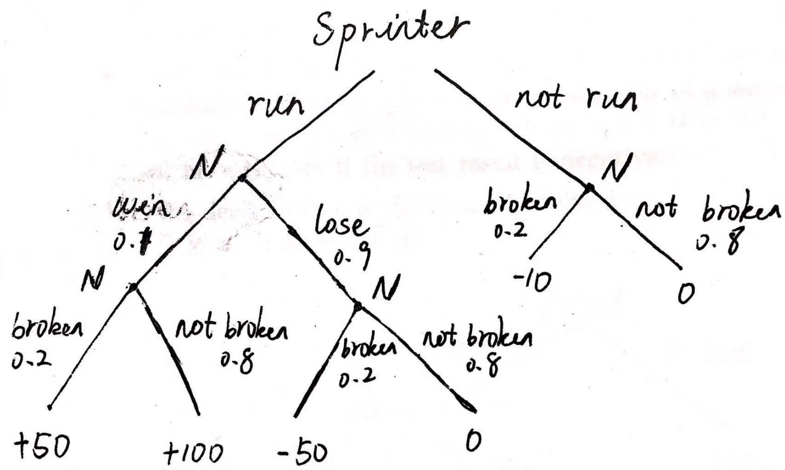
MATH 4321 - Game Theory

Mid-term Test Solution, 2019

1. (a) The decision tree is shown as follows:



The expected value of "run" is

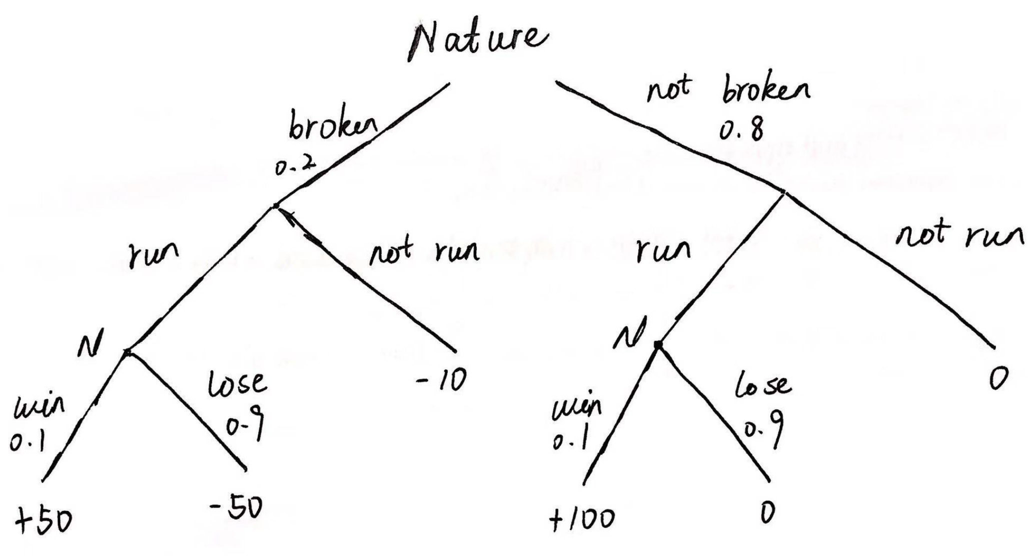
$$v(\text{run}) = 0.1 \times (50 \times 0.2 + 100 \times 0.8) + 0.9 \times (-50 \times 0.2 + 0 \times 0.8) = 0.$$

The expected value of "not run" is

$$v(\text{not run}) = -10 \times 0.2 + 0 \times 0.8 = -2 < 0.$$

Then the sprinter chooses to run with zero payoff, which is better than choosing not to run with negative payoff.

(b) Suppose the sprinter knows in advance from more tests the state of her leg, we have the following reformulated decision tree:



Given that she knows the leg is broken, we have the following expected payoffs:

$$v(\text{run}) = 50 \times 0.1 - 50 \times 0.9 = -40$$
$$v(\text{not run}) = -10 > -40.$$

The sprinter will choose not to run and gets expected payoff -10 if she knows in advance that the leg is already broken. Similarly, given that she knows the leg is not broken, we have

$$v(\text{run}) = 100 \times 0.1 + 0 \times 0.9 = 10$$
$$v(\text{not run}) = 0 < 10.$$

The sprinter will choose to run and gets expected payoff 10 if she knows in advance that the leg is broken. In conclusion, her expected payoff with perfect information of the state of her leg is given by

$$v = 0.2 \times (-10) + 0.8 \times 10 = 6.$$

Without the information, she will have expected payoff 0 (shown in (a)), Then the value of information about the state of her leg is equal to $6 - 0 = 6$.

