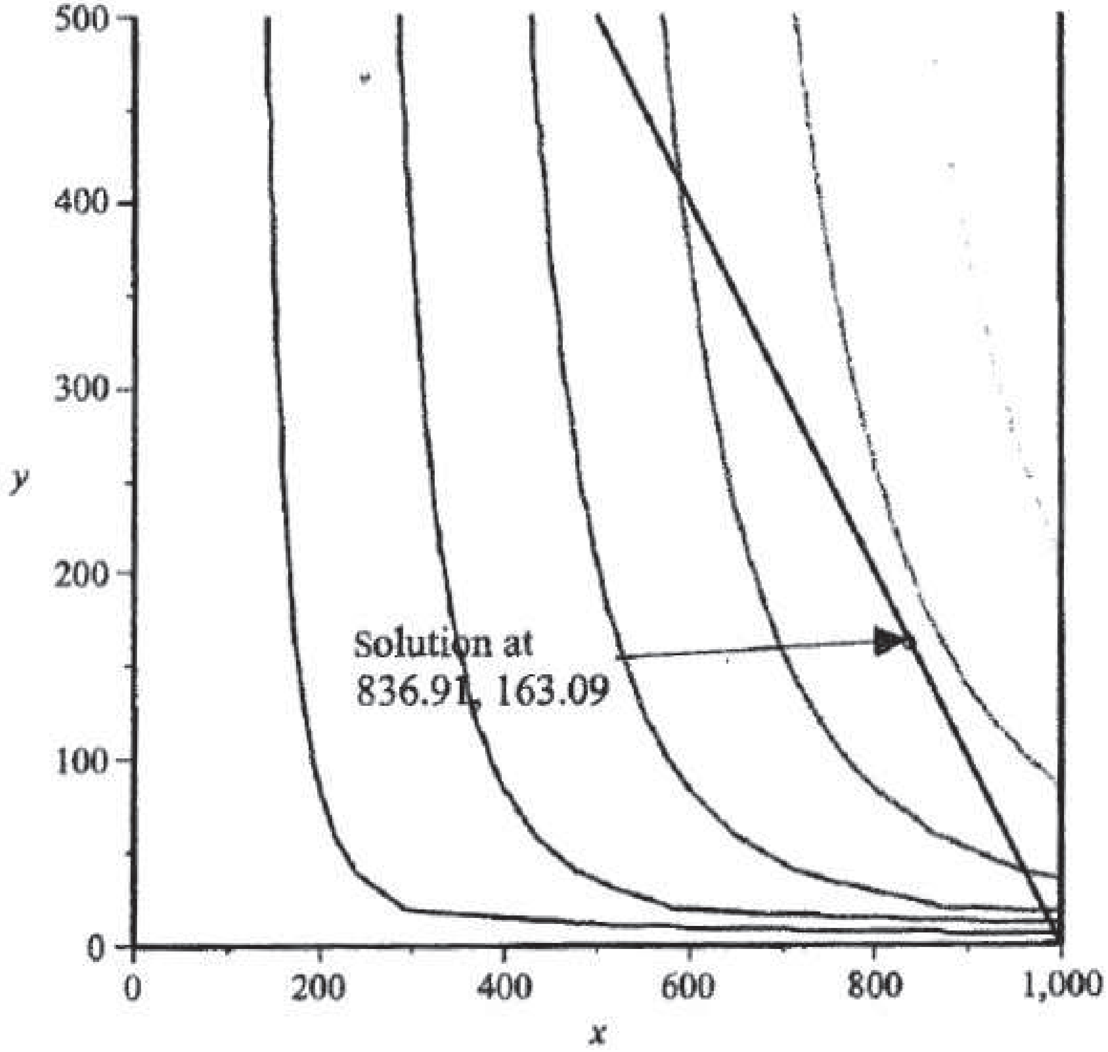
The other player is poor, so his payoff function is

$$u_2(x,y) = \ln(y+1),\ 0 \leq x+y \leq 1000,$$

because small amounts of money mean a lot but the money has less and less impact as he gets more but no more than \$1000. Note that $\ln(y+1)$ increases at high rate when $y$ is small and the rate of increase slows down when $y$ is large.

We want to find the bargained solution. The safety points are taken as $(0,0)$ because that is what they get if they cannot agree on a split. The feasible set is $S = \{(x,y) | 0 \leq x, y \leq 1000, x+y \leq 1000\}$.

*Plot of the feasible set and the contours of the objective function*




Rich and poor split $1000: solution at (836.91, 163.09).

The Nash bargaining solution is given by solving the non-linear programming problem

$$\text{Maximize } (u_1 - 0)(u_2 - 0) = (\frac{x}{2} - 0)[\ln(y + 1) - 0]$$

subject to

$$0 \leq x \leq 1000,\ 0 \leq y \leq 1000,\ x + y \leq 1000.$$

Since the solution lies on the line $x+y = 1000$, we substitute $x = 1000-y$. If we take the derivative of $f(y) = \frac{1}{2}(1000 - y)\ln(y + 1)$ and set to zero, we solve the equation

$$\frac{1000 - y}{y + 1} = \ln(y + 1),$$

which is found to be $y = 163.09$.

The maximum is achieved at $x = 836.91$ and $y = 163.09$, so the poor man gets $\$163$ while the rich woman gets $\$837$. The utility (or value of this money) to each player is $u_1 = 418.5$ to the rich guy, and $u_2 = 5.10$ to the poor guy.

The figure on P.107 shows the feasible set as well as the level curves of $f(x, y) = \frac{x}{2} \ln(1 + y) = k$, $k$ is constant. The optimal solution is obtained by increasing $k$ until the curve is tangent to the Pareto-optimal boundary. That occurs at the point $(836.91, 163.09)$.

**Threat strategies**

A player may be able to force the opposing player to play a certain strategy by threatening to use a strategy that will be very detrimental for the opponent. The security levels $(u^*, v^*)$ may be replaced by $E_{\mathrm{I}}(X_t, Y_t)$ and $E_{\mathrm{II}}(X_t, Y_t)$ if both use their respective threat strategies $X_t$ and $Y_t$.

We reformulate the Nash model as follows:

$$\begin{aligned}&\text{Maximize } g(u,v) := (u - X_t A Y_t^T)(v - X_t B Y_t^T)\\&\text{subject to } (u,v) \in S, u \geq X_t A Y_t^T, v \geq X_t B Y_t^T.\end{aligned}$$

For each player, how to find the best threat strategy to be used? The optimal bargaining solution on the Pareto-optimal boundary depends on the threat security point: $(X_t A Y_t^T, X_t B Y_t^T)$. The determination of $X_t$ and $Y_t$ becomes part of the solution procedure.

**Example**

Consider the two-person game

| | $\mathrm{II}_1$ | $\mathrm{II}_2$ |
|---|---|---|
| $\mathrm{I}_1$ | $(2,4)$ | $(-3,-10)$ |
| $\mathrm{I}_2$ | $(-8,-2)$ | $(10,1)$ |

The payoff matrices are

$$A = \begin{pmatrix} 2 & -3 \\ -8 & 10 \end{pmatrix} \qquad B^T = \begin{pmatrix} 4 & -2 \\ -10 & 1 \end{pmatrix}.$$

The corresponding security point is given by

$$\text{value}(A) = -\frac{4}{23},\ \text{value}(B^T) = -\frac{16}{17}.$$

The Pareto-optimal boundary is the line joining $(2,4)$ and $(10,1)$, and it is found to be

$$\frac{v-1}{u-10} = \frac{1-4}{10-2} = -\frac{3}{8} \text{ or } v = -\frac{3}{8}u + \frac{38}{8}.$$

With the security point $\left(-\frac{4}{23}, -\frac{16}{17}\right)$, we solve the Nash bargaining problem

$$\text{Maximize } g(u,v) = \left(u + \frac{4}{23}\right)\left(v + \frac{16}{17}\right)$$
$$\text{subject to } u \geq -\frac{4}{23}, v \geq -\frac{16}{17}, v \geq \frac{11}{13}u - \frac{97}{13},$$
$$v \leq -\frac{3}{8}u + \frac{38}{8}, v \leq \frac{6}{10}u + \frac{28}{10}.$$

We seek the bargaining solution along the Pareto-optimal line:

$$v = -\frac{3}{8}u + \frac{38}{8}.$$

We maximize $\left(u + \frac{4}{23}\right)\left(-\frac{3}{8}u + \frac{38}{8} + \frac{16}{17}\right)$. Calculus exercise gives the solution $\bar{u} = 7.501$, $\bar{v} = 1.937$. This is achieved by I and II agreeing to play the pure strategies $(\mathrm{I}_1, \mathrm{II}_1)$ 31.2% of the time and pure $(\mathrm{I}_2, \mathrm{II}_2)$ 68.8% of the time.

Player II may be in a stronger position than Player I. Why?

Player II can always threaten Player I with playing $II_1$. Under this threat:

- Suppose Player I continues to play $I_2$, his payoff becomes $-8$, which is much lower than $-2$; here both players lose.

- When Player I plays $I_1$, Player II gets $4 > 1.937$ while Player I gets $2 < 7.501$.

Is the threat posed by Player II credible?

Suppose the threat strategies are $X_t = (0,1)$ (player I plays $\text{I}_2$) and $Y_t = (1,0)$ (player II plays $\text{II}_1$). The new safety point

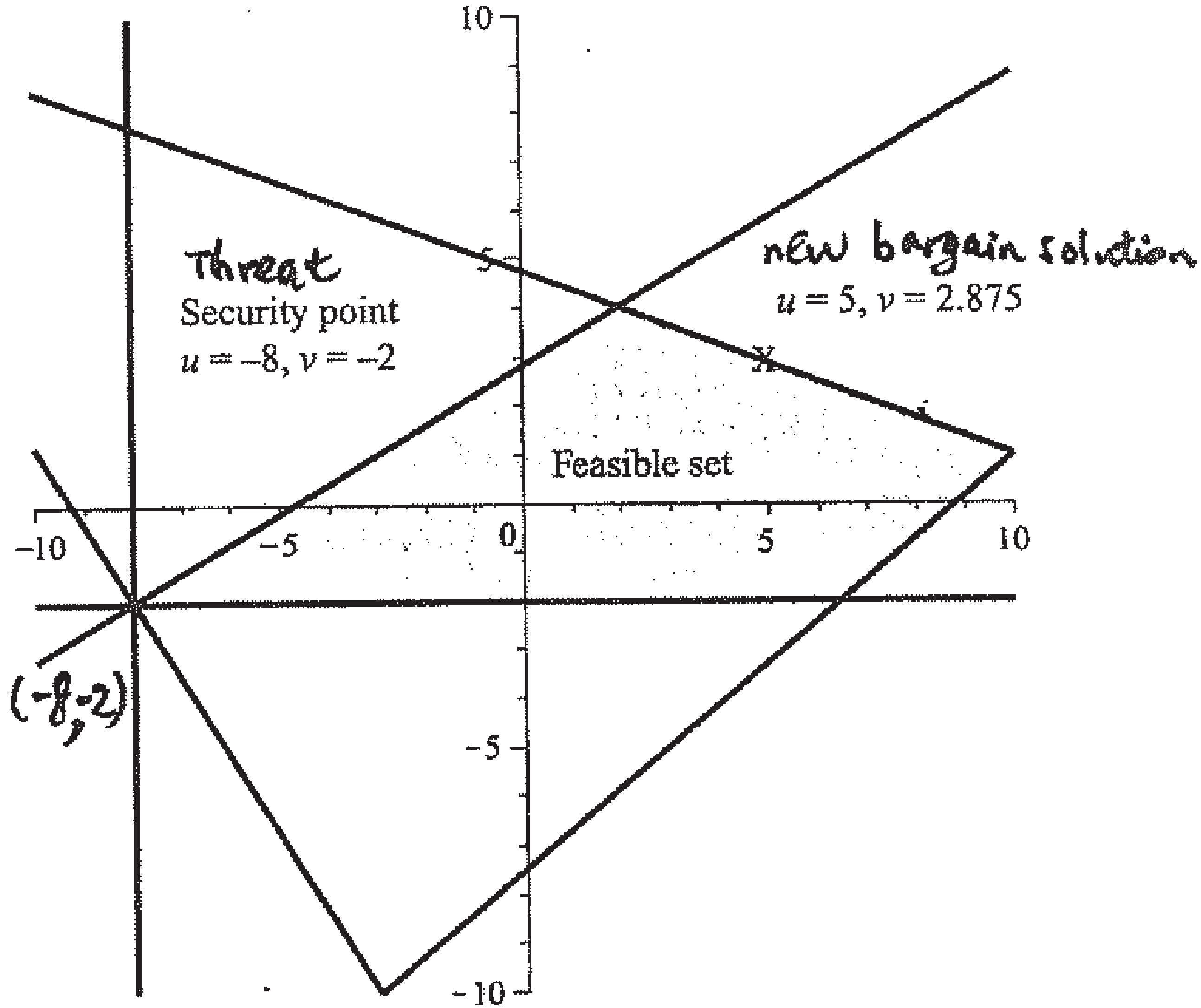
$$u^* = X_t^T A Y_t = -8, \quad v^* = X_t B Y_t^T = -2.$$

Changing the security point increases the size of the feasible set and changes the objective function to $g(u,v) = (u+8)(v+2)$.

The solution of the threat problem is

$$\bar{u} = 5 < 7.501 \text{ and } \bar{v} = 2.875 > 1.937.$$

Player II gets more with the threat, which is credible.




Feasible set with security point (−8, −2) using threat strategies.

## Lemma

If $(\overline{u}, \overline{v})$ is the solution of the Nash bargaining problem with any security point $(u_0, v_0)$ and the Pareto-optimal boundary through $(\overline{u}, \overline{v})$ is a straight line with slope $m_p$. Provided that $(\overline{u}, \overline{v})$ is not at an end point of a line segment along the Pareto-optimal boundary so that $(\overline{u}, \overline{v})$ is an interior maximum point, we then have

$$\frac{\overline{v} - v_0}{\overline{u} - u_0} = -m_p.$$

That is, the slope of the line through $(u_0, v_0)$ and $(\overline{u}, \overline{v})$ must be the negative of the slope of the Pareto-optimal boundary at the point $(\overline{u}, \overline{v})$.

*Remark*

The maximum of the objective function in the bargaining formulation may end at the intersection of two line segments of the Pareto-optimal boundary. Under this degenerate case, the above property of "negative slope" fails.

*Why the line joining the threat point and the optimal bargain solution has slope that is negative to the line segment of the Pareto-optimal boundary?*

Along the Pareto-optimal boundary with slope $m_p$, if we increase one unit for Player I, we need to decrease $-m_p$ units of Player II, in the ratio of $1 : -m_p$. The same ratio is retained whey they negotiate the bargain solution. When the ratio of increases of payoffs of I and II from the security point $(u_0, v_0)$ observes the same ratio $1 : -m_p$, the line thus has slope $-m_p$.