2. (a) First we find that Column 3 is strictly dominated by Column 1 since $-4 < -1$, $-4 < 4$ and $0 < 5$, so we eliminate Column 3 and get

$$\begin{pmatrix} -4 & 2 \\ -4 & 1 \\ 0 & -1 \end{pmatrix}.$$

For $\lambda = \frac{3}{4}$, we have

$$-4 < -4(\lambda) + 0(1-\lambda) = -3, \quad 1 < 2(\lambda) - 1(1-\lambda) = \frac{5}{4}.$$

Then Row 2 is strictly dominated by a convex combination of Row 1 and Row 3, so we eliminate Row 2 and finally get a 2×2 reduced game matrix:

$$B = \begin{pmatrix} -4 & 2 \\ 0 & -1 \end{pmatrix}.$$

It is seen that $v^+ = 0$ and $v^- = -1$, so there is no pure strategy saddle point. Then the optimal strategies X^* and Y^* must be completely mixed for a 2×2 matrix. B^{-1} is found to be $B^{-1} = \begin{pmatrix} -\frac{1}{4} & -\frac{1}{2} \\ 0 & -1 \end{pmatrix}$. The value of the game and optimal mixed strategies are calculated by the method of invertible matrix game:

$$v = \frac{1}{J_2 B^{-1} J_2^T} = \frac{1}{(1 \quad 1)\begin{pmatrix} -\frac{1}{4} & -\frac{1}{2} \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}} = -\frac{4}{7},$$

$$Y^{*T} = vB^{-1}J_2^T = -\frac{4}{7}\begin{pmatrix} -\frac{1}{4} & -\frac{1}{2} \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{3}{7} \\ \frac{4}{7} \end{pmatrix},$$

$$X^* = vJ_2B^{-1} = -\frac{4}{7}(1 \quad 1)\begin{pmatrix} -\frac{1}{4} & -\frac{1}{2} \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} \frac{1}{7} & \frac{6}{7} \end{pmatrix}.$$

Then for the initial game matrix A, the optimal mixed strategies are $X^* = \left(\frac{1}{7}, 0, \frac{6}{7}\right)$ and $Y^* = \left(\frac{3}{7}, \frac{4}{7}, 0\right)$ and the value of the game is $v = -\frac{4}{7}$.

(b) Observing that the two players choose Row 1, Row 3, Column 1 and Column 2 with positive probability, we verify the Equality of Payoff Theorem as follows:

$$
\begin{aligned}
&x_1 = \frac{1}{7} > 0 \implies E(1, Y^*) = -4 \times \frac{3}{7} + 2 \times \frac{4}{7} + 0 = -\frac{4}{7} = v,\\
&x_3 = \frac{6}{7} > 0 \implies E(3, Y^*) = 0 - 1 \times \frac{4}{7} + 0 = -\frac{4}{7} = v,\\
&y_1 = \frac{3}{7} > 0 \implies E(X^*, 1) = -4 \times \frac{1}{7} + 0 + 0 = -\frac{4}{7} = v,\\
&y_2 = \frac{4}{7} > 0 \implies E(X^*, 2) = 2 \times \frac{1}{7} + 0 - 1 \times \frac{6}{7} = -\frac{4}{7} = v,\\
&E(2, Y^*) = -4 \times \frac{3}{7} + 1 \times \frac{4}{7} + 0 = -\frac{8}{7} < -\frac{4}{7} = v \implies x_2 = 0,\\
&E(X^*, 3) = -1 \times \frac{1}{7} + 0 + 5 \times \frac{6}{7} = \frac{29}{7} > -\frac{4}{7} = v \implies y_3 = 0.
\end{aligned}
$$

The Equality of Payoff Theorem holds for all the rows and columns. The last two inequalities are strict, which implies that Row 2 and Column 3 will not be played by the two players.

3. (a) The statement is true.

Proof by contradiction:

Suppose a strategy profile (s_i^*, s_{-i}^*) weakly Pareto-dominates all other strategy profiles and is not a Nash equilibrium, then $\exists s_i'$ such that

$$\pi_i(s_i^*, s_{-i}^*) < \pi_i(s_i', s_{-i}^*).$$

It implies that strategy profile (s_i', s_{-i}^*) is not Pareto-dominated by strategy profile (s_i^*, s_{-i}^*) since there is one player i who achieves higher payoff under (s_i', s_{-i}^*). There is a contradiction.