**Proof**

Suppose $(\overline{u}, \overline{v})$ is an interior maximum point (not an end point of the Pareto-optimal boundary), by Nash's theorem, $(\overline{u}, \overline{v})$ maximizes $f(u) = (u - u_0)(m_p u + b - v_0)$. Taking the derivative and setting to zero, the first order condition gives

$$b - v_0 + m_p u + m_p(u - u_0) = 0$$

giving

$$u = \frac{b - v_0 - m_p u_0}{-2m_p}.$$

Therefore, for an arbitrary security point $(u_0, v_0)$, the maximizing point is given by

$$\overline{u} = \frac{-m_p u_0 + b - v_0}{-2m_p},$$
$$\overline{v} = m_p \overline{u} + b = \frac{b + m_p u_0 + v_0}{2}.$$

We calculate the slope of the line through $(u_0, v_0)$ and $(\overline{u}, \overline{v})$:

$$\frac{\overline{v} - v_0}{\overline{u} - u_0} = \frac{\frac{b+m_p u_0+v_0}{2} - v_0}{\frac{-m_p u_0+b-v_0}{-2m_p} - u_0} = -m_p \frac{b + m_p u_0 - v_0}{m_p u_0 + b - v_0} = -m_p.$$

Note that $b$ and $m_p < 0$ are fixed while $u_0$ and $v_0$ are the control variables.

- Since $\overline{u} = \frac{-m_p u_0 - v_0 + b}{-2m_p}$, so Player I maximizes $\overline{u}$ via maximization of $(-m_p u_0 - v_0)$;

- Since $\overline{v} = \frac{b+m_p u_0 + v_0}{2}$, so Player II maximizes $\overline{v}$ via minimization of $(-m_p u_0 - v_0)$.

This is a game on the choices of $u_0$ and $v_0$, where $u_0 = X_t A Y_t^T$ and $v_0 = X_t B Y_t^T$. We find the optimal strategies of the zero sum game with matrix $-m_p A - B$ since

$$-m_p u_0 - v_0 = -m_p(X_t A Y_t^T) - X_t B Y_t^T = X_t(-m_p A - B)Y_t^T.$$

## Summary approach for bargaining with threat strategies

1. Identify the Pareto-optimal boundary of the feasible payoff set and find the slope of that line, call it $m_p$. This slope should be negative. The equation of the Pareto-optimal boundary is $v = m_p u + b$, where $b$ is the $v$-intercept.

2. Construct the new matrix for a zero sum game

$$-m_p u^t - v^t = -m_p(X_t A Y_t^T) - X_t B Y_t^T = X_t(-m_p A - B)Y_t^T$$

   with matrix $-m_p A - B$.

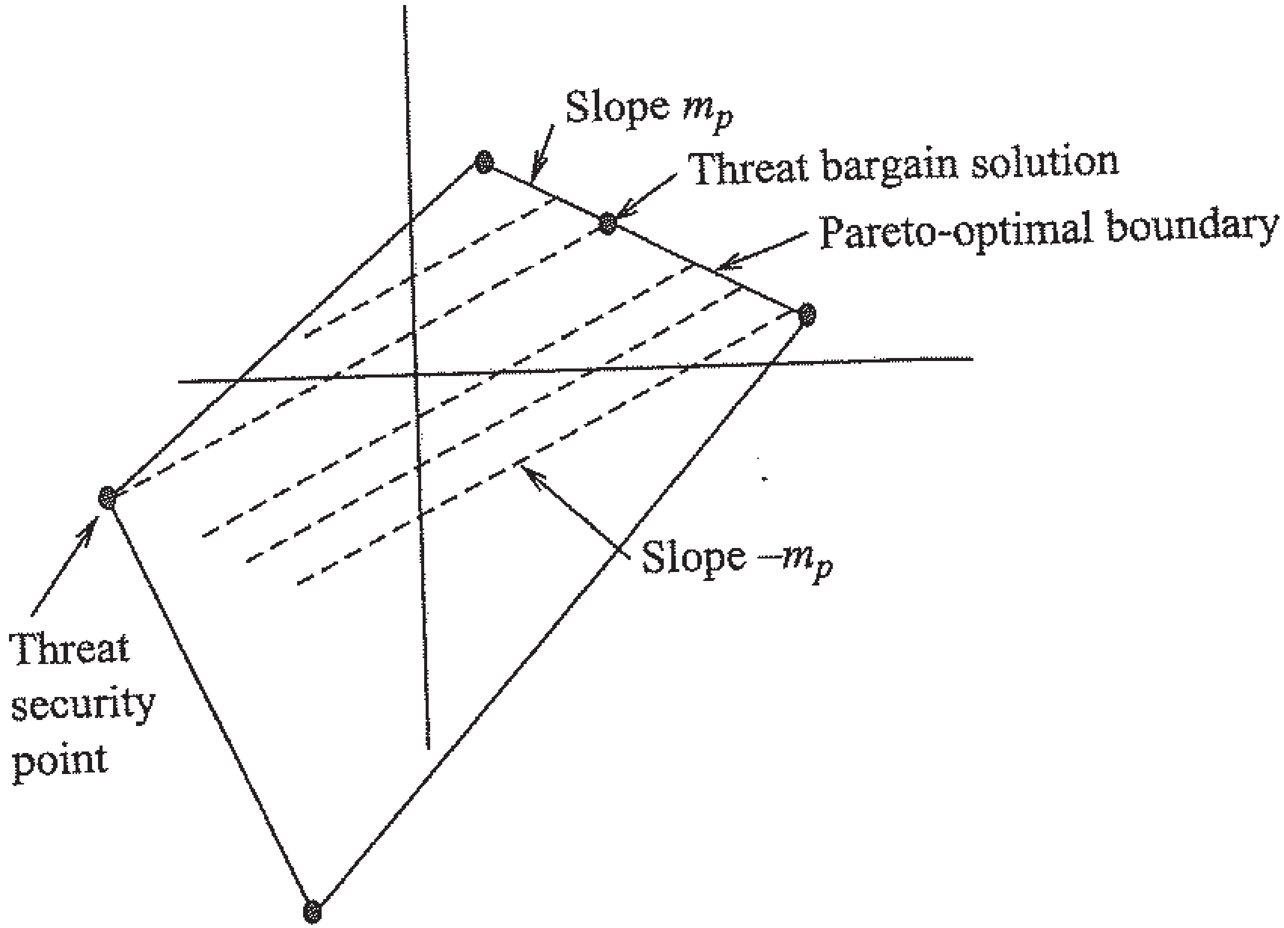
3. Find the optimal strategies $X_t$ and $Y_t$ for the above zero sum game and compute $u^t = X_t A Y_t^T$ and $v^t = X_t B Y_t^T$. This $(u^t, v^t)$ is the threat security point to be used to solve the bargaining problem.

4. Once we know the threat security point $(u^t, v^t)$, we may use the following formulas to find $(\overline{u}, \overline{v})$:

$$\overline{u} = \frac{m_p u^t + v^t - b}{2m_p}, \quad \overline{v} = \frac{1}{2}(m_p u^t + v^t + b).$$

   The above formula for $(\overline{u}, \overline{v})$ is valid provided that $(\overline{u}, \overline{v})$ is an interior maximum with the line segment with slope $m_p$ along the Pareto-optimal boundary. Be aware that the Pareto-optimal boundary may consist of several line segments.

## Different choices of security points under various threat strategies




Lines through possible threat security points.

The threat security point is $(u_0, v_0)$, where $u_0 = X_t A Y_t^T$ and $v_0 = X_t B Y_t^T$.