Then we conclude that if a strategy profile weakly Pareto-dominates all other strategy profiles, it must be a Nash equilibrium.

(b) (i) True. Proof by contradiction. Suppose an iterated dominance equilibrium (s_i^*, s_{-i}^*) is not a Nash equilibrium, then $\exists\, s_i'$, such that

$$\pi_i(s_i^*, s_{-i}^*) < \pi_i(s_i', s_{-i}^*).$$

By the definition of iterated dominance equilibrium, the strategy profile (s_i^*, s_{-i}^*) must be the unique survivor after elimination of weakly dominated strategies. It implies that strategy s_i' must have been dominated by s_i^* for some reduced game matrix and must have been eliminated. However, the above inequality suggests that s_i' is not dominated by s_i^* for any reduced matrix containing strategy profile (s_i^*, s_{-i}^*). There is a contradiction. Therefore, if any strategy profile (s_i^*, s_{-i}^*) is an iterated dominance equilibrium, it must be a Nash equilibrium.

(ii) False. Consider the following game matrix as an example:

	Col1	Col2
Row1	(5,5)	(1,1)
Row2	(3,3)	(4,4)

We observe that the strategy profile (Row1, Col1) Pareto-dominates all other strategy profiles and is a Nash equilibrium. However, it is not an iterated dominance equilibrium observing that there is no weakly dominated row or column and we cannot do any elimination in this case.

4. (a) The game matrix (with the police officer being the row player) can be represented by

	prowl	hide
street	$(20, -10)$	$(0, 0)$
coffee	$(10, 10)$	$(10, 0)$

(b) Let the strategy of the police officer be $X = (x, 1-x)$ and the strategy of the robber be $Y = (y, 1-y)$.

- For the police officer:
  If the police officer chooses to patrol the street, his expected payoff should be

$$E_1(1, Y) = 20y + 0(1-y) = 20y.$$

  If the police officer chooses to hang out at the coffee shop, his expected payoff will always be

$$E_1(2, Y) = 10.$$

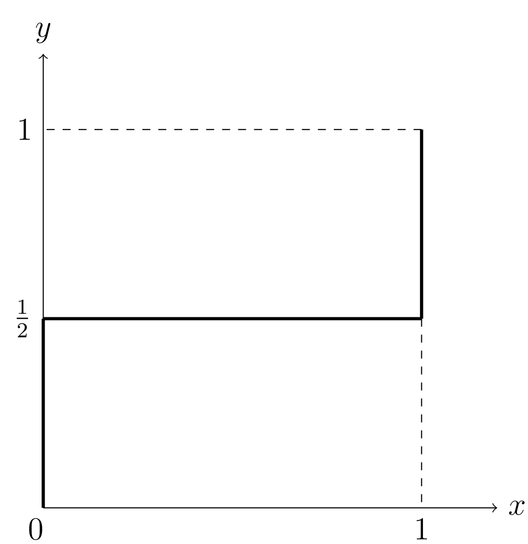
  The police officer feels indifferent between patrolling the street and hanging out at the coffee shop when

$$E_1(1, Y) = E_1(2, Y) \implies 20y = 10 \implies y = \frac{1}{2}.$$

  Then we get

$$BR_1(y) = \begin{cases} 0 & \text{if } y < \frac{1}{2} \\ [0,1] & \text{if } y = \frac{1}{2} \\ 1 & \text{if } y > \frac{1}{2} \end{cases},$$

  which is plotted as following:



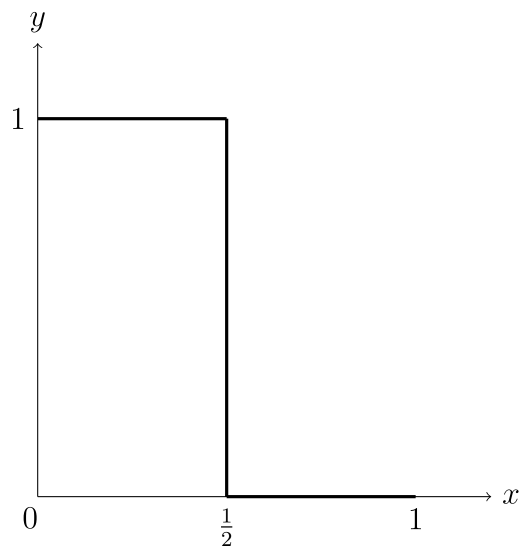
- For the robber:
  The robber will feel indifferent between prowling the streets and staying hidden when