In the earlier example, the Pareto-optimal line is $v = -\frac{3}{8}u + \frac{38}{8}$, so $m_p = -\frac{3}{8}$, $b = \frac{38}{8}$. The matrix for the zero-sum game associated with the threat strategies is

$$\frac{3}{8}A - B = \begin{pmatrix} -\frac{26}{8} & \frac{71}{8} \\ -1 & \frac{22}{8} \end{pmatrix}.$$

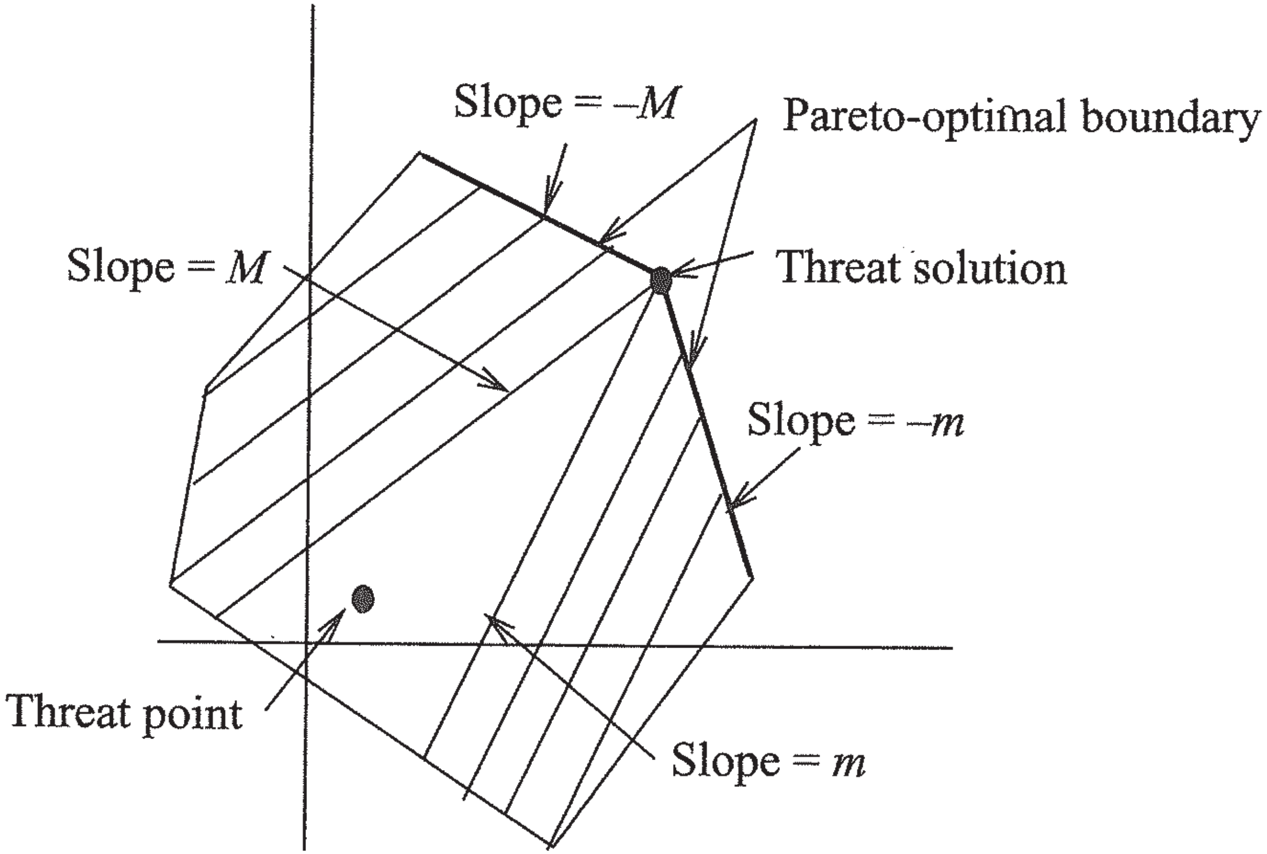
We find $value(\frac{3}{8}A - B) = -1$ since there is a saddle point at the second row and first column, the optimal threat strategies are $X_t = (0, 1)$, $Y_t = (1, 0)$. Then $u^t = X_t A Y_t^T = -8$, and $v^t = X_t B Y_t^T = -2$.

The above calculations verify that $(-8, -2)$ is indeed the *optimal threat* security point. Once we know that, we can use the formulas above to get

$$\begin{aligned} \overline{u} &= \frac{-\frac{3}{8}(-8) + (-2) - \frac{38}{8}}{2(-\frac{3}{8})} = 5, \\ \overline{v} &= \frac{1}{2}\left[-\frac{3}{8}(-8) + (-2) + \frac{38}{8}\right] = 2.875. \end{aligned}$$

The line joining $(u_t, v_t) = (-8, -2)$ and $(\overline{u}, \overline{v}) = (5, 2.875)$ has slope$= \frac{3}{8}$.

## Multiple line segments in the Pareto-optimal boundary




Threat solution for vertex

When the threat point is within the cone, the threat solution lies at the intersection of the two line segments of the Pareto-optimal boundary.

**Bargaining solution for threats when the threat point is in the cone**

Consider the cooperative game with bimatrix

| | $II_1$ | $II_2$ |
|---|---|---|
| $I_1$ | $(-1,-1)$ | (1,1) |
| $I_2$ | $(2,-2)$ | (-2,2) |

The individual matrices are as follows:

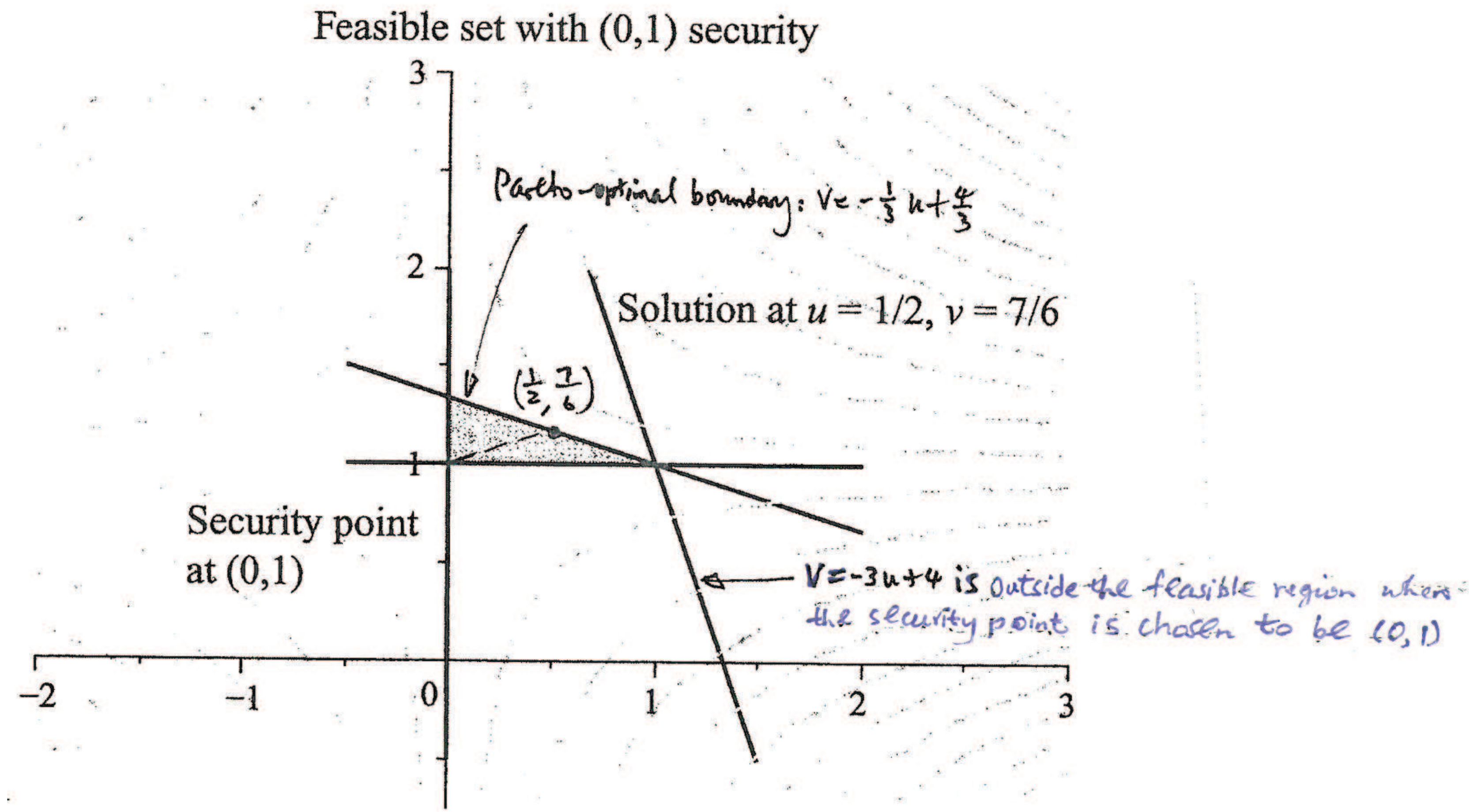
$$A = \begin{pmatrix} -1 & 1 \\ 2 & -2 \end{pmatrix}, \qquad B = \begin{pmatrix} -1 & 1 \\ -2 & 2 \end{pmatrix}.$$