$$E_2(X, 1) = E_2(X, 2) \implies -10x + 10(1-x) = 0 \implies x = \frac{1}{2}.$$

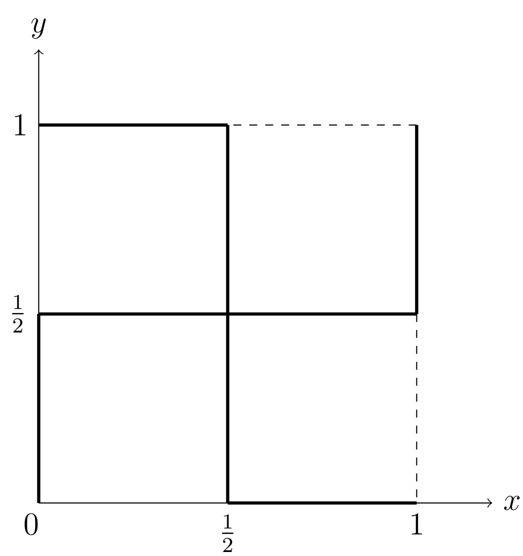
Then we get

$$BR_2(x) = \begin{cases} 0 & \text{if } x > \dfrac{1}{2} \\ [0,1] & \text{if } x = \dfrac{1}{2} \\ 1 & \text{if } x < \dfrac{1}{2} \end{cases},$$

which is plotted as following:



(c) Intuitively, a strategy profile s^* is a Nash equilibrium if and only if every player's action is a best response to the other player's strategy choices so that no one has incentive to deviate from the equilibrium given that the strategy of her opponent stays unchanged. It implies that a Nash equilibrium can be found as the intersection point of the best response functions for both players. We plot the two best response functions in (b) together and get



The Nash equilibrium corresponds to the intersection point of the two best response functions, which is $(x^*, y^*) = (\frac{1}{2}, \frac{1}{2})$. Then the unique Nash equilibrium should be $X^* = (\frac{1}{2}, \frac{1}{2})$ and $Y^* = (\frac{1}{2}, \frac{1}{2})$.

Given the Nash equilibrium, values of the game for both players are given by

$$E_1(X^*, Y^*) = E_1(2, Y^*) = 10 \quad \text{and} \quad E_2(X^*, Y^*) = E(X^*, 2) = 0$$

for the police officer and the robber, respectively.

5. (a) Let E denote immediate exit, T denote exit this quarter, and N denote exit next quarter. The game matrix is represented as below:

I \ II	E	T	N
E	$(0, 0)$	$(0, 2)$	$(0, 4)$
T	$(2, 0)$	$(-1, -1)$	$(-1, 1)$
N	$(4, 0)$	$(1, -1)$	$(-2, -2)$

(b) There are no strictly dominated strategy but there is a weakly dominated one: T. To see this, we consider a convex combination of the payoffs for strategy E and N. When we have

$$2 \leq 0(\lambda) + 4(1-\lambda), \quad -1 \leq 0(\lambda) + 1(1-\lambda) \quad \text{and} \quad -1 \leq 0(\lambda) - 2(1-\lambda),$$

which is equivalent to $\frac{1}{2} \leq \lambda \leq \frac{1}{2}$, there is only one value of $\lambda = \frac{1}{2}$ satisfying the condition above. It corresponds to payoffs

$$0(\lambda) + 4(1-\lambda) = 2, \quad 0(\lambda) + 1(1-\lambda) = \frac{1}{2} > -1 \quad \text{and} \quad 0(\lambda) - 2(1-\lambda) = -1.$$

Therefore, strategy T is weakly dominated by a convex combination of E and N for both players.

(c) Consider the best response of Player I:

- When Player II chooses E, Player I will choose N and get the maximized payoff 4.
- When Player II chooses T, Player I will still choose N for the maximized payoff 1.
- When Player II chooses N, Player I will choose E and get zero payoff to avoid losing more.

Since the two players are identical, the best response for Player II is similar to Player I. From the above discussion, we conclude that the pure strategy Nash equilibria should be (E, N) and (N, E).