It is easy to calculate that value$(A) = 0$, value$(B^T) = 1$ and so the status quo security point for this game is at $(u^*, v^*) = (0, 1)$. The problem we then need to solve is

Maximize $u(v-1)$ subject to $(u, v) \in S^*$, where

$$S^* = \{(u,v) | v \leq \left(-\frac{1}{3}\right) u + \frac{4}{3}, v \leq -3u + 4, u \geq 0, v \geq 1\}.$$

Apparently, there are two line segments in the convex hull containing the 4 pure strategy points that constitute the Pareto-optimal boundary. However, the line segment, $v = -3u+4$, is outside the feasible region when the security point is chosen to be $(0, 1)$. The solution of the bargaining problem is at the unique point $(\overline{u}, \overline{v}) = (\frac{1}{2}, \frac{7}{6})$.



The solution of threat strategies is complicated by the fact that the Pareto-optimal boundary may consist of two line segments: (i) $m_p = -\frac{1}{3}$, $b = \frac{4}{3}$ and (ii) $m_p = -3$, $b = 4$.

When one seeks the bargaining solution along $v = -3u + 4$, we consider

$$3A - B = \begin{pmatrix} -2 & 2 \\ 8 & -8 \end{pmatrix}$$

with value$(3A - B) = 0$. The optimal threat strategies are

$$X_t = \left(\frac{1}{2}, \frac{1}{2}\right) = Y_t;$$

giving

$$u^t = X_t A Y_t^T = 0 \quad \text{and} \quad v^t = X_t B Y_t^T = 0.$$

The new security point is $(0, 0)$.

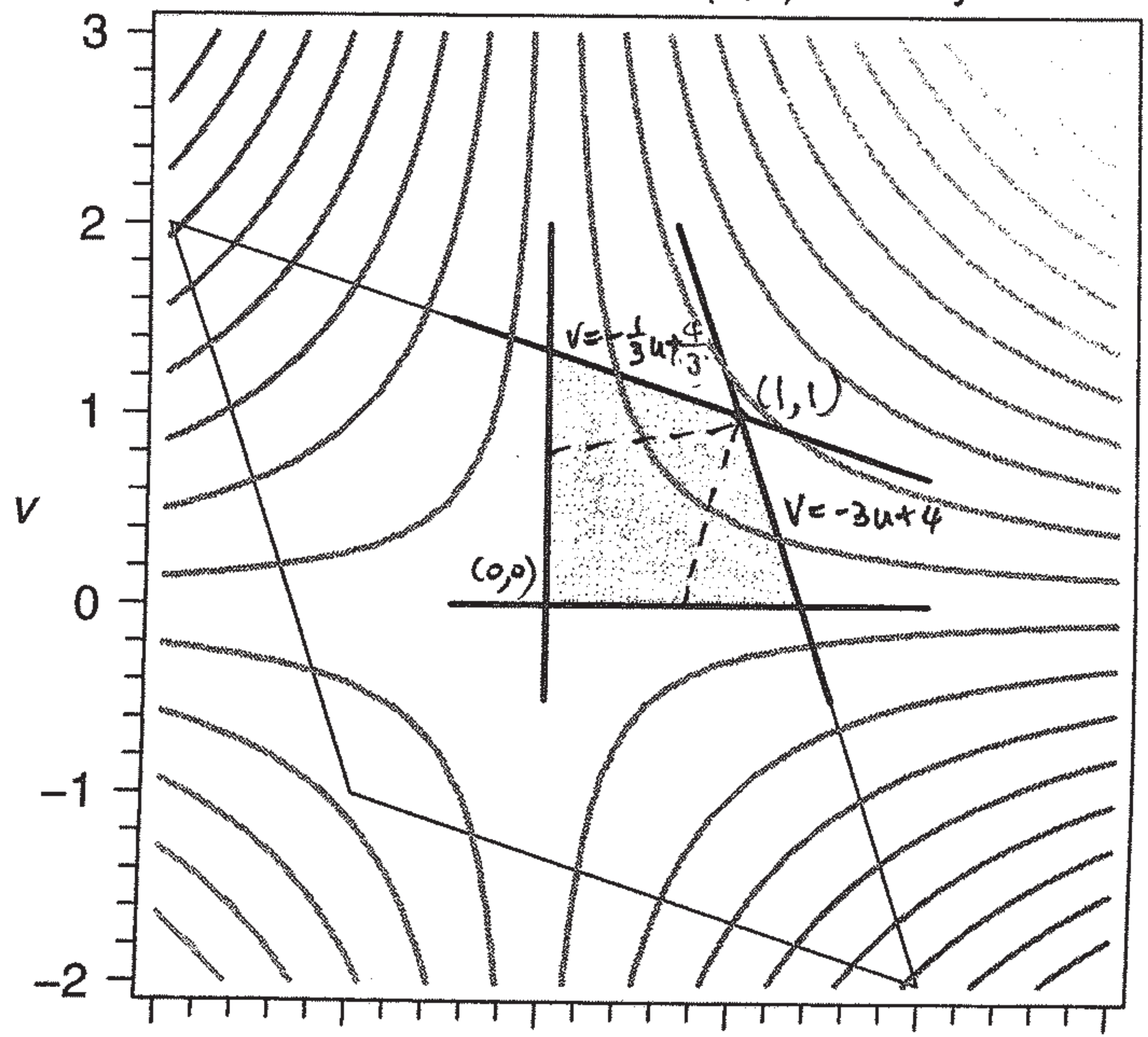
We seek the bargaining solution along $v = -3u + 4$, $1 \leq u \leq 2$, that maximizes the objective function $uv$. Along $v = -3u + 4$, we consider

$$uv = u(-3u + 4) = -3u^2 + 4u$$

whose local maximum occurs at $u = 2/3$, which is outside $[1, 2]$. The maximum of $uv$ along $v = -3u + 4$ is seen to occur at the end point $u = 1$.

The bargaining solution lies at the intersection point of the two line segments of the Pareto-optimal boundary. The security point $(0, 0)$ lies inside the cone bounded by the two dotted line through the intersection point $(1, 1)$.

Feasible set with (0,0) security



For the first case: $m_p = -\frac{1}{3}$, $b = \frac{4}{3}$, the associated matrix for the zero-sum game for the threat strategies is given by

$$\frac{1}{3}A - B = \begin{pmatrix} \dfrac{2}{3} & -\dfrac{2}{3} \\ \\ \dfrac{8}{3} & -\dfrac{8}{3} \end{pmatrix}.$$

Since $-\frac{2}{3}$ happens to be row min and column max, so value$(\dfrac{1}{3}A - B) = -\dfrac{2}{3}$.

The optimal threat strategies: $X_t = (1, 0)$, $Y_t = (0, 1)$.

The security threat points are as follows:

$$u^t = X_t A Y_t^T = 1 \quad \text{and} \quad v^t = X_t B Y_t^T = 1.$$

This security threat point is exactly at a vertex of the feasible set.

Maximize $(u-1)(v-1)$ subject to $(u,v) \in S^t$.

$$S^* = \{(u,v) | v \leq \left(-\frac{1}{3}\right) u + \frac{4}{3}, v \leq -3u + 4, u \geq 1, v \geq 1\}.$$

But this set has exactly one point and it is $(1,1)$, so we immediately get the solution $(\overline{u} = 1, \overline{v} = 1)$. This is the same result as in the earlier case.

Under the threat strategies, Player I achieves $u = 1$ compared to $u = 1/2$ under the status quo security point.

## Example

Find the Nash bargaining solution and the threat solution to the game with bimatrix

$$\begin{bmatrix} (-3,-1) & (0,5) & (1,\frac{19}{4}) \\ (2,\frac{7}{2}) & (\frac{5}{2},\frac{3}{2}) & (-1,-3) \end{bmatrix}.$$

## Solution

The matrices are as follows:

$$A = \begin{pmatrix} -3 & 0 & 1 \\ 2 & \frac{5}{2} & -1 \end{pmatrix} \quad B = \begin{pmatrix} -1 & 5 & \frac{19}{4} \\ \frac{7}{2} & \frac{3}{2} & -3 \end{pmatrix}.$$

We have value$(A) = -\frac{1}{7}$, value$(B^T) = \frac{19}{8}$. That is our safety point. The Pareto-optimal boundary has three line segments:

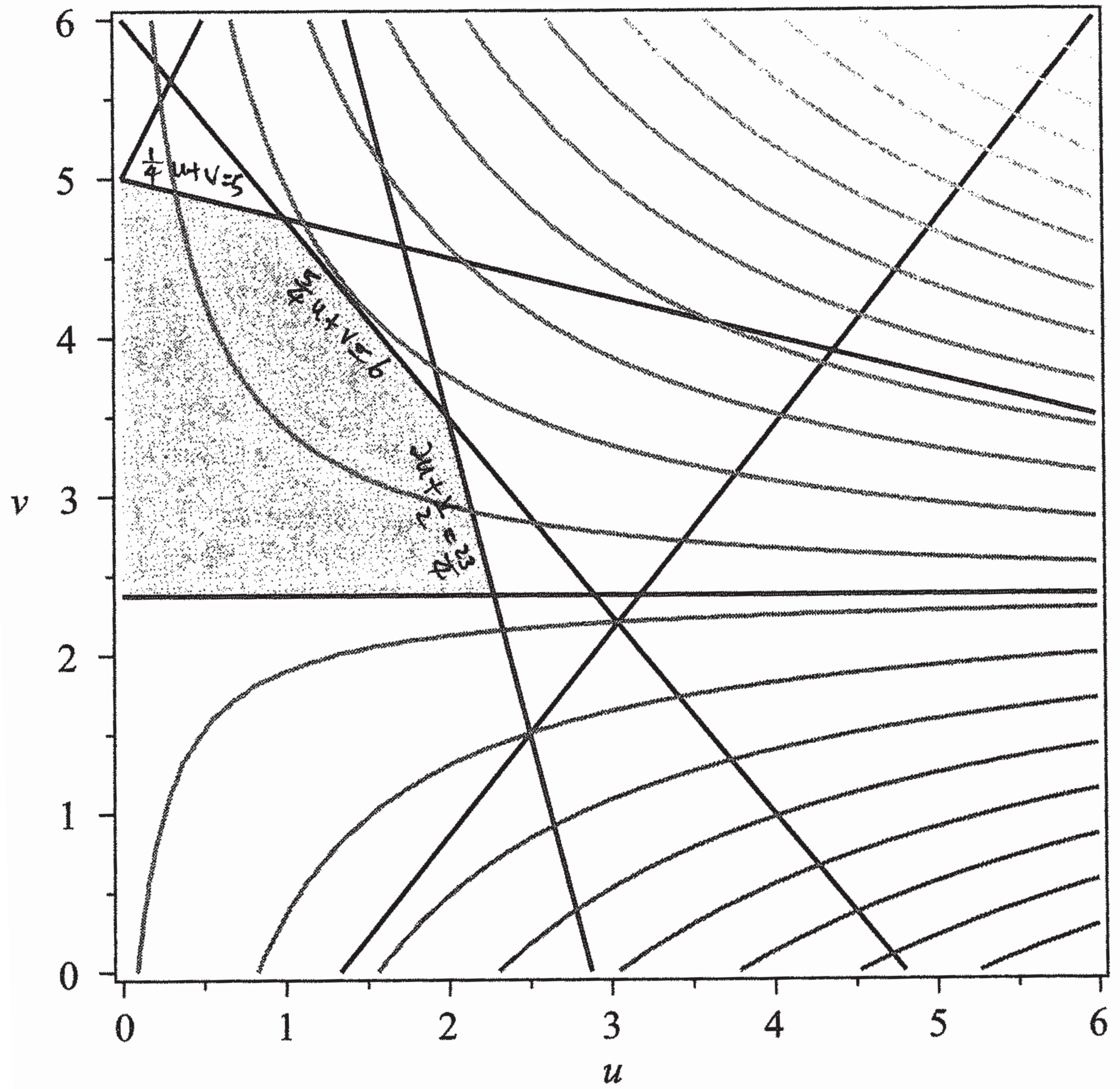
$$\begin{cases} \frac{1}{4}u + v = 5, & \text{if } 0 \le u \le 1; \\ \frac{5}{4}u + v = 6, & \text{if } 1 \le u \le 2; \\ 2u + \frac{1}{2}v = \frac{23}{4}, & \text{if } 2 \le u \le \frac{5}{2}. \end{cases}$$

The Nash bargaining problem is

$$\begin{aligned}&\text{Maximize } \left(u+\frac{1}{7}\right)\left(v-\frac{19}{8}\right)\\&\text{subject to } (u,v)\in S.\end{aligned}$$

The part of the Pareto-optimal boundary for this problem is the line segment $\frac{5}{4}u+v=6$, $1\le u\le 2$. Using calculus, we find

$$\overline{u}=\frac{193}{140},\quad \overline{v}=\frac{479}{112}.$$



1/4 u+v=5
5/4 u+v=6
3u+v/2=23/4
v
u
0
1
2
3
4
5
6

Next, we consider the threat solution. We have to find the threat strategies for all three line segments of the Pareto-optimal boundary.

1. $v = -\frac{1}{4}u + 5$, $0 \le u \le 1$. Then $m_p = -\frac{1}{4}$, $b = 5$, and

$$\text{value}\left(\frac{1}{4}A - B\right) = -\frac{487}{236}, X_t = \left(\frac{17}{59}, \frac{42}{59}\right), Y_t = \left(\frac{33}{59}, \frac{26}{59}, 0\right),$$

and

$$u^t = X_t A Y_t^T = 1.097, v^t = X^t B Y_t^T = 2.34, \Rightarrow \overline{u} = 5.87, \overline{v} = 3.53.$$

Since $5.87 \notin [0, 1]$, this is not an admissible threat solution.

2. $v = -\frac{5}{4}u + 6$, $1 \le u \le 2$. Then $m_p = -\frac{5}{4}$, $b = 6$, and

$$\text{value}\left(\frac{5}{4}A - B\right) = -1, \quad X_t = (0,1), \quad Y_t = (1,0,0).$$

The threat point is given by

$$u^t = X_t A Y_t^T = 2, v^t = X^t B Y_t^T = \frac{7}{2}.$$

Along $v = -\frac{5}{4}u + 6$, the objective function $(u-2)(v-\frac{7}{2})$ becomes

$$(u-2)\left(-\frac{5}{4}u + 6 - \frac{7}{2}\right).$$

To find the maximum point, the first order condition gives

$$\left(-\frac{5}{4}u + \frac{5}{2}\right) - \frac{5}{4}(u-2) = 0 \Rightarrow u = 2.$$

This happens to be at the end of the range $[1,2]$, we then obtain

$$\overline{u} = 2 \quad \text{and} \quad \overline{v} = \frac{7}{2}.$$

3. $v = -4u + \frac{23}{2}$, $2 \leq u \leq \frac{5}{2}$. In this case $m_p = -4$, $b = \frac{23}{2}$, and

$$\text{value}(4A - B) = -\frac{115}{126}, \quad X_t = \left(\frac{22}{63}, \frac{41}{63}\right), \quad Y_t = \left(\frac{1}{63}, 0, \frac{62}{63}\right).$$

The safety point is then

$$u^t = X_t A Y_t^T = -0.294, v^t = -0.258 \Rightarrow \overline{u} = 1.32, \overline{v} = 6.206.$$

Since $1.32 \notin [2, \frac{5}{2}]$, this too is not the threat solution.

We conclude that the threat solution is $\overline{u} = 2$, $\overline{v} = \frac{7}{2}$ and player 1 threatens to always play the second row; player 2 threatens to use the first column.

**Sequential Bargaining**

There are offers and counter-offers that can go several rounds until an agreement is reached or negotiations break down.

*Example*

For any item, there is a reserve price. The seller offers the item for sale at the *ask price*. Spread = asked − reserve, and the spread is the negotiating range.

Let $x$ be the fraction of the spread going to the buyer,
$1 - x$ be the fraction of the spread going to the seller; $0 \leq x \leq 1$.

The final price will be: reserve + $(1 - x)$ spread.

## One-stage bargaining: Ultimatum game

In a one-shot bargaining, the payoffs for the players are

$$u_1(x, 1-x) = x \text{ and } u_2(x, 1-x) = 1-x.$$

Let $(d_1, d_2)$ be a safety point that determines the worth to each player if no deal is made. The Nash bargaining problem consists of maximizing

$$g(x, 1-x) = (x-d_1)(1-x-d_2) \text{ over } 0 \leq x \leq 1.$$

The solution $x^*$ is

$$x^* = \frac{1+d_1-d_2}{2}.$$

In order to have $x^* > d_1$, we require $\frac{1+d_1-d_2}{2} > d_1 \Leftrightarrow d_1 + d_2 < 1$.

If $d_1 = d_2$, then the optimal split is $\frac{1}{2}$, so the transaction takes place at the midpoint of the spread.

We extend one-stage bargaining to multi-stage bargaining. Delay imposes some loss on the player. In each round of bargaining, $\delta$ of the pie disappears, where $\delta$ is called the discount factor. If this round fails to reach an acceptance of the bargaining bid offered by player 1, then the next round continues with the bargaining bid offered by player 2. This is assumed to continue with either finite number of rounds or perpetually.

With discount factors $\delta_1$ and $\delta_2$ for player 1 and player 2, respectively, the payoff functions for the players are

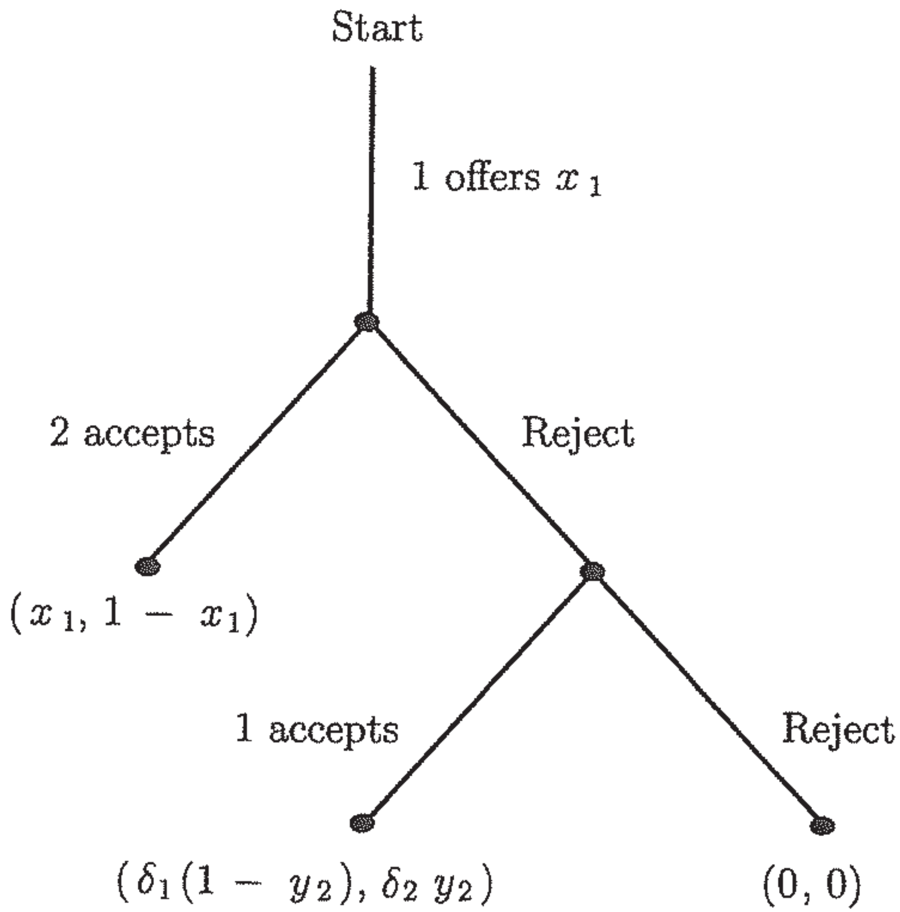
$$u_1(x_n, 1 - x_n) = \delta_1^{n-1} x_n \text{ and } u_2(x_n, 1 - x_n) = \delta_2^{n-1}(1 - x_n),\ n = 1, 2.$$

## Two-stage bargaining

By the backward induction procedure, we start at the end of the second period, where player 1 makes the final decision. If $\delta_1(1-y_2) > 0$, that is, if $y_2 \neq 1$, then player 1 receives the larger payoff $\delta_1(1-y_2)$ when compared with zero payoff if rejection is chosen. Naturally, player 2 will choose $y_2$ extremely close to 1 since player 2 knows that *any* $y_2 < 1$ will give player 1 a positive payoff. Thus, at the beginning of the last stage where player 2 offers $y_2 \approx 1$ but less than 1, player 2 gets $\approx \delta_2$ and player 1 gets $\approx 0$ (but positive).

At the start of the first stage, player 2 will compare $1 - x_1$ with $\delta_2 y_2$. If $1 - x_1 > \delta_2 y_2 \approx \delta_2$, player 2 will accept player 1's first offer. Player 1's offer needs to satisfy $x_1 \leq 1 - \delta_2$, but as large as possible. That means player 1 will play $x_1 = 1 - \delta_2$, and that should be the offer 1 makes at the start of the game.

Player 1 (player 2) proposes the offer in the first (second) stage. The strategic space of the proposer is the split that lies between 0 and 1. The strategy space of the counterparty is {accept, reject}.




Two-stage bargaining. Here, $\delta_1$ is the discount factor for player 1 in the second stage.

*Summary*

Player 1 begins the game by offering $x_1 = 1-\delta_2$. Then, if player 2 accepts the offer, the payoff to player 1 is $1 - \delta_2$ and the payoff to player 2 is $\delta_2$. If player 2 rejects and counter offers $y_2 \approx 1$, $y_2 < 1$, then player 1 will accept the offer, receiving $\delta_1(1 - y_2) > 0$. If player 2 counters with $y_2 = 1$, then player 1 is indifferent between acceptance or rejection (and so will reject).

- At any fixed node, a player will choose to play according to a Nash equilibrium no matter at which stage of the game (there is nothing that can be done about the past). A good Nash equilibrium for a player is one which is a Nash equilibrium for every subgame. This is called the subgame perfect equilibrium.

**Perpetual game**

First, we need to make a change in the three-stage game if the final offer is rejected. Instead of $(0, 0)$ going to each player, let $s$ denote the total spread available and assume that when player 2 rejects player 1's counter at stage 3, the payoff to each player is $(\delta^2 s, \delta^2(1-s))$. Here, $s$ is to be determined later.

Player 2 in stage 3 will accept the counter of player 1, $x_3$, if $\delta^2(1-x_3) \geq \delta^2(1-s)$, which is true if $x_3 \leq s$. Since player 1 gets $\delta^2 x_3$ if the offer is accepted, player 1 makes the offer $x_3$ as large as possible and so $x_3^* = s$.

Working back to the second stage, player 2's offer of $y_2$ will be accepted by player 1 if $\delta(1 - y_2) \geq \delta^2 s$, that is, if $y_2 \leq 1 - \delta s$. Since player 2 receives $\delta y_2$, player 2 wants $y_2$ to be as large as possible and hence offers $y_2 = 1 - \delta s$ in the second stage.

In the first stage, player 2 will accept player 1's offer of $x_1$ if $1 - x_1 > \delta y_2 = \delta(1 - \delta s)$. Simplifying, this requires $x_1 < 1 - \delta(1 - \delta s)$, or, making $x_1$ as large as possible, player 1 should offer $x_1 = 1 - \delta + \delta^2 s$ in the first stage.

We may summarize the subgame perfect equilibrium (every subgame is a Nash equilibrium) by writing this down in reverse order

| **Player 1** | **Player 2** |
| --- | --- |
| 1. Offer player 2 $x_1 = 1 - \delta + \delta^2 s$. | 1. Accept if $x_1 = 1 - \delta + \delta^2 s$. |
| 2. If player 2 offers $y_2 = 1 - \delta s$, accept. | 2. Offer $y_2 = 1 - \delta s$. |
| 3. Offer $x_3 = s$. | 3. Accept if $x_3 = s$. |

**What should $s$ be?**

If the bargaining goes on for three-stages and ends, $s$ is 0 if no agreement can be achieved. The way to choose $s$ is to observe that when we are in a three-stage game, at the third stage we are back to the original conditions at the start of the game for both players except for the fact that time has elapsed.

In other words, player 1 will now begin bargaining just as she did in the beginning of the game. This implies that for the original three-stage game, $s$ should be chosen so that the offer at the first stage $x_1$ is the same as what she should offer at stage 3. This results in

$$1 - \delta + \delta^2 s = s \Rightarrow s = \frac{1}{1 + \delta}.$$

*Remark*

In sequential bargaining problems, we may view the discount factor not as the time value of money but as the probability the bargaining problem will end with a rejection of the latest offer.

This would make $\delta$ very subjective, which is often the case with bargaining. In particular, when you are negotiating to buy a car or house, you have to assess the chances that an offer will be rejected, and with no possibility of revisiting the offer.

If player 1's payoff is an increasing function of $\delta_1$, it may be beneficial to signal "patience" to the counterparty that the discount factor $\delta_1$ is very close to 1.

**Example - sale of a car**

The sellers know that the car is worth at least $\$2000$ and will not take a penny under that. She advertises it for sale at $\$2800$.

The buyer looks like he will give an offer but may not negotiate if the offer is turned down. The buyer thinks the same of the seller but he knows the car is worth at least $\$2000$.

Let's take $\delta = 0.5$ to account for the uncertainty in the continuation of bargaining. Assuming indefinite stages of bargaining, the buyer should offer the sale price

$$x^* = 2000 + \frac{1}{1+\delta}800 = 2000 + \frac{2}{3}800 = 2533.33.$$

The seller should accept this offer. There should be no delay for further stages of bargaining.

**Proposition**

There is a unique subgame perfect equilibrium in the perpetual sequential bargaining game described as follows. We assume that player 1 is the player who first makes an offer.

Whenever player 1 proposes, she suggests a split $(x, 1-x)$ with $x = \frac{1-\delta_2}{1-\delta_1\delta_2}$. Player 2 accepts any division giving her at least $1-x$.

Whenever player 2 proposes, she suggests a split $(y, 1-y)$ with $y = \frac{\delta_1(1-\delta_2)}{1-\delta_1\delta_2}$. Player 1 accepts any division giving her at least $y$.

The bargaining ends immediately with a split $(x, 1-x)$.

*Proof*

It suffices to check that no player can make a profitable deviation from her equilibrium strategy in one single period.

Consider a period when player 1 offers. According to the equilibrium strategies of $x = \frac{1-\delta_2}{1-\delta_1\delta_2}$ and $y = \frac{\delta_1(1-\delta_2)}{1-\delta_1\delta_2} = \delta_1 x$, it suffices to show that player 1 has no profitable deviation. She cannot make an acceptable offer that will get her more than $x$. And if makes an offer that will be rejected, she will get $y = \delta_1 x$ the next period, or $\delta_1^2 x$ in present terms, which is worse than $x$.

Player 2 also has no profitable deviation. If she accepts, she gets $1 - x$. If she rejects, she will get $1 - y$ the next period, or in present terms $\delta_2(1 - \delta_1 x)$. Given $x = \frac{1-\delta_2}{1-\delta_1\delta_2}$, it is easy to check that $1 - x = \delta_2 - \delta_1\delta_2 x$.

A similar argument applies to periods when player 2 offers.

*Uniqueness of the equilibrium*

We now show that the equilibrium is unique. To do this, let $\underline{v}_1$ and $\bar{v}_1$ denote the lowest and highest payoffs that player 1 could conceivably get in any subgame perfect equilibrium starting at a date where he gets to make an offer.

To begin, consider a date where player 2 makes an offer. Player 1 will certainly accept any offer greater than $\delta_1 \bar{v}_1$ and reject any offer less than $\delta_1 \underline{v}_1$. Thus, starting from a period in which she offers, player 2 can secure *at least* $1 - \delta_1 \bar{v}_1$ by proposing a split $(\delta_1 \bar{v}_1, 1 - \delta_1 \bar{v}_1)$. On the other hand, she can secure *at most* $1 - \delta_1 \underline{v}_1$.

Now, consider a period when player 1 makes an offer. To get player 2 to accept, he must offer her *at least* $\delta_2(1-\delta_1\bar{v}_1)$ to get an agreement, so

$$\bar{v}_1 \leq 1-\delta_2(1-\delta_1\bar{v}_1).$$

At the same time, player 2 will certainly accept if offered more than $\delta_2(1-\delta_1\underline{v}_1)$, so

$$\underline{v}_1 \geq 1-\delta_2(1-\delta_1\underline{v}_1).$$

Combining the inequalities, we obtain

$$\underline{v}_1 \geq \frac{1-\delta_2}{1-\delta_1\delta_2} \geq \bar{v}_1.$$

Since $\bar{v}_1 \geq \underline{v}_1$ by definition, we know that in any *subgame perfect equilibrium*, player 1 receives $v_1 = \frac{1-\delta_2}{1-\delta_1\delta_2}$. Making the same argument for player 2 completes the proof.

**Properties of the solution**

1. Note that player 1's payoff $\frac{1-\delta_2}{1-\delta_1\delta_2}$, is increasing in $\delta_1$ and decreasing in $\delta_2$. Therefore, signaling "patience" helps player 1.

2. The first player to make an offer has an advantage. With identical discount factors $\delta$, the model predicts a split
$$\left(\frac{1}{1+\delta}, \frac{\delta}{1+\delta}\right)$$
which is better for player 1. However, as $\delta \to 1$, this first mover advantage goes away. The limiting split is $(\frac{1}{2}, \frac{1}{2})$.

3. There is no delay. Player 2 accepts player 1's first offer.